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Research Article

Complete Convergence of the Maximum Partial Sums for Arrays of Rowwise of AANA Random Variables

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The limiting behavior of the maximum partial sums $(1/a_n)\max_{1\le j\le n}|\sum_{i=1}^{j} X_{ni}|$ is investigated, and some new results are obtained, where $\{X_{ni}, i \ge 1, n \ge 1\}$ is an array of rowwise AANA random variables and $\{a_n, n \ge 1\}$ is a sequence of positive real numbers. As an application, the Chung-type strong law of large numbers for arrays of rowwise AANA random variables is obtained. The results extend and improve the corresponding ones of Hu and Taylor (1997) for arrays of rowwise independent random variables.

1. Introduction

Let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$ with value in a real space $\mathbb{R}$. We say that the sequence $\{X_n, n \ge 1\}$ satisfies the general strong law of large numbers if there exist some increasing sequence $\{b_n, n \ge 1\}$ and some sequence $\{a_n, n \ge 1\}$ such that

$$\frac{1}{b_n}\sum_{i=1}^{n}(X_i - a_i) \longrightarrow 0 \text{ a.s.} \quad \text{as } n \longrightarrow \infty. \tag{1}$$

For the definition of general strong law of large numbers, one can refer to Chow and Teicher [1] or Kuczmaszewska [2]. Many authors have extended the strong law of large numbers for sequences of random variables to the case of triangular array of rowwise random variables and arrays of rowwise random variables. In the case of independence, Hu and Taylor [3] proved the following strong law of large numbers.

Theorem A. *Let $\{X_{ni}: 1 \le i \le n, n \ge 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $g(t)$ be a positive, even function such that $g(|t|)/|t|^p$ is an increasing function of $|t|$ and $g(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{g(|t|)}{|t|^p}\uparrow, \quad \frac{g(|t|)}{|t|^{p+1}}\downarrow, \quad \text{as } |t|\uparrow \tag{2}$$

for some nonnegative integer p. If $p \ge 2$ and

$$\begin{gathered} EX_{ni} = 0, \\ \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{g(|X_{ni}|)}{g(a_n)} < \infty, \\ \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2\right)^{2k} < \infty, \end{gathered} \tag{3}$$

where k is a positive integer, then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \text{ a.s.} \tag{4}$$

Gan and Chen [4] extended and improved the result of Theorem A for rowwise independent random variable arrays to the case of rowwise negatively associated (NA, in short) random variable arrays. For the details about negatively associated random variables, one can refer to Joag-Dev and Proschan [5], Wu and Jiang [6, 7], Wu [8], Wang et al. [9], and so forth. In this paper, we will further study the strong law of large numbers for arrays of rowwise asymptotically almost negatively associated random variables based on some different conditions.

The concept of asymptotically almost negatively associated random variables was introduced by Chandra and Ghosal [10] as follows.

Definition 1. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $u(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &\operatorname{Cov}\left(f\left(X_n\right), g\left(X_{n+1}, X_{n+2}, \ldots, X_{n+k}\right)\right) \\ &\quad \leq u\left(n\right)\left[\operatorname{Var}\left(f\left(X_n\right)\right) \operatorname{Var}\left(g\left(X_{n+1}, X_{n+2}, \ldots, X_{n+k}\right)\right)\right]^{1/2} \end{aligned} \tag{5}$$

for all n, $k \geq 1$ and for all coordinatewise nondecreasing continuous functions f and g whenever the variances exist.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise AANA random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of AANA random variables.

It is easily seen that the family of AANA sequence contains NA (in particular, independent) sequences (with $u(n) = 0$, $n \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated. An example of an AANA sequence which is not NA was constructed by Chandra and Ghosal [10]. Hence, extending the limit properties of independent or NA random variables to the case of AANA random variables is highly desirable in the theory and application.

