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Research Article

On a System of Difference Equations

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We have investigated the periodical solutions of the system of rational difference equations $x_{n+1} = y_{n-2}/(-1 \pm y_{n-2}x_{n-1}y_n)$, $y_{n+1} = x_{n-2}/(-1 \pm x_{n-2}y_{n-1}x_n)$, and $z_{n+1} = (x_{n-2} + y_{n-2})/(-1 \pm x_{n-2}y_{n-1}x_n)$, where $y_0, y_{-1}, y_{-2}, x_0, x_{-1}, x_{-2}, z_0, z_{-1}, z_{-2} \in \mathbb{R}$.

1. Introduction

Recently, a great interest has arisen on studying difference equation systems. One of the reasons for that is the necessity for some techniques which can be used in investigating equations which originate in mathematical models to describe real-life situations such as population biology, economics, probability theory, genetics, and psychology. There are many papers related to the difference equations system.

In [1], Kurbanli et al. studied the periodicity of solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1} + x_n}{x_n y_{n-1} - 1}. \tag{1}$$

In [2], Çinar studied the solutions of the systems of difference equations

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}. \tag{2}$$

In [3, 4], Özban studied the positive solutions of the system of rational difference equations

$$x_n = \frac{a}{y_{n-3}}, \qquad y_n = \frac{b y_{n-3}}{x_{n-q} y_{n-q}},$$
$$x_{n+1} = \frac{1}{y_{n-k}}, \qquad y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}. \tag{3}$$

In [5–16], Elsayed studied a variety of systems of rational difference equations; for more, see references.

In this paper, we have investigated the periodical solutions of the system of difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2} x_{n-1} y_n}, \qquad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n},$$
$$z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}, \quad n \in \mathbb{N}_0, \tag{4}$$

where the initial conditions are arbitrary real numbers.

2. Main Results

Theorem 1. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, and $z_{-2} = q$ be arbitrary real numbers, and let $\{x_n, y_n, z_n\}$ be a solution of the system*

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2} x_{n-1} y_n}, \qquad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2} y_{n-1} x_n},$$
$$z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 + x_{n-2} y_{n-1} x_n}, \quad n \in \mathbb{N}_0. \tag{5}$$

Also, assume that $b \neq 0$, $e \neq 0$, $fbd \neq 1$, and $cea \neq 1$. Then, all six-period solutions of (5) are as follows:

$$x_{6n+1} = \frac{c}{1 - cea}, \qquad y_{6n+1} = \frac{f}{fbd - 1},$$
$$z_{6n+1} = -\frac{f + c}{fbd - 1},$$

$$x_{6n+1}=b\left(fbd-1\right),\qquad y_{6n+2}=e\left(cea-1\right),$$
$$z_{6n+2}=-\left(e+b\right)\left(cea+1\right),$$
$$x_{6n+1}=\frac{a}{1-cea},\qquad y_{6n+3}=\frac{d}{fbd-1},$$
$$z_{6n+3}=-\frac{d+a}{fbd+1},$$
$$x_{6n+1}=f,\qquad y_{6n+4}=c,$$
$$z_{6n+4}=\frac{c\left(fbd+1\right)+f\left(cea+1\right)}{fbd+1},$$
$$x_{6n+1}=e,\qquad y_{6n+5}=b,$$
$$z_{6n+5}=\frac{b\left(fbd+1\right)+e\left(cea+1\right)}{fbd+1},$$
$$x_{6n+1}=d,\qquad y_{6n+6}=a,$$
$$z_{6n+6}=\frac{a\left(fbd+1\right)+d\left(cea+1\right)}{fbd+1},$$
$$n\in\mathbb{N}_0. \tag{6}$$

