Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 936351, 9 pages
http://dx.doi.org/10.1155/2013/936351

Research Article

Oscillations of Numerical Solutions for Nonlinear Delay Differential Equations in the Control of Erythropoiesis

Qi Wang and Jiechang Wen

School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China

Correspondence should be addressed to Qi Wang; bmwzwq@126.com

Received 10 August 2012; Accepted 24 December 2012

Academic Editor: Vimal Singh

We consider the oscillations of numerical solutions for the nonlinear delay differential equations in the control of erythropoiesis. The exponential θ-method is constructed and some conditions under which the numerical solutions oscillate are presented. Moreover, it is proven that every nonoscillatory numerical solution tends to the equilibrium point of the continuous system. Numerical examples are given to illustrate the main results.

1. Introduction

The oscillatory and asymptotic behavior of solutions of delay differential equations has been the subject of intensive investigations during the past decades. A large number of articles has appeared in the literature, we refer to [1–4] and the references therein. The strong interest in this study is motivated by the fact that it has many useful applications in some mathematical models, such as ecology, biology, spread of some infectious diseases in humans, and so on. For more information on this investigation, the reader can see [5, 6] and the references therein.

By contrast with the research on the oscillations of the analytic solutions, much studies have been focused on the oscillations of the numerical solutions for delay differential equations. In [7, 8], oscillations of numerical solutions in θ-methods and Runge-Kutta methods for a linear differential equation with piecewise constant arguments (EPCA, a special type of Delay Differential Equations) $x'(t) + ax(t) + a_1 x([t-1]) = 0$ were considered, respectively. More recently, Wang et al. [9] studied numerical oscillations of alternately advanced and retarded linear EPCA, the conditions of oscillations for the θ-methods are obtained. To the best of our knowledge, until now less attention had been paid for the oscillations of the numerical solutions for nonlinear delay differential equations except for [10]. Differently from [10], in our paper, we will investigate another nonlinear delay differential equation in the control of erythropoiesis and obtain some new results.

Consider the following nonlinear delay differential equation:

$$x'(t) = \frac{\beta_0 \omega^\mu}{\omega^\mu + x^\mu (t-\tau)} - \gamma x(t), \tag{1}$$

with conditions

$$\mu, \omega, \beta_0, \gamma > 0, \qquad \tau \geq 0. \tag{2}$$

Mackey and Glass [11] have proposed (1) as model of hematopoiesis (blood cell production). In (1), $x(t)$ denotes the density of mature cells in blood circulation, τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating blood stream, and the production is a monotonic decreasing function of $x(t-\tau)$. Equation (1) has been recently studied by many authors. Gopalsamy et al. [12] obtained sufficient and also necessary and sufficient conditions for all positive solutions to oscillate about their positive steady states. They also obtained sufficient conditions for the positive equilibrium to be a global attractor. Using the linearization method, Zaghrout et al. [13] considered (1) and gave a sufficient condition for oscillations of all solutions about the positive steady state x^* and proved that every nonoscillatory positive solution of (1) tends to x^* as $t \to \infty$. For more details of (1), we refer to Mackey and Milton [14] and Mackey [15]. Up to now, few results on the properties of numerical solutions for (1) were obtained. In the present paper, our main goal is to investigate some sufficient conditions under which the numerical solutions are oscillatory. We also consider the asymptotic behavior of nonoscillatory numerical solutions.

The contents of this paper are as follows. In Section 2, some necessary definitions and results for oscillations of the analytic solutions are given. In Section 3, we obtain a recurrence relation by applying the θ-methods to the simplified form which comes from making two transformations on (1). Moreover, the oscillations of the numerical solutions are discussed and conditions under which the numerical solutions oscillate are obtained. In Section 4, we study the asymptotic behavior of nonoscillatory solutions. In Section 5, we present numerical examples that illustrate the theoretical results for the numerical methods. Finally, Section 6 gives conclusions and issues for future research.

2. Preliminaries

In this section, we start by introducing some definitions, lemmas and theorems that will be employed throughout the work.

Definition 1. A function $x(t)$ of (1) is said to oscillate about K^* if $x(t)-K^*$ has arbitrarily large zeros. Otherwise, $x(t)$ is called nonoscillatory. When $K^* = 0$, we say that $x(t)$ oscillates about zero or simply oscillates.

Definition 2. A sequence $\{x_n\}$ is said to oscillate about $\{y_n\}$ if $\{x_n - y_n\}$ is neither eventually positive nor eventually negative. Otherwise, $\{x_n\}$ is called nonoscillatory. If $\{y_n\} = \{y\}$ is a constant sequence, we simply say that $\{x_n\}$ oscillates about $\{y\}$. When $\{y_n\} = \{0\}$, we say that $\{x_n\}$ oscillates about zero or simply oscillates.

Definition 3. We say (1) oscillates if all of its solutions are oscillatory.