Since the concept of AANA sequence was introduced by Chandra and Ghosal [10], many applications have been found. See, for example, Chandra and Ghosal [10] who derived the Kolmogorov-type inequality and the strong law of large numbers of Marcinkiewicz-Zygmund; Chandra and Ghosal [11] obtained the almost sure convergence of weighted averages; Wang et al. [12] established the law of the iterated logarithm for product sums; Ko et al. [13] studied the Hájek-Rényi-type inequality; Yuan and An [14] established some Rosenthal-type inequalities for maximum partial sums of AANA sequence; Wang et al. [15] obtained some strong growth rate and the integrability of supremum for the partial sums of AANA random variables; Wang et al. [16] established some maximal inequalities and strong law of large numbers for AANA sequences; Wang et al. [17, 18] studied complete convergence for arrays of rowwise AANA random variables and weighted sums of arrays of rowwise AANA random variables, respectively; Hu et al. [19] studied the strong convergence properties for AANA sequence; Yang et al. [20] investigated the complete convergence, complete moment convergence, and the existence of the moment of supremum of normed partial sums for the moving average process for AANA sequence; and so forth.

The main purpose of this paper is to study the limiting behavior of the maximum partial sums for arrays of rowwise AANA random variables. As an application, the Chung-type strong law of large numbers for arrays of rowwise AANA random variables is obtained. We will give some sufficient conditions for the complete convergence for an array of rowwise AANA random variables without assumptions of identical distribution and stochastic domination. The results presented in this paper are obtained by using the truncated method and the maximal Rosenthal-type inequality of AANA random variables.

Throughout this paper, let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise of AANA random variables with the mixing coefficients $\{u(i), i \geq 1\}$ in each row. The symbol C denotes a positive constant which may be different in various places.

2. Preliminaries

To prove the main results of the paper, we need the following two lemmas.

Lemma 2 (see [18, Lemma 2.2]). *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{u(n), n \geq 1\}$, and let $f_1, f_2, \ldots$ be all nondecreasing (or all nonincreasing) continuous functions, and then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{u(n), n \geq 1\}$.*

Lemma 3 (see [14, Theorem 2.1]). *Let $p > 1$ and let $\{X_n, n \geq 1\}$ be a sequence of zero mean random variables with mixing coefficients $\{u(n), n \geq 1\}$.*

If $\sum_{n=1}^{\infty} u^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that for all $n \geq 1$ and $1 < p \leq 2$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_p \sum_{i=1}^{n} E\left|X_i\right|^p. \tag{6}$$

If $\sum_{n=1}^{\infty} u^{1/(p-1)}(n) < \infty$ for some $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where the integer number $k \geq 1$, then there exists a positive constant D_p depending only on p such that for all $n \geq 1$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq D_p \left\{\sum_{i=1}^{n} E\left|X_i\right|^p + \left(\sum_{i=1}^{n} E X_i^2\right)^{p/2}\right\}. \tag{7}$$

3. Main Results and Proofs

In this section, we will investigate the limiting behavior of the maximum partial sums for arrays of rowwise of AANA random variables. The first three theorems consider different conditions from Hu and Taylor's [3].

Theorem 4. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Let $\{g_n(x), n \geq 1\}$ be a sequence of nonnegative, even functions such that $g_n(|x|)$ is an increasing function of $|x|$ for every $n \geq 1$. Assume that there exists a constant $\delta > 0$ such that $g_n(x) \geq \delta x$ for $0 < x \leq 1$. If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E g_i\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{8}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j} X_{ni}\right| > \varepsilon\right) < \infty. \tag{9}$$

Proof. For fixed $n \geq 1$, define

$$X_i^{(n)} = -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n), \quad i \geq 1,$$
$$T_j^{(n)} = \frac{1}{a_n}\sum_{i=1}^{j}\left(X_i^{(n)} - EX_i^{(n)}\right), \quad j = 1, 2, \ldots, n. \tag{10}$$

By Lemma 2, we can see that for fixed $n \geq 1$, $\{X_i^{(n)}, i \geq 1\}$ is still a sequence of AANA random variables. It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\right| > \varepsilon\right) \subset \left(\max_{1\leq i\leq n}|X_{ni}| > a_n\right) \bigcup \left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_i^{(n)}\right| > \varepsilon\right), \tag{11}$$

which implies that

$$\begin{aligned} P&\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\right| > \varepsilon\right) \\ &\leq P\left(\max_{1\leq i\leq n}|X_{ni}| > a_n\right) + P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_i^{(n)}\right| > \varepsilon\right) \\ &\leq \sum_{i=1}^{n}P(|X_{ni}| > a_n) + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right|\right). \end{aligned} \tag{12}$$

