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Research Article

On Fibonacci Functions with Period k

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A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function if $f(x+2) = f(x+1) + f(x)$ for all $x \in \mathbb{R}$. In 2012, some properties on the Fibonacci functions were presented. In this paper, for any positive integer k, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period k if $f(x+2k) = f(x+k) + f(x)$ for all $x \in \mathbb{R}$; we present some properties on the Fibonacci functions with period k.

1. Introduction

Presently, there are many research articles about Fibonacci numbers (see [1]). Fibonacci numbers are also involved in the golden ratio (see [2]). In 2008, Kim and Neggers [3] studied Fibonacci means. In 2009, Jung [4] studied Hyers-Ulam stability of Fibonacci functional equation. In 2010, Han et al. [5] studied a Fibonacci norm of positive integers. In 2012, Han et al. [6] studied Fibonacci sequences in groupoids. Moreover, they [7] gave some properties on Fibonacci functions; a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function if $f(x+2) = f(x+1) + f(x)$, for all $x \in \mathbb{R}$, using the concept of f-even and f-odd functions. They also showed that if f is a Fibonacci function, then $\lim_{x\to\infty} f(x+1)/f(x) = (1+\sqrt{5})/2$.

In this paper, for any positive integer k, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period k if $f(x+2k) = f(x+k) + f(x)$ for all $x \in \mathbb{R}$; we present some properties on the Fibonacci functions with period k using the concept of f-even and f-odd functions with period k. Moreover, we also present some properties on the odd Fibonacci functions with period k.

2. Fibonacci Functions with Period k

Definition 1. Let k be a positive integer. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period k if $f(x+2k) = f(x+k) + f(x)$ for all $x \in \mathbb{R}$.

Example 2. Let $f(x) = a^{x/k}$ be a Fibonacci function with period $k \in \mathbb{N}$, where $a > 0$. It follows that $a^{(x/k)+2} = a^{(x/k)+1} + a^{x/k}$ for all $x \in \mathbb{R}$, so $a^2 = a + 1$. Then $a = (1+\sqrt{5})/2$. Thus, $f(x) = ((1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$.

Proposition 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$. Assume that f is differentiable. Then f' is also a Fibonacci function with period k.*

Proof. Let $x \in \mathbb{R}$. Since $f(x+2k) = f(x+k) + f(x)$, it follows that $f'(x+2k) = f'(x+k) + f'(x)$. □

Proposition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$, and define $g_t(x) = f(x+t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then g_t is also a Fibonacci function with period k.*

Proof. Let $x \in \mathbb{R}$. Then $g_t(x+2k) = f(x+2k+t) = f(x+t+k) + f(x+t) = g_t(x+k) + g_t(x)$. □

Example 5. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by $g_t(x) = ((1+\sqrt{5})/2)^{(x+t)/k}$ for all $x \in \mathbb{R}$. Then g_t is a Fibonacci function with period k.

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$, and let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of Fibonacci numbers with $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(x+nk) = F_n f(x+k) + F_{n-1} f(x)$.*

Proof. Let $x \in \mathbb{R}$. We note that $f(x+k) = F_1 f(x+k) + F_0 f(x)$ and $f(x+2k) = F_2 f(x+k) + F_1 f(x)$. Now, we assume that $f(x+nk) = F_n f(x+k) + F_{n-1} f(x)$ and $f(x+(n+1)\text{k}) = F_{n+1} f(x+k) + F_n f(x)$, where $n \in \mathbb{N}$. Then

$$\begin{aligned} f(x+(n+2)k) &= f(x+(n+1)k) + f(x+nk) \\ &= F_{n+1} f(x+k) + F_n f(x) + F_n f(x+k) + F_{n-1} f(x) \\ &= (F_{n+1} + F_n) f(x+k) + (F_n + F_{n-1}) f(x) \\ &= F_{n+2} f(x+k) + F_{n+1} f(x). \end{aligned} \quad (1)$$

This proof is completed. □

3. Odd Fibonacci Functions with Period k

Definition 7. Let k be a positive integer. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an odd Fibonacci function with period k if $f(x+2k) = -f(x+k) + f(x)$ for all $x \in \mathbb{R}$.