Theorem 4 (see [16]). *Consider the difference equation*

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0, \tag{3}$$

assume that $k, l \in \mathbf{N}$ and $q_j \in \mathbf{R}$ for $j = -k, \ldots, l$. Then the following statements are equivalent:

(i) *every solution of* (3) *oscillates;*

(ii) *the characteristic equation $\lambda - 1 + \sum_{j=-k}^{l} q_j \lambda^j = 0$ has no positive roots.*

Theorem 5 (see [16]). *Consider the difference equation*

$$a_{n+1} - a_n + p a_{n-k} + q a_n = 0, \tag{4}$$

where $k > 0$, $p > 0$ and $q > 0$. Then the necessary and sufficient conditions for the oscillation of all solutions of (4) *are $q \in (0, 1)$ and*

$$p \frac{(k+1)^{k+1}}{k^k} > (1-q)^{k+1}. \tag{5}$$

Lemma 6. *The inequality $\ln(1+x) > x/(1+x)$ holds for $x > -1$ and $x \neq 0$.*

Lemma 7. *The inequality $e^x < 1/(1-x)$ holds for $x < -1$ and $x \neq 0$.*

Lemma 8 (see [17]). *For all $m \geq M$,*

(i) *$(1 + a/(m - \theta a))^m \geq e^a$ if and only if $1/2 \leq \theta \leq 1$ for $a > 0$, $\varphi(-1) \leq \theta \leq 1$ for $a < 0$;*

(ii) *$(1 + a/(m - \theta a))^m < e^a$ if and only if $0 \leq \theta < 1/2$ for $a < 0$, $0 \leq \theta \leq \varphi(1)$ for $a > 0$,*

where $\varphi(x) = 1/x - 1/(e^x - 1)$ and M is a positive constant.

3. Oscillations of Numerical Solutions

3.1. Transformation. For (1), we take an initial condition of the form

$$x(t) = \psi(t), \quad -\tau \leq t \leq 0, \tag{6}$$

where $\psi \in C([-\tau, 0], (0, \infty))$, $\psi(0) > 0$.

In order to simplify (1), we introduce a similar method in [13]. The change of variables

$$x(t) = \omega y(t) \tag{7}$$

transforms (1) to the delay differential equation

$$y'(t) = \frac{a}{1 + y^\mu(t-\tau)} - \gamma y(t), \tag{8}$$

where $a = \beta_0/\omega$. One can see that (8) has a unique equilibrium K and that

$$\frac{a}{1 + K^\mu} = \gamma K. \tag{9}$$

The following result concerning oscillations of the analytic solution of (8) is given in [13].

Theorem 9. *Assume that*

$$\frac{a\mu K^{\mu-1}\tau}{(1+K^\mu)^2} e^{a\tau/K(1+K^\mu)} > \frac{1}{e}, \tag{10}$$

then every positive solution of (8) *oscillates about its positive equilibrium K.*

Therefore, we obtain the following corollary naturally.

Corollary 10. *Assume that the condition*

$$\frac{a\mu K^{\mu-1}\tau}{(1+K^\mu)^2} e^{a\tau/K(1+K^\mu)} > \frac{1}{e} \tag{11}$$

holds, then every positive solution of (1) *oscillates about its positive equilibrium $K^* = \omega K$.*

Next, we also introduce an invariant oscillation transformation $y(t) = Ke^{z(t)}$, then (8) can be written as

$$z'(t) + \frac{a}{K(1+K^{\mu})}\left[f_1(z(t))\, f_2(z(t-\tau)) - f_1(z(t)) - f_2(z(t-\tau)) + 2\right] = 0, \tag{12}$$

where

$$f_1(u) = 1 - e^{-u}, \qquad f_2(u) = 1 + \frac{1+K^{\mu}}{1+K^{\mu}e^{\mu u}}. \tag{13}$$

Then $y(t)$ oscillates about K if and only if $z(t)$ oscillates about zero. Moreover, for the sake of brevity, let

$$T = \frac{a}{K(1+K^{\mu})}, \tag{14}$$

then (12) becomes

$$z'(t) = -Tf_1(z(t))\, f_2(z(t-\tau)) + Tf_1(z(t)) + Tf_2(z(t-\tau)) - 2T. \tag{15}$$

For convenience, denote

$$L = \frac{a\mu K^{\mu-1}}{(1+K^{\mu})^2}, \tag{16}$$

then the inequality (11) yields

$$L\tau e^{T\tau} > \frac{1}{e}. \tag{17}$$

3.2. The Difference Scheme. In this subsection we consider the adaptation of the θ-methods. Let $h = \tau/m$ be a given step size with integer $m > 1$. The adaptation of the linear θ-method and the one-leg θ-method to (15) leads to the same numerical process of the following type:

$$\begin{aligned} z_{n+1} = z_n &- Th\left(\theta f_1(z_{n+1})\, f_2(z_{n+1-m}) + (1-\theta)\, f_1(z_n)\, f_2(z_{n-m})\right) \\ &+ Th\left(\theta f_1(z_{n+1}) + (1-\theta)\, f_1(z_n)\right) \\ &+ Th\left(\theta f_2(z_{n+1-m}) + (1-\theta)\, f_2(z_{n-m})\right) - 2Th, \end{aligned} \tag{18}$$

where $0 \le \theta \le 1$, z_{n+1} and z_{n+1-m} are approximations to $z(t)$ and $z(t-\tau)$ of (15) at t_{n+1}, respectively.

Let $z_n = \ln(x_n/K\omega)$, and take into account of the expressions of f_1 and f_2 we have

$$x_{n+1} = x_n \exp\left(hTK\omega^{\mu+1}(1+K^{\mu}) \times \left(\frac{\theta}{x_{n+1}(\omega^{\mu} + x^{\mu}_{n+1-m})} + \frac{1-\theta}{x_n(\omega^{\mu}+x^{\mu}_{n-m})}\right) - hT\right). \tag{19}$$

Definition 11. We call the iteration formula (19) as the exponential θ-method for (1), where x_{n+1} and x_{n+1-m} are approximations to $x(t)$ and $x(t-\tau)$ of (1) at t_{n+1}, respectively.

The following theorem gives the convergence of exponential θ-method. We can easily prove it by the method of steps which is used in [18].