Firstly, we will show that

$$\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{13}$$

In fact, by the conditions $g_n(x) \geq \delta x$ for $0 < x \leq 1$ and (8), we have

$$\begin{aligned} \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| &\leq \sum_{i=1}^{n}P(|X_{ni}| > a_n) + \sum_{i=1}^{n}E\left(\frac{|X_{ni}|}{a_n}I(|X_{ni}| \leq a_n)\right) \\ &\leq \frac{1}{\delta}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right) + \frac{1}{\delta}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right)I(|X_{ni}| \leq a_n) \\ &\leq \frac{2}{\delta}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{14}$$

which implies (13).

It follows from (12) and (13) that for n large enough,

$$P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\right| > \varepsilon\right) \leq \sum_{i=1}^{n}P(|X_{ni}| > a_n) + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right). \tag{15}$$

Hence, to prove (9), we only need to show that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}| > a_n) < \infty, \tag{16}$$

$$\sum_{n=1}^{\infty}P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) < \infty. \tag{17}$$

When $|X_{ni}| > a_n > 0$, we have $g_i(X_{ni}/a_n) \geq g_i(1) \geq \delta$, which yields that

$$P(|X_{ni}| > a_n) = EI(|X_{ni}| > a_n) \leq \frac{1}{\delta}Eg_i\left(\frac{X_{ni}}{a_n}\right). \tag{18}$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}| > a_n) \leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{19}$$

which implies (16).

By Markov's inequality, Lemma 3 (for $p = 2$), $g_n(x) \geq \delta x$ for $0 < x \leq 1$, and (8), we can get that

$$\begin{aligned} \sum_{n=1}^{\infty}&P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) \\ &\leq C\sum_{n=1}^{\infty}E\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right|^2\right) \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}E\left|X_i^{(n)} - EX_i^{(n)}\right|^2 \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}E\left|X_i^{(n)}\right|^2 \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}| > a_n) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}| \leq a_n)}{a_n^2} \\ &\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}| I(|X_{ni}| \leq a_n)}{a_n} \\ &\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right)I(|X_{ni}| \leq a_n) \\ &\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty, \end{aligned} \tag{20}$$

which implies (17). This completes the proof of the theorem. □

Corollary 5. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. If there exists a constant $\beta \in (0,1]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{|a_n|^{\beta} + |X_{ni}|^{\beta}}\right) < \infty, \tag{21}$$

then (9) holds for any $\varepsilon > 0$.

Proof. In Theorem 4, we take

$$g_n(x) \equiv \frac{|x|^{\beta}}{1 + |x|^{\beta}}, \quad 0 < \beta \leq 1,\ n \geq 1. \tag{22}$$

It is easy to check that $\{g_n(x), n \geq 1\}$ is a sequence of nonnegative, even functions such that $g_n(|x|)$ is an increasing function of $|x|$ for every $n \geq 1$. And

$$g_n(x) \geq \frac{1}{2}x^{\beta} \geq \frac{1}{2}x, \quad 0 < x \leq 1,\ 0 < \beta \leq 1. \tag{23}$$

Therefore, by Theorem 4, we can easily get (9). □

Corollary 6. *Under the conditions of Theorem 4 or Corollary 5,*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \ a.s. \tag{24}$$

Theorem 7. *Let $\{X_{ni}: i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. $EX_{ni} = 0$, $i \geq 1$, $n \geq 1$. Let $\{g_n(x), n \geq 1\}$ be a sequence of nonnegative, even functions. Assume that there exist $\beta \in (1,2]$ and $\delta > 0$ such that $g_n(x) \geq \delta x^{\beta}$ for $0 < x \leq 1$ and there exists a $\delta > 0$ such that $g_n(x) \geq \delta x$ for $x > 1$. If (8) satisfies, then (9) holds for any $\varepsilon > 0$.*

Proof. We use the same notations as those in Theorem 4. The proof is similar to that of Theorem 4.