Example 8. Let $f(x) = a^{x/k}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, where $a > 0$. It follows that $a^{(x/k)+2} = -a^{(x/k)+1} + a^{x/k}$ for all $x \in \mathbb{R}$, so $a^2 = -a + 1$. Then $a = (-1+\sqrt{5})/2$. Thus, $f(x) = ((-1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$.

Proposition 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$. Assume that f is differentiable. Then f' is also an odd Fibonacci function with period k.*

Proof. Let $x \in \mathbb{R}$. Since $f(x+2k) = -f(x+k) + f(x)$, it follows that $f'(x+2k) = -f'(x+k) + f'(x)$. □

Proposition 10. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, and define $g_t(x) = f(x+t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then g_t is also an odd Fibonacci function with period k.*

Proof. Let $x \in \mathbb{R}$. Then $g_t(x+2k) = f(x+2k+t) = -f(x+t+k) + f(x+t) = -g_t(x+k) + g_t(x)$. □

Example 11. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by $g_t(x) = ((-1+\sqrt{5})/2)^{(x+t)/k}$ for all $x \in \mathbb{R}$. Then g_t is an odd Fibonacci function with period k.

Theorem 12. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, and let $\{F_{-n}\}_{n\in\mathbb{N}}$ be a sequence of Fibonucci numbers with $F_0 = 0$, $F_{-1} = 1$, and $F_{-n-1} = -F_{-n} + F_{-n+1}$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(x+nk) = F_{-n} f(x+k) + F_{-n+1} f(x)$.*

Proof. Let $x \in \mathbb{R}$. We note that $f(x+k) = F_{-1} f(x+k) + F_0 f(x)$ and $f(x+2k) = F_{-2} f(x+k) + F_{-1} f(x)$. Now, we assume that $f(x+nk) = F_{-n} f(x+k) + F_{-n+1} f(x)$ and $f(x+(n+1)k) = F_{-n-1} f(x+k) + F_{-n} f(x)$, where $n \in \mathbb{N}$. Then

$$\begin{aligned} f(x+(n+2)k) &= -f(x+(n+1)k) + f(x+nk) \\ &= -(F_{-n-1} f(x+k) + F_{-n} f(x)) \\ &\quad + F_{-n} f(x+k) + F_{-n+1} f(x) \\ &= (-F_{-n-1} + F_{-n}) f(x+k) \\ &\quad + (-F_{-n} + F_{-n+1}) f(x) \\ &= F_{-n-2} f(x+k) + F_{-n-1} f(x). \end{aligned} \quad (2)$$

This proof is completed. □

4. f-Even Functions with Period k

Definition 13. Let k be a positive integer and let $\alpha : \mathbb{R} \to \mathbb{R}$ be such that if $\alpha h = 0$, where $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h = 0$. The function α is said to be an f-even function with period k if $\alpha(x+k) = \alpha(x)$ for all $x \in \mathbb{R}$.

Example 14. Define $\alpha(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\alpha h = 0$. For any $x \notin \mathbb{Z}$, we have $\alpha(x) \neq 0$, so $h(x) = 0$. Since $\mathbb{R} \setminus \mathbb{Z}$ is dense in $\mathbb{R}$ and h is continuous, it follows that $h = 0$. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\alpha(x+k) = x + k - \lfloor x+k \rfloor = x + k - \lfloor x \rfloor - k = x - \lfloor x \rfloor = \alpha(x)$. Hence, α is an f-even function with period k.

Theorem 15. *Let $k \in \mathbb{N}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ be an f-even function with period k and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then g is a Fibonacci function with period k if and only if αg is a Fibonacci function with period k.*

Proof. First, we assume that g is a Fibonacci function with period k. For any $x \in \mathbb{R}$, we have

$$\begin{aligned} (\alpha g)(x+2k) &= \alpha(x+2k) g(x+2k) \\ &= \alpha(x+k)(g(x+k) + g(x)) \\ &= \alpha(x+k) g(x+k) + \alpha(x+k) g(x) \\ &= \alpha(x+k) g(x+k) + \alpha(x) g(x) \\ &= (\alpha g)(x+k) + (\alpha g)(x). \end{aligned} \quad (3)$$

Hence, αg is a Fibonacci function with period k.