Theorem 12. *The exponential θ-method* (19) *is convergent with order*

$$\begin{aligned} &1, \quad \text{if } \theta \neq \frac{1}{2}, \\ &2, \quad \text{if } \theta = \frac{1}{2}. \end{aligned} \tag{20}$$

3.3. Oscillation Analysis. It is not difficult to know that x_n oscillates about K^* if and only if z_n is oscillatory. In order to study oscillations of (19), we only need to consider the oscillations of (18). The following conditions which are taken from [13] will be used in the next analysis:

$$\begin{aligned} &uf_1(u) > 0, \quad \text{for } u \neq 0, \quad \lim_{u\to 0}\frac{f_1(u)}{u} = 1, \\ &f_2(u) > 0, \quad \text{any } u, \quad \lim_{u\to 0} f_2(u) = 2, \\ &f_1(u) \le u \quad \text{for } u > 0, \qquad f_1(0) = 0, \\ &f_2(u) \le 2 \quad u \ge 0, \qquad f_2(0) = 2. \end{aligned} \tag{21}$$

For (18), its linearized form is given by

$$z_{n+1} = z_n - h\theta Tz_{n+1} - h(1-\theta)Tz_n - h\theta Lz_{n+1-m} - h(1-\theta)z_{n-m}, \tag{22}$$

which is equivalent to

$$z_{n+1} = \frac{1-h(1-\theta)T}{1+h\theta T}z_n - \frac{h\theta L}{1+h\theta T}z_{n+1-m} - \frac{h(1-\theta)L}{1+h\theta T}z_{n-m}. \tag{23}$$

It follows from [16] that (18) oscillates if (23) oscillates under the condition (21).

Definition 13. Equation (19) is said to be oscillatory if all of its solutions are oscillatory.

Definition 14. We say that the exponential θ-method preserves the oscillations of (1) if (1) oscillates, then there is a $h_0 > 0$ or $h_0 = \infty$, such that (19) oscillates for $h < h_0$. Similarly, we say that the exponential θ-method preserves the nonoscillations of (1) if (1) nonoscillates, then there is a $h_0 > 0$ or $h_0 = \infty$, such that (19) nonoscillates for $h < h_0$.

In the following, we will study whether the exponential θ-method preserves the oscillations of (1). That is, when Corollary 10 holds, we will investigate the conditions under which (19) is oscillatory.

Lemma 15. *The characteristic equation of* (22) *is given by*

$$\xi = R\left(-h\left(T + L\xi^{-m}\right)\right). \tag{24}$$

Proof. Let $z_n = \xi^n z_0$ in (22), we have

$$\begin{aligned}\xi^{n+1} z_0 &= \xi^n z_0 - h\theta T \xi^{n+1} z_0 - h(1-\theta) T \xi^n z_0 \\ &\quad - h\theta L \xi^{n+1-m} z_0 - h(1-\theta) L \xi^{n-m} z_0,\end{aligned} \tag{25}$$

that is

$$\xi = 1 - h\theta\left(T + L\xi^{-m}\right)\xi - h(1-\theta)\left(T + L\xi^{-m}\right), \tag{26}$$

which is equivalent to

$$\xi = \frac{1 - h(1-\theta)\left(T + L\xi^{-m}\right)}{1 + h\theta\left(T + L\xi^{-m}\right)} = 1 - \frac{h\left(T + L\xi^{-m}\right)}{1 + h\theta\left(T + L\xi^{-m}\right)}. \tag{27}$$

In view of [19], we know that the stability function of the θ-method is

$$R(x) = 1 + \frac{x}{1 - \theta x}, \tag{28}$$

thus the characteristic equation of (22) is given by (24). This completes the proof of the lemma. □

Lemma 16. *If $L\tau e^{T\tau} > 1/e$, then the characteristic equation* (24) *has no positive roots for $0 \le \theta \le 1/2$.*

Proof. Let $V(\xi) = \xi - R(-h(T + L\xi^{-m}))$. By Lemma 8, we have

$$\begin{aligned}&R\left(-h\left(T + L\xi^{-m}\right)\right) \\ &\quad \le \exp\left(-h\left(T + L\xi^{-m}\right)\right), \quad \xi > 0, 0 \le \theta \le 1/2.\end{aligned} \tag{29}$$

Now we will prove that $W(\xi) = \xi - \exp(-h(T + L\xi^{-m})) > 0$ for $\xi > 0$. Suppose the opposite, that is, there exists a $\xi_0 > 0$ such that $W(\xi_0) \le 0$, then we have $\xi_0 \le \exp(-hT - hL\xi_0^{-m})$, and

$$\xi_0^m \le \exp\left(-T\tau - L\tau\xi_0^{-m}\right). \tag{30}$$

Multiplying both sides of the inequality (30) by $L\tau e^{T\tau} e \xi_0^{-m}$, we obtain

$$L\tau e^{T\tau} e \xi_0^{-m} \xi_0^m \le L\tau e^{T\tau} e \xi_0^{-m} \exp\left(-T\tau - L\tau\xi_0^{-m}\right), \tag{31}$$

which gives

$$L\tau e^{T\tau} e \le L\tau\xi_0^{-m} \exp\left(1 - L\tau\xi_0^{-m}\right), \tag{32}$$

therefore we have the following two cases.

Case 1. If $1 - L\tau\xi_0^{-m} = 0$, then $L\tau e^{T\tau} e \le 1$, which contradicts the condition $L\tau e^{T\tau} > 1/e$.