Firstly, we will show that (13) holds true. Actually, by the conditions $EX_{ni} = 0$, $g_n(x) \geq \delta x$ for $x > 1$, and (8), we have

$$\begin{aligned}\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j} EX_i^{(n)}\right| &\leq \sum_{i=1}^{n} P(|X_{ni}| > a_n) \\ &\quad + \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j} EX_{ni} I(|X_{ni}| > a_n)\right| \\ &\leq 2\sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| > a_n)\right) \\ &\leq \frac{2}{\delta}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| > a_n) \\ &\leq \frac{2}{\delta}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{25}$$

which implies (13). Hence, to prove (9), we only need to show that (16) and (17) hold true.

The conditions $g_n(x) \geq \delta x$ for $x > 1$ and (8) yield that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} EI(|X_{ni}| > a_n) \\ &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| > a_n)\right) \\ &\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| > a_n) \\ &\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty,\end{aligned} \tag{26}$$

which implies (16).

By Markov's inequality, Lemma 3 (for $p = 2$), $g_n(x) \geq \delta x^{\beta}$ for $1 < \beta \leq 2$, $0 < x \leq 1$, and (8), we can get that

$$\begin{aligned}&\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) \\ &\quad \leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) \\ &\quad\quad + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^2 I(|X_{ni}| \leq a_n)}{a_n^2} \\ &\quad \leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^{\beta} I(|X_{ni}| \leq a_n)}{a_n^{\beta}} \\ &\quad \leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \leq a_n) \\ &\quad \leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty,\end{aligned} \tag{27}$$

which implies (17). This completes the proof of the theorem. □

Corollary 8. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. $EX_{ni} = 0$, $i \geq 1$, $n \geq 1$. If there exists a constant $\beta \in (1,2]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{a_n|X_{ni}|^{\beta-1} + a_n^{\beta}}\right) < \infty, \tag{28}$$

then (9) holds for any $\varepsilon > 0$.

Proof. In Theorem 7, we take

$$g_n(x) \equiv \frac{|x|^{\beta}}{1 + |x|^{\beta-1}}, \quad 1 < \beta \leq 2,\ n \geq 1. \tag{29}$$

It is easy to check that $\{g_n(x), n \geq 1\}$ is a sequence of nonnegative, even functions satisfying

$$g_n(x) \geq \frac{1}{2}x^{\beta}, \quad 0 < x \leq 1,\ 1 < \beta \leq 2, \\ g_n(x) \geq \frac{1}{2}x, \quad x > 1. \tag{30}$$

Therefore, by Theorem 7, we can easily get (9). □

Furthermore, by Corollaries 5 and 8, we can get the following important Chung-type strong law of large numbers for arrays of rowwise AANA random variables.

Corollary 9. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. If there exists some $\beta \in (0, 2]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^{\beta}}{a_n^{\beta}} < \infty, \tag{31}$$

and $EX_{ni} = 0$, $i \geq 1$, $n \geq 1$ if $\beta \in (1, 2]$, then (9) holds for any $\varepsilon > 0$ and $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ a.s.

For $\beta \geq 2$, we have the following result.

Theorem 10. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with $\sum_{n=1}^{\infty} u^2(n) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Let $\{g_n(x), n \geq 1\}$ be a sequence of nonnegative, even functions. Assume that there exists some $\beta \geq 2$ such that $g_n(x) \geq \delta x^{\beta}$ for $x > 0$. If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right)\right]^{1/\beta} < \infty, \tag{32}$$

then (9) holds for any $\varepsilon > 0$ and $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ a.s.

