Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 742912, 3 pages
http://dx.doi.org/10.1155/2013/742912

Research Article

The Periodicity of Positive Solutions of the Nonlinear Difference Equation $x_{n+1} = \alpha + (x_{n-k}^{p})/(x_n^{q})$

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Received 30 November 2012; Revised 4 February 2013; Accepted 6 February 2013

Academic Editor: M. De la Sen

We give a remark about the periodic character of positive solutions of the difference equation $x_{n+1} = \alpha + x_{n-k}^{p}/x_n^{q}$, $n = 0, 1, \ldots$, where $k > 1$ is an odd integer, $\alpha, p, q \in (0, \infty)$, and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive numbers.

1. Introduction

In this paper, we consider the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^{p}}{x_n^{q}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $k > 1$ is an odd integer, α is positive, $p, q \in (0, \infty)$, and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive numbers. Equation (1) was studied by many authors for different cases of k, p, q.

In [1], the authors studied the special case of $k = 1$ of (1), that is, the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}^{p}}{x_n^{q}}, \quad n = 0, 1, \ldots, \tag{2}$$

where α, p, q are positive and the initial values x_{-1}, x_0 are positive numbers.

Recently, in [2], the authors obtained the periodicity results of positive solutions of (1). They investigated the existence of a prime periodic solution of (1). But they did not investigate the positive solutions of (1) which converge to a prime two-periodic solution.

There exist many other papers related with (1) and on its extensions (see [3–8]).

Our aim in this paper is to give a remark about the periodic character of all positive solutions of (1). We show that all positive solutions of (1), for k is odd, converge to a prime two-periodic solution.

We believe that difference equations, also referred to as recursive sequence, are a hot topic. There has been an increasing interest in the study of qualitative analysis of difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economics, physics, computer sciences, and so on.

Here, we recall some notations and results which will be useful in our proofs.

Let I be some interval of real numbers and let F be continuous function defined on I^{k+1}. Then, for initial conditions $x_{-k}, \ldots, x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{3}$$

has a unique solution $\{x\}_{n=-k}^{\infty}$.

We say that the equilibrium point $\overline{x}$ of (3) is the point that satisfies the condition

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{4}$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (3), or equivalently, $\overline{x}$ is a fixed point of F.

A solution $\{x\}_{n=-k}^{\infty}$ of (3) is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$

$$x_{n+p} = x_n, \quad \forall n \geq -k. \tag{5}$$

A solution $\{x\}_{n=-k}^{\infty}$ of (3) is said to be periodic with prime period p, or a p-cycle if it is periodic with period p and p is the least positive integer for which (5) holds.

The linearized equation for (1) about the positive equilibrium $\overline{x}$ is

$$y_{n+1} + q\overline{x}^{p-q-1} y_n - p\overline{x}^{p-q-1} y_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{6}$$

2. Main Results

In this section, we find solutions, for k is odd of (1), which converge to a prime two-periodic solution. The following three results are essentially proved in [1, 2]. Hence, we omit their proofs.

Lemma 1. *Suppose that*

$$p \in (0, 1), \tag{7}$$

then every positive solution of (1) *is bounded.*

Lemma 2. *Assume that k is odd. Then,* (1) *has prime two-periodic solutions if and only if*

$$0 < p < 1 < q \tag{8}$$

and there exists a sufficient small positive number ε_1, such that

$$\begin{gathered} \frac{1}{(\alpha+\varepsilon_1)^{q-p}} > \varepsilon_1, \\ (\alpha+\varepsilon_1)^{p/q} \varepsilon_1^{-1/q} < \alpha + \varepsilon_1^{-p/q} (\alpha+\varepsilon_1)^{p^2-q^2/q}. \end{gathered} \tag{9}$$

Lemma 3. *If either*

$$0 < q < p < 1, \tag{10}$$

or

$$0 < p < q, \quad q \nrightarrow \infty \tag{11}$$

hold, then (1) *has a unique equilibrium point $\overline{x}$.*

Theorem 4. *Consider* (1) *where k is odd, for every positive solution $\{x_n\}$ of* (1) *which satisfies any of the following initial conditions:*

(i) $x_{-k+2l} \geq x_{-k+2l-2}$ *for all* $l = 1, 2, \ldots, (k-1)/2$ *and* $x_{-k+2s+1} \leq x_{-k+2s-1}$ *for all* $s = 1, 2, \ldots, (k+1)/2$.