Case 2. If $1 - L\tau\xi_0^{-m} \ne 0$, then in view of Lemma 7, we get

$$\exp\left(1 - L\tau\xi_0^{-m}\right) < \frac{1}{1 - \left(1 - L\tau\xi_0^{-m}\right)} = \frac{1}{L\tau\xi_0^{-m}}, \tag{33}$$

that is,

$$L\tau\xi_0^{-m} \exp\left(1 - L\tau\xi_0^{-m}\right) < 1, \tag{34}$$

so $L\tau e^{T\tau} e < 1$, which is also a contradiction to $L\tau e^{T\tau} > 1/e$.

Combining both the cases, by (29) we obtain that for $\xi > 0$

$$\begin{aligned}V(\xi) &= \xi - R\left(-h\left(T + L\xi^{-m}\right)\right) \\ &\ge \xi - \exp\left(-h\left(T + L\xi^{-m}\right)\right) = W(\xi) > 0,\end{aligned} \tag{35}$$

which implies that the characteristic equation (24) has no positive roots. The proof of the lemma is complete. □

Without loss of generality, in the case of $1/2 < \theta \le 1$, we assume that $m > 1$.

Lemma 17. *If $L\tau e^{T\tau} > 1/e$ and $1/2 < \theta \le 1$, then the characteristic equation* (24) *has no positive roots for $h < h_0$, where*

$$h_0 = \begin{cases} \infty, & \text{for } L\tau \ge 1, \\ \dfrac{\tau(1 + T\tau + \ln L\tau)}{1 + T\tau(1 - \ln L\tau)}, & \text{for } L\tau < 1. \end{cases} \tag{36}$$

Proof. Since $R(-h(T + L\xi^{-m}))$ is an increasing function of θ when $\xi > 0$, then for $\xi > 0$ and $1/2 < \theta \le 1$

$$\begin{aligned}R\left(-h\left(T + L\xi^{-m}\right)\right) &= \frac{1 - h(1-\theta)\left(T + L\xi^{-m}\right)}{1 + h\theta\left(T + L\xi^{-m}\right)} \\ &\le \frac{1}{1 + h\left(T + L\xi^{-m}\right)}.\end{aligned} \tag{37}$$

Next, we will prove that the inequality

$$\xi - \frac{1}{1 + h\left(T + L\xi^{-m}\right)} > 0, \quad \xi > 0 \tag{38}$$

holds under certain conditions.

From (38), it follows that

$$\xi - \frac{1}{1 + h\left(T + L\xi^{-m}\right)} = \frac{(1 + hT)\xi^{1-m}}{1 + h\left(T + L\xi^{-m}\right)}\lambda(\xi), \tag{39}$$

where

$$\lambda(\xi) = \xi^m - \frac{1}{1 + hT}\xi^{m-1} + \frac{hL}{1 + hT}, \tag{40}$$

so we only need to prove $\lambda(\xi) > 0$ for $\xi > 0$. It is not difficult to know that $\lambda(\xi)$ is the characteristic polynomial of the following difference scheme

$$z_{n+1} - z_n + \frac{hL}{1 + hT} z_{n+1-m} + \frac{hT}{1 + hT} z_n = 0. \tag{41}$$

In view of Theorems 4 and 5, we have that $\lambda(\xi)$ has no positive roots if and only if

$$\frac{hL}{1 + hT}\frac{m^m}{(m-1)^{m-1}} > \left(1 - \frac{hT}{1 + hT}\right)^m, \tag{42}$$

which is equivalent to

$$\ln L\tau + (m-1)\ln\left(1 + \frac{1+T\tau}{m-1}\right) > 0. \tag{43}$$

We examine two cases depending on the position of $L\tau$: Either $L\tau \geq 1$ or $L\tau < 1$.

Case 1. If $L\tau \geq 1$, by $m > 1$, then (43) holds true.

Case 2. If $L\tau < 1$ and

$$h < \frac{\tau(1 + T\tau + \ln L\tau)}{1 + T\tau(1 - \ln L\tau)}, \tag{44}$$

then according to Lemma 6 we have

$$\begin{aligned} &\ln\ L\tau + (m-1)\ln\left(1 + \frac{1+T\tau}{m-1}\right) \\ &\quad > \ln\ L\tau + (m-1)\frac{(1+T\tau)/(m-1)}{1 + (1+T\tau)/(m-1)} \\ &\quad = \ln\ L\tau + \frac{(m-1)(1+T\tau)}{m+T\tau} > 0. \end{aligned} \tag{45}$$

Therefore the inequality (38) holds for $h < h_0$, where

$$h_0 = \begin{cases} \infty, & \text{for } L\tau \geq 1, \\ \dfrac{\tau(1 + T\tau + \ln L\tau)}{1 + T\tau(1 - \ln L\tau)}, & \text{for } L\tau < 1. \end{cases} \tag{46}$$

So we get that the following inequality:

$$V(\xi) = \xi - R\left(-h\left(T + L\xi^{-m}\right)\right) \geq \xi - \frac{1}{1 + h\left(T + L\xi^{-m}\right)} > 0, \tag{47}$$

holds for $h < h_0$ and $\xi > 0$, which implies that the characteristic equation (24) has no positive roots. This completes the proof. □

Remark 18. By inequality (43) and condition $L\tau < 1$, we have that

$$\frac{\tau(1 + T\tau + \ln L\tau)}{1 + T\tau(1 - \ln L\tau)} > 0, \tag{48}$$

thus h_0 is meaningful.

In view of (21), Lemmas 16 and 17, and Theorem 4, we have the first main theorem of this paper.

Theorem 19. *If $L\tau e^{T\tau} > 1/e$, then* (19) *is oscillatory for*

$$h < \begin{cases} \infty, & \text{when } 0 \leq \theta \leq 1/2, \\ h_0, & \text{when } 1/2 < \theta \leq 1, \end{cases} \tag{49}$$

where h_0 is defined in Lemma 17.