Proof. We use the same notations as those in Theorem 4. The proof is similar to that of Theorem 4. It is easily seen that (32) implies that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{33}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right)\right]^{2/\beta} < \infty. \tag{34}$$

Firstly, we will show that (13) holds true. In fact, by Hölder's inequality, $g_n(x) \geq \delta x^{\beta}$ for $x > 0$, (32), and (33), we have

$$\begin{aligned}
&\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j} EX_i^{(n)}\right| \\
&\leq \sum_{i=1}^{n} P(|X_{ni}| > a_n) + \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| \leq a_n)\right) \\
&\leq \sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{a_n^{\beta}} I(|X_{ni}| > a_n)\right) \\
&\quad + \sum_{i=1}^{n}\left[E\left(\frac{|X_{ni}|^{\beta}}{a_n^{\beta}} I(|X_{ni}| \leq a_n)\right)\right]^{1/\beta} \\
&\leq C\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) \\
&\quad + C\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \leq a_n)\right]^{1/\beta} \\
&\leq C\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) \\
&\quad + C\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right)\right]^{1/\beta} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{35}$$

which implies (13). To prove (9), we only need to show that (16) and (17) hold true.

By the condition $g_n(x) \geq \delta x^{\beta}$ for $x > 0$ again and (33), we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} EI(|X_{ni}| > a_n) \\
&\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{a_n^{\beta}} I(|X_{ni}| > a_n)\right) \\
&\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_i\left(\frac{X_{ni}}{a_n}\right) < \infty,
\end{aligned} \tag{36}$$

which implies (16).

By Markov's inequality, Lemma 3 (for $p = 2$), $g_n(x) \geq \delta x^{\beta}$ for $x > 0$, and (34), we can get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) \\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) \\
&\quad + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}| \leq a_n)}{a_n^2} \\
&\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[E\left(\frac{|X_{ni}|^{\beta}}{a_n^{\beta}} I(|X_{ni}| \leq a_n)\right)\right]^{2/\beta} \\
&\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \leq a_n)\right]^{2/\beta} \\
&\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_i\left(\frac{X_{ni}}{a_n}\right)\right]^{2/\beta} < \infty,
\end{aligned} \tag{37}$$

which implies (17). This completes the proof of the theorem. □

The next two theorems extend and improve Theorem A for arrays of rowwise of independent random variables to the case of AANA random variables.

Theorem 11. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise AANA random variables with* $\sum_{n=1}^{\infty} u^2(n) < \infty$ *and let* $\{a_n, n \geq 1\}$ *be a sequence of positive real numbers. Let* $\{g_n(x), n \geq 1\}$ *be a sequence of nonnegative and even functions such that* $g_n(x) > 0$ *for* $x > 0$ *and every* $n \geq 1$*, and*

$$\frac{g_n(|x|)}{|x|} \uparrow, \quad \frac{g_n(|x|)}{|x|^p} \downarrow \quad \text{as } |x| \uparrow, \tag{38}$$

where $1 < p \leq 2$*. Assume further that*

$$EX_{ni} = 0, \quad i \geq 1, \ n \geq 1, \tag{39}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{Eg_i(X_{ni})}{g_i(a_n)} < \infty. \tag{40}$$

Then (9) *holds for any* $\varepsilon > 0$ *and* $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *a.s.*

Proof. We use the same notations as those in Theorem 4. The proof is similar to that of Theorem 4.

Firstly, we will show that (13) holds true. Actually, by the conditions $EX_{ni} = 0$ and (38)–(40), we have

$$\begin{aligned}
\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_i^{(n)} \right| &\leq \sum_{i=1}^{n} P(|X_{ni}| > a_n) \\
&\quad + \max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_{ni} I(|X_{ni}| > a_n) \right| \\
&\leq 2 \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| > a_n) \right) \\
&\leq 2 \sum_{i=1}^{n} \frac{Eg_i(X_{ni}) I(|X_{ni}| > a_n)}{g_i(a_n)} \\
&\leq 2 \sum_{i=1}^{n} \frac{Eg_i(X_{ni})}{g_i(a_n)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{41}$$

which implies (13). Hence, to prove (9), it suffices to show that (16) and (17) hold true.

The conditions (38) and (40) yield that

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > a_n) &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| > a_n) \right) \\
&\leq 2 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{Eg_i(X_{ni})}{g_i(a_n)} < \infty,
\end{aligned} \tag{42}$$

which implies (16).