Next, we assume that αg is a Fibonacci function with period k. Let $x \in \mathbb{R}$. Then

$$\begin{aligned} \alpha(x+k) g(x+2k) &= \alpha(x+2k) g(x+2k) \\ &= (\alpha g)(x+2k) \\ &= (\alpha g)(x+k) + (\alpha g)(x) \end{aligned}$$

$$\begin{aligned}
&= \alpha(x+k)\,g(x+k) + \alpha(x)\,g(x) \\
&= \alpha(x+k)\,g(x+k) + \alpha(x+k)\,g(x) \\
&= \alpha(x+k)\big(g(x+k) + g(x)\big).
\end{aligned} \tag{4}$$

By the assumption of α, we obtain that $g(x+2k) = g(x+k)+g(x)$. Hence, g is a Fibonacci function with period k. □

Example 16. Let $k \in \mathbb{N}$. Define $\alpha(x) = x - \lfloor x \rfloor$ and $g(x) = ((1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. For all $x \in \mathbb{R}$, we have $\alpha g(x) = (x - \lfloor x \rfloor)((1+\sqrt{5})/2)^{x/k}$. We recall that α is an f-even function with period k, and g is a Fibonacci function with period k. Hence, αg is a Fibonacci function with period k.

Theorem 17. *Let $k \in \mathbb{N}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ be an f-even function with period k and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then g is an odd Fibonacci function with period k if and only if αg is an odd Fibonacci function with period k.*

Proof. First, we assume that g is an odd Fibonacci function with period k. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&(\alpha g)(x+2k) \\
&\quad = \alpha(x+2k)\,g(x+2k) \\
&\quad = \alpha(x+k)\big(-g(x+k) + g(x)\big) \\
&\quad = -\alpha(x+k)\,g(x+k) + \alpha(x+k)\,g(x) \\
&\quad = -\alpha(x+k)\,g(x+k) + \alpha(x)\,g(x) \\
&\quad = -(\alpha g)(x+k) + (\alpha g)(x).
\end{aligned} \tag{5}$$

Hence, αg is an odd Fibonacci function with period k.

Next, we assume that αg is an odd Fibonacci function with period k. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}
&\alpha(x+k)\,g(x+2k) \\
&\quad = \alpha(x+2k)\,g(x+2k) \\
&\quad = (\alpha g)(x+2k) \\
&\quad = -(\alpha g)(x+k) + (\alpha g)(x) \\
&\quad = -\alpha(x+k)\,g(x+k) + \alpha(x)\,g(x) \\
&\quad = -\alpha(x+k)\,g(x+k) + \alpha(x+k)\,g(x) \\
&\quad = \alpha(x+k)\big(-g(x+k) + g(x)\big).
\end{aligned} \tag{6}$$

By the assumption of α, we obtain that $g(x+2k) = -g(x+k)+g(x)$. Hence, g is an odd Fibonacci function with period k. □

Example 18. Let $k \in \mathbb{N}$. Define $\alpha(x) = x - \lfloor x \rfloor$ and $g(x) = ((-1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. For all $x \in \mathbb{R}$, we have $\alpha g(x) = (x - \lfloor x \rfloor)((-1+\sqrt{5})/2)^{x/k}$. We recall that α is an f-even function with period k and g is an odd Fibonacci function with period k. Hence, αg is an odd Fibonacci function with period k.

5. f-Odd Functions with Period k

Definition 19. Let k be a positive integer and let $\alpha : \mathbb{R} \to \mathbb{R}$ be such that if $\alpha h = 0$ where $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h = 0$. The function α is said to be an f-odd function with period k if $\alpha(x+k) = -\alpha(x)$ for all $x \in \mathbb{R}$.

Example 20. Define $\alpha(x) = \sin(\pi x)$ for all $x \in \mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\alpha h = 0$. For any $x \notin \pi\mathbb{Z}$, we have $\alpha(x) \neq 0$, so $h(x) = 0$. Since $\mathbb{R} \setminus \pi\mathbb{Z}$ is dense in $\mathbb{R}$ and h is continuous, it follows that $h = 0$. Let k be a positive odd integer and $x \in \mathbb{R}$. Then $\alpha(x+k) = \sin(\pi(x+k)) = \sin(\pi x + \pi k) = -\sin(\pi x) = -\alpha(x)$. Hence, α is an f-even function with period k.