4. Asymptotic Behavior of Nonoscillatory Solutions

In this section, we will investigate the asymptotic behavior of nonoscillatory solutions of (19). The following lemma is an important result about asymptotic behavior of (8).

Lemma 20 (see [13]). *Let $y(t)$ be a positive solution of* (8), *which does not oscillate about K. Then* $\lim_{t\to\infty} y(t) = K$.

From the relationship between (8) and (12), we know that the nonoscillatory solution of (12) satisfies $\lim_{t\to\infty} z(t) = 0$ if Lemma 20 holds. Furthermore, $\lim_{t\to\infty} x(t) = K^*$ is also obtained. Next, we will prove that the numerical solution of (1) can inherit this property.

Lemma 21. *Let z_n be a nonoscillatory solution of* (18), *then* $\lim_{n\to\infty} z_n = 0$.

Proof. Without loss of generality, we may assume that $z_n > 0$ for sufficiently large n. Then by condition (21) we know that the following inequalities holds true:

$$f_1(z_i) > 0, \qquad f_2(z_i) - 1 > 0, \qquad f_2(z_i) - 2 < 0 \tag{50}$$

for sufficiently large i. Moreover, it is can be seen from (18) that

$$\begin{aligned} z_{n+1} - z_n = &-h\theta T f_1(z_{n+1})\left[f_2(z_{n+1-m}) - 1\right] \\ &- h(1-\theta) T f_1(z_n)\left[f_2(z_{n-m}) - 1\right] \\ &+ h\theta T\left[f_2(z_{n+1-m}) - 2\right] \\ &+ h(1-\theta) T\left[f_2(z_{n-m}) - 2\right], \end{aligned} \tag{51}$$

which gives

$$\begin{aligned} &z_{n+1} - z_n - h\theta T\left[f_2(z_{n+1-m}) - 2\right] \\ &\quad - h(1-\theta) T\left[f_2(z_{n-m}) - 2\right] < 0. \end{aligned} \tag{52}$$

Thus

$$\begin{aligned} &z_{n+1} - z_n < h\theta T\left[f_2(z_{n+1-m}) - 2\right] \\ &\quad + h(1-\theta) T\left[f_2(z_{n-m}) - 2\right] < 0, \end{aligned} \tag{53}$$

then the sequence $\{z_n\}$ is decreasing, and therefore

$$\lim_{n\to\infty} z_n = \eta \in [0, \infty). \tag{54}$$

Now we will prove that $\eta = 0$. If $\eta > 0$, then there exists $N \in \mathbf{N}$ and $\varepsilon > 0$ such that for $n - m > N$, $0 < \eta - \varepsilon < z_n < \eta + \varepsilon$. Hence $\eta - \varepsilon < z_{n-m}$ and $\eta - \varepsilon < z_{n-1+m}$. So inequality (52) yields

$$\begin{aligned} &z_{n+1} - z_n - h\theta T\left[\frac{1+K^\mu}{1+K^\mu e^{\mu(\eta-\varepsilon)}} - 1\right] \\ &\quad - h(1-\theta) T\left[\frac{1+K^\mu}{1+K^\mu e^{\mu(\eta-\varepsilon)}} - 1\right] < 0, \end{aligned} \tag{55}$$

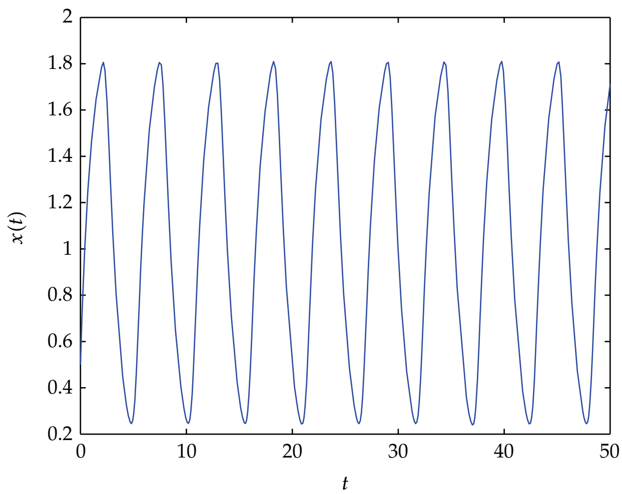

FIGURE 1: The analytic solution of (57).

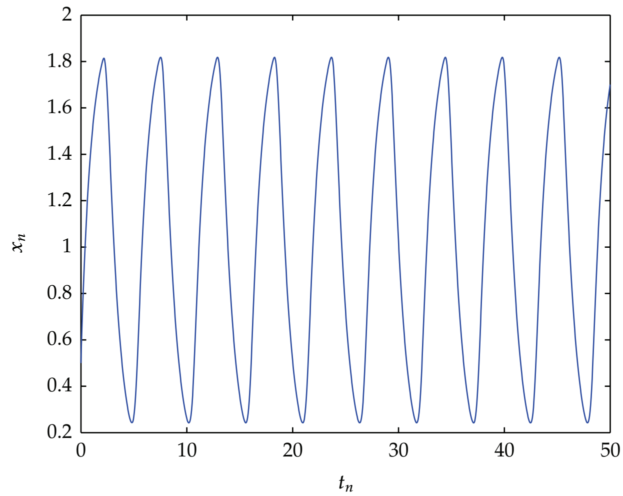

FIGURE 2: The numerical solution of (57) with $m = 40$ and $\theta = 0.2$.

which implies that $z_{n+1} - z_n < B < 0$, where

$$B = \frac{hTK^{\mu}\left(1 - e^{\mu(\eta-\varepsilon)}\right)}{1 + K^{\mu}e^{\mu(\eta-\varepsilon)}}. \tag{56}$$

Thus $z_n \to -\infty$ as $n \to \infty$, which is a contradiction to (54). This completes the proof. □

Therefore, the second main theorem of this paper is as follows.