By Markov's inequality, Lemma 3, (38), and (40), we can get that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} \left| T_j^{(n)} \right| > \frac{\varepsilon}{2} \right) &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > a_n) \\
&\quad + C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|X_{ni}|^p I(|X_{ni}| \leq a_n)}{a_n^p} \\
&\leq C + C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{Eg_i(X_{ni})}{g_i(a_n)} \\
&\quad \times I(|X_{ni}| \leq a_n) < \infty,
\end{aligned} \tag{43}$$

which implies (17). This completes the proof of the theorem. □

Theorem 12. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise AANA random variables and let* $\{a_n, n \geq 1\}$ *be a sequence of positive real numbers. Let* $p > 2$, $0 < r \leq 2$, $s > 0$, *and* $\delta \doteq \max\{p, 2s\} \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$*, where the integer number* $k \geq 1$*. Let* $\{g_n(x), n \geq 1\}$ *be a sequence of nonnegative and even functions such that* $g_n(x) > 0$ *for* $x > 0$ *and every* $n \geq 1$*. Assume that conditions* (38)–(40) *are satisfied. If* $\sum_{n=1}^{\infty} u^{1/(\delta-1)}(n) < \infty$ *and*

$$\sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} \frac{E|X_{ni}|^r}{a_n^r} \right]^s < \infty, \tag{44}$$

then (9) *holds for any* $\varepsilon > 0$ *and* $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *a.s.*

Proof. We use the same notations as those in Theorem 4. The proof is similar to that of Theorems 4 and 11.

Following the methods of the proof in Theorems 4 and 11, (13) and (16) hold. So we need only to show (17). Denote $\delta = \max\{p, 2s\}$. We have by Markov's inequality, Lemma 3, and C_r's inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} \left| T_j^{(n)} \right| > \frac{\varepsilon}{2} \right) \\
&\leq C \sum_{n=1}^{\infty} E\left(\max_{1\leq j\leq n} \left| T_j^{(n)} \right|^\delta \right) \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^\delta} \left\{ \sum_{i=1}^{n} E\left| X_i^{(n)} - EX_i^{(n)} \right|^\delta \right. \\
&\qquad \left. + \left[\sum_{i=1}^{n} E\left| X_i^{(n)} - EX_i^{(n)} \right|^2 \right]^{\delta/2} \right\} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^\delta} \left\{ \sum_{i=1}^{n} E\left| X_i^{(n)} \right|^\delta + \left[\sum_{i=1}^{n} E\left| X_i^{(n)} \right|^2 \right]^{\delta/2} \right\}.
\end{aligned} \tag{45}$$

Since $\delta \geq p$ and $(g_n(|x|)/|x|) \uparrow$, $(g_n(|x|)/|x|^p) \downarrow$ as $|x| \uparrow$, it follows that $g_n(|x|) \uparrow$ and $(g_n(|x|)/|x|^\delta) \downarrow$ as $|x| \uparrow$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n^\delta}\sum_{i=1}^{n}E\left|X_i^{(n)}\right|^\delta &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_i\left(\left|X_i^{(n)}\right|\right)}{g_i(a_n)}\\ &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_i\left(\left|X_{ni}\right|\right)}{g_i(a_n)}\\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_i\left(X_{ni}\right)}{g_i(a_n)} < \infty.\end{aligned} \quad (46)$$

It follows by $0 < r \leq 2$, $s > 0$, $\delta \geq 2s$, and (44) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n^\delta}\left[\sum_{i=1}^{n}E\left|X_i^{(n)}\right|^2\right]^{\delta/2} &= \sum_{n=1}^{\infty}\left[\left(\sum_{i=1}^{n}\frac{E\left|X_i^{(n)}\right|^2}{a_n^2}\right)^s\right]^{\delta/2s}\\ &\leq \left[\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\left|X_i^{(n)}\right|^2}{a_n^2}\right)^s\right]^{\delta/2s}\\ &\leq \left[\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\left|X_i^{(n)}\right|^r}{a_n^r}\right)^s\right]^{\delta/2s}\\ &\leq \left[\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\left|X_{ni}\right|^r}{a_n^r}\right)^s\right]^{\delta/2s} < \infty.\end{aligned} \quad (47)$$

Therefore, (17) follows from (45)–(47) immediately. This completes the proof of the theorem. □

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