Theorem 21. *Let $k \in \mathbb{N}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ be an f-odd function with period k and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then g is a Fibonacci function with period k if and only if αg is an odd Fibonacci function with period k.*

Proof. First, we assume that g is a Fibonacci function with period k. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&(\alpha g)(x+2k) \\
&\quad = \alpha(x+2k)\,g(x+2k) \\
&\quad = -\alpha(x+k)\big(g(x+k) + g(x)\big) \\
&\quad = -\alpha(x+k)\,g(x+k) - \alpha(x+k)\,g(x) \\
&\quad = -\alpha(x+k)\,g(x+k) + \alpha(x)\,g(x) \\
&\quad = -(\alpha g)(x+k) + (\alpha g)(x).
\end{aligned} \tag{7}$$

Hence, αg is an odd Fibonacci function with period k.

Next, we assume that αg is an odd Fibonacci function with period k. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}
&\alpha(x+k)\,g(x+2k) \\
&\quad = -\alpha(x+2k)\,g(x+2k) \\
&\quad = -(\alpha g)(x+2k) \\
&\quad = -\big(-(\alpha g)(x+k) + (\alpha g)(x)\big) \\
&\quad = (\alpha g)(x+k) - (\alpha g)(x) \\
&\quad = \alpha(x+k)\,g(x+k) - \alpha(x)\,g(x) \\
&\quad = \alpha(x+k)\,g(x+k) + \alpha(x+k)\,g(x) \\
&\quad = \alpha(x+k)\big(g(x+k) + g(x)\big).
\end{aligned} \tag{8}$$

By the assumption of α, we obtain that $g(x+2k) = g(x+k)+g(x)$. Hence, g is a Fibonacci function with period k. □

Example 22. Let k be a positive odd integer. Define $\alpha(x) = \sin(\pi x)$ and $g(x) = ((1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. We have $\alpha g(x) = (\sin(\pi x))((1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. We recall that α is an f-odd function with period k and g is a Fibonacci function with period k. Hence, αg is an odd Fibonacci function with period k.

Theorem 23. *Let $k \in \mathbb{N}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ be an f-odd function with period k and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then g is an odd Fibonacci function with period k if and only if αg is a Fibonacci function with period k.*

Proof. First, we assume that g is an odd Fibonacci function with period k. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}(\alpha g)(x+2k) &= \alpha(x+2k)\,g(x+2k) \\ &= -\alpha(x+k)\left(-g(x+k)+g(x)\right) \\ &= \alpha(x+k)\,g(x+k) - \alpha(x+k)\,g(x) \\ &= \alpha(x+k)\,g(x+k) + \alpha(x)\,g(x) \\ &= (\alpha g)(x+k) + (\alpha g)(x).\end{aligned} \tag{9}$$

Hence, αg is a Fibonacci function with period k.

Next, we assume that αg is a Fibonacci function with period k. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}\alpha(x+k)\,g(x+2k) &= -\alpha(x+2k)\,g(x+2k) \\ &= -(\alpha g)(x+2k) \\ &= -\left((\alpha g)(x+k)+(\alpha g)(x)\right) \\ &= -(\alpha g)(x+k) - (\alpha g)(x) \\ &= -\alpha(x+k)\,g(x+k) - \alpha(x)\,g(x) \\ &= -\alpha(x+k)\,g(x+k) + \alpha(x+k)\,g(x) \\ &= \alpha(x+k)\left(-g(x+k)+g(x)\right).\end{aligned} \tag{10}$$

By the assumption of α, we obtain that $g(x+2k) = -g(x+k)+g(x)$. Hence, g is an odd Fibonacci function with period k. □

Example 24. Let k be a positive odd integer. Define $\alpha(x) = \sin(\pi x)$ and $g(x) = ((-1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. We have $\alpha g(x) = (\sin(\pi x))((-1+\sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$. We recall that α is an f-odd function with period k and g is an odd Fibonacci function with period k. Hence, αg is a Fibonacci function with period k.

6. Open Problems

Conjecture 25. *If f is a Fibonacci function with period $k \in \mathbb{N}$, then*

$$\lim_{x\to\infty} \frac{f(x+k)}{f(x)} = \frac{1+\sqrt{5}}{2}. \tag{11}$$

Conjecture 26. *If f is an odd Fibonacci function with period $k \in \mathbb{N}$, then*

$$\lim_{x\to\infty} \frac{f(x+k)}{f(x)} = \frac{-1-\sqrt{5}}{2}. \tag{12}$$

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