Theorem 22. *Let x_n be a positive solution of* (19), *which does not oscillate about K^*, then $\lim_{n\to\infty} x_n = K^*$.*

5. Numerical Experiments

In this section, we will give some numerical examples to illustrate our results.

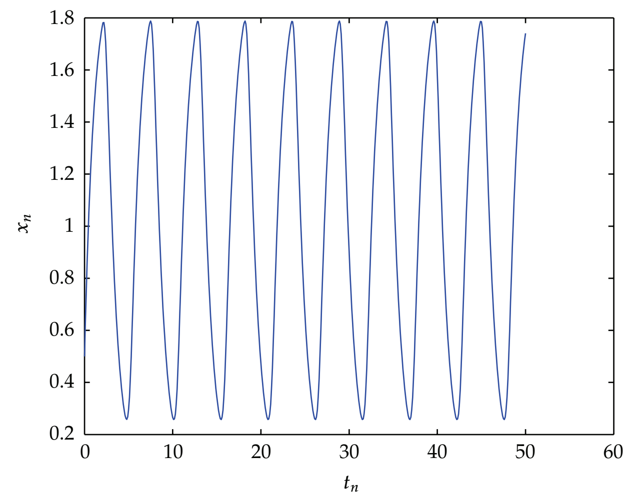

FIGURE 3: The numerical solution of (57) with $m = 20$ and $\theta = 0.8$.

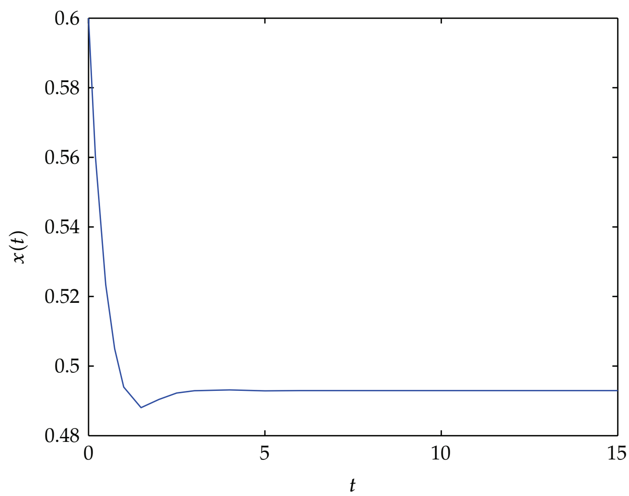

FIGURE 4: The analytic solution of (58).

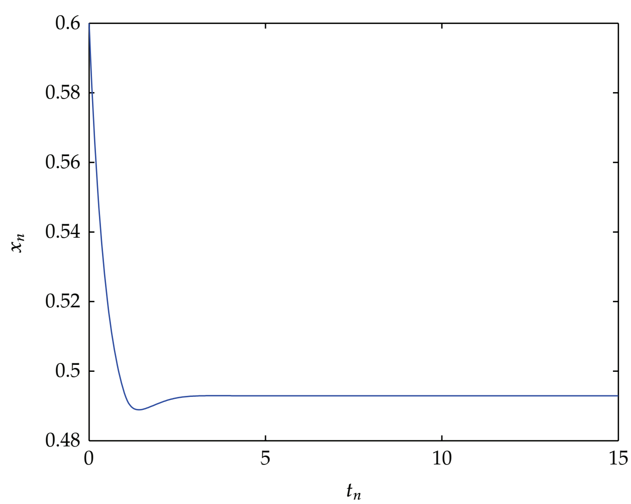

FIGURE 5: The numerical solution of (58) with $m = 25$ and $\theta = 0.3$.

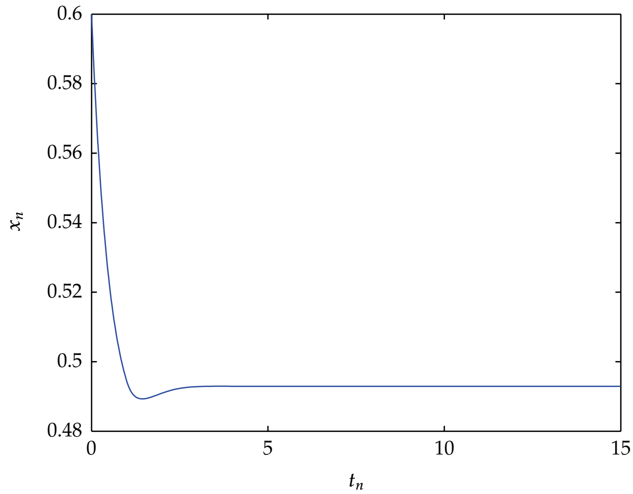

FIGURE 6: The numerical solution of (58) with $m = 50$ and $\theta = 0.6$.