(ii) $x_{-k+2l} \leq x_{-k+2l-2}$ *for all* $l = 1, 2, \ldots, (k-1)/2$ *and* $x_{-k+2s+1} \geq x_{-k+2s-1}$ *for all* $s = 1, 2, \ldots, (k+1)/2$.

(iii) $x_{-k+2l} \geq x_{-k+2l-2}$ *for all* $l = 1, 2, \ldots, (k-1)/2$ *and* $x_{-k+2s+1} \geq x_{-k+2s-1}$ *for all* $s = 1, 2, \ldots, (k+1)/2$.

(iv) $x_{-k+2l} \leq x_{-k+2l-2}$ *for all* $l = 1, 2, \ldots, (k-1)/2$ *and* $x_{-k+2s+1} \leq x_{-k+2s-1}$ *for all* $s = 1, 2, \ldots, (k+1)/2$.

Then, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone.

Proof. We have

$$\begin{aligned} & x_{2n+1} - x_{2n-1} \\ & \quad = \frac{x_{2n-2}^q \left(x_{2n-k}^p - x_{2n-k-2}^p\right) + x_{2n-k-2}^p \left(x_{2n-2}^q - x_{2n}^q\right)}{\left(x_{2n} x_{2n-2}\right)^q}, \end{aligned} \tag{12}$$

$$\begin{aligned} & x_{2n+2} - x_{2n} \\ & \quad = \frac{x_{2n-1}^q \left(x_{2n-k+1}^p - x_{2n-k-1}^p\right) + x_{2n-k-1}^p \left(x_{2n-1}^q - x_{2n+1}^q\right)}{\left(x_{2n+1} x_{2n-1}\right)^q}. \end{aligned} \tag{13}$$

If $x_{-k+2l} \geq x_{-k+2l-2}$ for all $l = 1, 2, \ldots, (k-1)/2$ and $x_{-k+2s+1} \leq x_{-k+2s-1}$ for all $s = 1, 2, \ldots, (k+1)/2$, from (12) and (13) we obtain $x_3 \geq x_1$ and consequently $x_4 \leq x_2$. By induction we obtain

$$\begin{gathered} x_0 \geq x_2 \geq \cdots \geq x_{2n} \geq \cdots \text{ and } \cdots \\ \geq x_{2n+1} \geq x_{2n-1} \geq \cdots \geq x_1. \end{gathered} \tag{14}$$

Similarly if $x_{-k+2l} \leq x_{-k+2l-2}$ for all $l = 1, 2, \ldots, (k-1)/2$ and $x_{-k+2s+1} \geq x_{-k+2s-1}$ for all $s = 1, 2, \ldots, (k+1)/2$, from (12) and (13) using induction we obtain

$$\begin{gathered} x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq \cdots \text{ and } \cdots \\ \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_1. \end{gathered} \tag{15}$$

If $x_{-k+2l} \geq x_{-k+2l-2}$ for all $l = 1, 2, \ldots, (k-1)/2$ and $x_{-k+2s+1} \geq x_{-k+2s-1}$ for all $s = 1, 2, \ldots, (k+1)/2$ and $x_1 \geq x_3$ we can obtain from (12) and (13)

$$\begin{gathered} x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq \cdots \text{and } \cdots \\ \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_1. \end{gathered} \tag{16}$$

Therefore, the result follows immediately. □

The following result is essentially proved in [1] for $k = 1$ of (1). We obtain the same result, for $k > 1$ is an odd integer.

Lemma 5. *Consider* (1) *where* (8) *and* (9) *hold and k is odd. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of* (1) *such that either*

$$\begin{gathered} \alpha < x_{-k}, x_{-k+2}, \ldots, x_{-1} < \alpha + \varepsilon_1, \\ (\alpha+\varepsilon_1)^{p/q} \varepsilon_1^{-1/q} < x_{-k+1}, x_{-k+3}, \ldots, x_0 \end{gathered} \tag{17}$$

or

$$\begin{gathered} \alpha < x_{-k+1}, x_{-k+3}, \ldots, x_0 < \alpha + \varepsilon_1, \\ (\alpha+\varepsilon_1)^{p/q} \varepsilon_1^{-1/q} < x_{-k}, x_{-k+2}, \ldots, x_{-1}. \end{gathered} \tag{18}$$