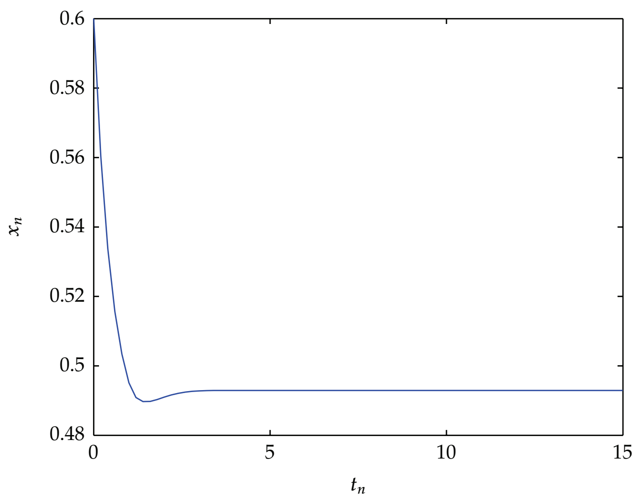

FIGURE 7: The numerical solution of (58) with $m = 5$ and $\theta = 0.6$.

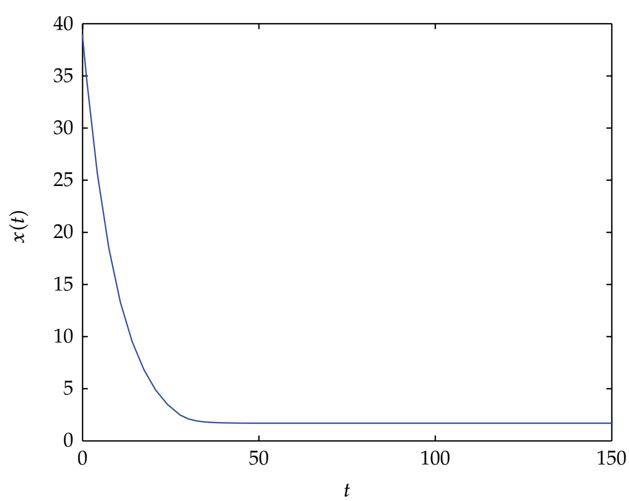

FIGURE 8: The analytic solution of (59).

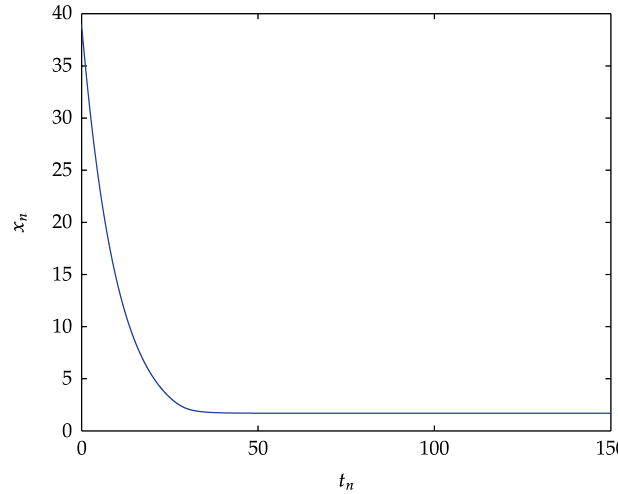

FIGURE 9: The numerical solution of (59) with $m = 10$ and $\theta = 0.4$.

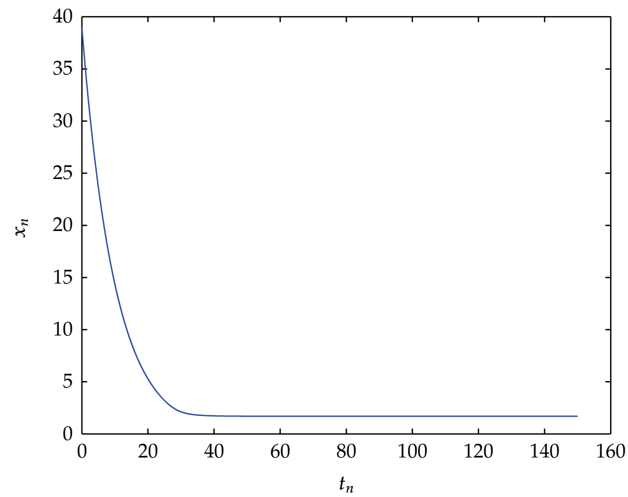

FIGURE 10: The numerical solution of (59) with $m = 15$ and $\theta = 0.75$.

Firstly, we consider the equation

$$x'(t) = \frac{2}{1 + x^7(t-2)} - x(t), \tag{57}$$

with initial value $x(t) = 0.5$ for $t \leq 0$. In (57), it is easy to see that condition (11) holds true and $L\tau \approx 7 > 1$. That is, the analytic solutions of (57) are oscillatory. In Figures 1–3, we draw the figures of the analytic solutions and the numerical solutions of (57), respectively. The parameters $m = 40$, $\theta = 0.2$ in Figure 2 and $m = 20$, $\theta = 0.8$ in Figure 3. From the two figures, we can see that the numerical solutions of (57) oscillate about $K^* = 1$, which are in agreement with Theorem 19.

Secondly, we consider

$$x'(t) = \frac{1}{1 + x^6(t-1)} - 2x(t), \tag{58}$$

with initial value $x(t) = 0.6$ for $t \leq 0$. In (58), it is not difficult to see that condition (11) is fulfilled. That is, the analytic solutions of (58) are oscillatory. In Figures 4–7, we draw the figures of the analytic solutions and the numerical solutions of (58), respectively. The parameters $m = 25$, $\theta = 0.3$ in Figure 5, $m = 50$, $\theta = 0.6$ in Figure 6 and $m = 5$, $\theta = 0.6$ in Figure 7. We can see from the three figures that the numerical solutions of (58) oscillate about $K^* \approx 0.4929$, which are consistent with Theorem 19. On the other hand, by direct calculation, we get $h_0 \approx 0.1882$. We notice that $h = 0.02 < h_0$ and $h = 0.2 > h_0$ in Figures 6 and 7, respectively, so the stepsize h_0 is not optimal.

Thirdly, we consider another equation,

$$x'(t) = \frac{0.2 \times 2.1^8}{2.1^8 + x^8(t - 0.15)} - 0.1x(t), \tag{59}$$

with initial value $x(t) = 39$ for $t \leq 0$. For (59), it is easy to see that $L\tau e^{T\tau+1} \approx 0.0505 < 1$, so the condition (11) is not satisfied. That is, the analytic solutions of (59) are nonoscillatory. In Figures 8–10, we draw the figures of the analytic solutions and the numerical solutions of (59), respectively. In Figure 8, we can see that $x(t) \rightarrow K^* \approx 1.6949$ as $t \rightarrow \infty$. From Figures 9 and 10, we can also see that the numerical solutions of (59) satisfy $x_n \rightarrow K^* \approx 1.6949$ as $n \rightarrow \infty$. That is, the numerical method preserves the asymptotic behavior of nonoscillatory solutions of (59), which coincides with Theorem 22.

Finally, according to Definition 14, we can see from these figures that the exponential θ-method preserves the oscillations of (57) and (58) and the nonoscillations of (59), respectively.

All the above numerical examples confirm our theoretical findings.

6. Conclusions

In this paper, we discuss the oscillations of the numerical solutions of a nonlinear delay differential equation in the control of erythropoiesis. The convergent exponential θ-method, namely the linear θ-method and the one-leg θ-method in exponential form, is constructed. We establish some conditions under which the numerical solutions oscillate in the case of oscillations of the analytic solutions. We also prove that nonoscillatory numerical solutions can inherit the corresponding properties of analytic solutions. It is pointed out that the stepsize h_0 in Lemma 17 is not optimal. Therefore, our future work will be devoted to investigating this problem.

Acknowledgments

The authors would like to thank Professor Mingzhu Liu and D. Zhanwen Yang for their useful suggestions. Q. Wang's work is supported by the National Natural Science Foundation of China (no. 11201084).

References

[1] L. Liu and Y. Bai, "New oscillation criteria for second-order nonlinear neutral delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 231, no. 2, pp. 657–663, 2009.

[2] E. M. Bonotto, L. P. Gimenes, and M. Federson, "Oscillation for a second-order neutral differential equation with impulses," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 1–15, 2009.

[3] C. Zhang, T. Li, B. Sun, and E. Thandapani, "On the oscillation of higher-order half-linear delay differential equations," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1618–1621, 2011.

[4] A. Weng and J. Sun, "Oscillation of second order delay differential equations," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 930–935, 2008.

[5] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, vol. 74 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[6] D. D. Baĭnov and D. P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, Bristol, UK, 1991.

[7] M. Z. Liu, J. Gao, and Z. W. Yang, "Oscillation analysis of numerical solution in the θ-methods for equation $x'(t)+ax(t)+a_1x([t-1]) = 0$," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 566–578, 2007.

[8] M. Z. Liu, J. F. Gao, and Z. W. Yang, "Preservation of oscillations of the Runge-Kutta method for equation $x'(t)+ax(t)+a_1x([t-1]) = 0$," *Computers & Mathematics with Applications*, vol. 58, no. 6, pp. 1113–1125, 2009.

[9] Q. Wang, Q. Zhu, and M. Liu, "Stability and oscillations of numerical solutions for differential equations with piecewise continuous arguments of alternately advanced and retarded type," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1542–1552, 2011.

[10] J. Gao, M. Song, and M. Liu, "Oscillation analysis of numerical solutions for nonlinear delay differential equations of population dynamics," *Mathematical Modelling and Analysis*, vol. 16, no. 3, pp. 365–375, 2011.

[11] M. C. Mackey and L. Glass, "Oscillations and chaos in physiological control system," *Science*, vol. 197, no. 4300, pp. 287–289, 1977.

[12] K. Gopalsamy, M. R. S. Kulenović, and G. Ladas, "Oscillations and global attractivity in models of hematopoiesis," *Journal of Dynamics and Differential Equations*, vol. 2, no. 2, pp. 117–132, 1990.

[13] A. Zaghrout, A. Ammar, and M. M. A. El-Sheikh, "Oscillations and global attractivity in delay differential equations of population dynamics," *Applied Mathematics and Computation*, vol. 77, no. 2-3, pp. 195–204, 1996.

[14] M. C. Mackey and J. G. Milton, "Dynamical diseases," *Annals of the New York Academy of Sciences*, vol. 316, pp. 214–235, 1979.

[15] M. C. Mackey, "Some models in hemopoiesis: predictions and problems," in *Biomathematics and Cell Kinetics*, pp. 23–38, Elsevier/North-Holland, Amsterdam, The Netherlands, 1979.

[16] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, NY, USA, 1991.

[17] M. H. Song, Z. W. Yang, and M. Z. Liu, "Stability of θ-methods for advanced differential equations with piecewise continuous arguments," *Computers & Mathematics with Applications*, vol. 49, no. 9-10, pp. 1295–1301, 2005.

[18] J. Hale, *Theory of Functional Differential Equations*, vol. 3, Springer, New York, NY, USA, 2nd edition, 1977.

[19] Z. Yang, M. Liu, and M. Song, "Stability of Runge-Kutta methods in the numerical solution of equation $u'(t) = au(t) + a_0u([t]) + a_1u([t-1])$," *Applied Mathematics and Computation*, vol. 162, no. 1, pp. 37–50, 2005.