Then if (17) *holds, we have*

$$\begin{aligned} \alpha < x_{2n-1} < \alpha + \varepsilon_1, \quad x_{2n} > (\alpha+\varepsilon_1)^{p/q} \varepsilon_1^{-1/q}, \\ n = 0, 1, \ldots,. \end{aligned} \tag{19}$$

Also if (18) *holds, we have*

$$\begin{aligned} \alpha < x_{2n} < \alpha + \varepsilon_1, \quad x_{2n-1} > (\alpha+\varepsilon_1)^{p/q} \varepsilon_1^{-1/q}, \\ n = 0, 1, \ldots,. \end{aligned} \tag{20}$$

Proof. Firstly, assume that (17) holds. Then, from (9) we have

$$\alpha < x_1 = \alpha + \frac{x_{-k}^p}{x_0^q} < \alpha + \varepsilon_1 \frac{(\alpha+\varepsilon_1)^p}{(\alpha+\varepsilon_1)^p} = \alpha + \varepsilon_1,$$
$$x_2 = \alpha + \frac{x_{-k+1}^p}{x_1^q} > \alpha + (\alpha+\varepsilon_1)^{(p^2-q^2)/q}\varepsilon_1^{-p/q} \tag{21}$$
$$> (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q}.$$

Working inductively, we can get

$$\alpha < x_k = \alpha + \frac{x_{-1}^p}{x_{k-1}^q} < \alpha + \varepsilon_1 \frac{(\alpha+\varepsilon_1)^p}{(\alpha+\varepsilon_1)^p} = \alpha + \varepsilon_1,$$
$$x_{k+1} = \alpha + \frac{x_0^p}{x_k^q} > \alpha + (\alpha+\varepsilon_1)^{(p^2-q^2)/q}\varepsilon_1^{-p/q} \tag{22}$$
$$> (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q}.$$

Therefore, we can easily prove relations (19). Similiarly if (18) holds, we can prove that (20) is satisfied. □

Lemma 6. *If $\alpha > 1$, then every positive solution $\{x_n\}_{n=-k}^{\infty}$ of* (1) *satisfies the following inequalities:*

$$\alpha < x_{2n+k} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{2n-1}^p, \quad n = 1,2,\ldots,$$
$$\alpha < x_{2n+k+1} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{2n}^p, \quad n = 1,2,\ldots \tag{23}$$

and here the cases

$$\alpha < x_{2n} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{-k+1}^p, \quad n = 1,2,\ldots,$$
$$\alpha < x_{2n-1} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{-k}^p, \quad n = 1,2,\ldots, \tag{24}$$

$$\beta = \frac{1}{\alpha^q} \tag{25}$$

hold.

Proof. We have $\alpha < x_{n+1} < \alpha + \beta x_{n-k}^p$ for all $n = 1,2,\ldots$. By induction, we obtain (23) and (24). If here $\alpha > 1$, we also see that

$$\alpha < x_{2n} < \frac{\alpha}{1-\beta} + x_{-k+1}^p, \quad n = 1,2,\ldots$$
$$\alpha < x_{2n-1} < \frac{\alpha}{1-\beta} + x_{-k}^p, \quad n = 1,2,\ldots \tag{26}$$

□

Now, we are ready for the main result of this paper.

Theorem 7. *Consider* (1) *where* (8) *and* (9) *hold and k is odd. Suppose that*

$$\alpha + \varepsilon_1 < 1, \tag{27}$$

then every positive solution $\{x_n\}_{n=-k}^{\infty}$ with initial values $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, which satisfy conditions of Theorem 4 and either (17) *or* (18), *converges to a prime two-periodic solution.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution with initial values $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, which satisfy conditions of Theorem 4, and either (17) or (18). Using Lemma 1 and Theorem 4, we have that there exist

$$\lim_{n\to\infty} x_{2n+1} = L, \qquad \lim_{n\to\infty} x_{2n} = S. \tag{28}$$

Besides, from Lemma 5 we have that either L or S belongs to the interval $(\alpha, \alpha+\varepsilon_1)$. Furthermore, from Lemma 3 we have that (1) has a unique equilibrium $\overline{x}$ such that $1 < \overline{x} < \infty$. Therefore, from (27) we have that $L \neq S$. So $\{x_n\}_{n=-k}^{\infty}$ converges to a prime two-periodic solution. The proof is complete. □

Acknowledgments

The authors would like to thank the referees for their helpful suggestions. This research is supported by TUBITAK and Bulent Ecevit University Research Project Coordinatorship.

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