Proof. For $n=0,1,2,3,4,5$, we have

$$x_1=\frac{y_{-2}}{-1+y_{-2}x_{-1}y_0}=\frac{c}{-1+cea},$$
$$y_1=\frac{x_{-2}}{-1+x_{-2}y_{-1}x_0}=\frac{f}{-1+fbd},$$
$$z_1=\frac{x_{-2}+y_{-2}}{-1+y_{-2}x_{-1}y_0}=\frac{f+c}{-1+fbd},$$
$$x_2=\frac{y_{-1}}{-1+y_{-1}x_0y_1}=\frac{b}{-1+bd\left(f/\left(fbd-1\right)\right)}$$
$$=\frac{b}{1/\left(fbd-1\right)}=b\left(fbd-1\right),$$
$$y_2=\frac{x_{-1}}{-1+x_{-1}y_0x_1}=\frac{e}{-1+ea\left(c/\left(cea-1\right)\right)}$$
$$=\frac{e}{1/\left(cea-1\right)}=e\left(cea-1\right),$$
$$z_2=\frac{x_{-1}+y_{-1}}{-1+x_{-1}y_0x_1}=\frac{e+b}{-1+ea\left(c/\left(cea-1\right)\right)}$$
$$=\frac{e+b}{1/\left(cea-1\right)}=\left(e+b\right)\left(cea-1\right),$$
$$x_3=\frac{y_0}{-1+y_0x_1y_2}=\frac{a}{-1+a\left(c/\left(cea-1\right)\right)e\left(cea-1\right)}$$
$$=\frac{a}{cea-1},$$
$$y_3=\frac{x_0}{-1+x_0y_1x_2}=\frac{d}{-1+d\left(f/\left(fbd-1\right)\right)b\left(fbd-1\right)}$$
$$=\frac{d}{fbd-1},$$
$$z_3=\frac{x_0+y_0}{-1+x_0y_1x_2}$$
$$=\frac{d+a}{-1+d\left(f/\left(fbd-1\right)\right)b\left(fbd-1\right)}=\frac{d+a}{fbd-1},$$
$$x_4=\frac{y_1}{-1+y_1x_2y_3}$$
$$=\frac{f/\left(fbd-1\right)}{-1+\left(f/\left(fbd-1\right)\right)b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)}$$
$$=\frac{f/\left(fbd-1\right)}{-1+\left(fbd/\left(fbd-1\right)\right)}=\frac{f/\left(fbd-1\right)}{1/\left(fbd-1\right)}=f,$$
$$y_4=\frac{x_1}{-1+x_1y_2x_3}$$
$$=\frac{c/\left(cea-1\right)}{-1+\left(c/\left(cea-1\right)\right)e\left(cea-1\right)\left(a/\left(cea-1\right)\right)}$$
$$=\frac{c/\left(cea-1\right)}{-1+\left(cea/\left(cea-1\right)\right)}=\frac{c/\left(cea-1\right)}{1/\left(cea-1\right)}=c,$$
$$z_4=\frac{x_1+y_1}{-1+x_1y_2x_3}$$
$$=\frac{\left(c/\left(cea-1\right)\right)+\left(f/\left(fbd-1\right)\right)}{-1+\left(c/\left(cea-1\right)\right)e\left(cea-1\right)\left(a/\left(cea-1\right)\right)}$$
$$=\frac{c\left(fbd-1\right)+f\left(cea-1\right)}{fbd-1},$$
$$x_5=\frac{y_2}{-1+y_2x_3y_4}$$
$$=\frac{e\left(cea-1\right)}{-1+e\left(cea-1\right)\left(a/\left(cea-1\right)\right)c}=e,$$
$$y_5=\frac{x_2}{-1+x_2y_3x_4}$$
$$=\frac{b\left(fbd-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f}=b,$$
$$z_5=\frac{x_2+y_2}{-1+x_2y_3x_4}$$
$$=\frac{b\left(fbd-1\right)+e\left(cea-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f}$$
$$=\frac{b\left(fbd-1\right)+e\left(cea-1\right)}{fbd-1},$$
$$x_6=\frac{y_3}{-1+y_3x_4y_5}=\frac{d/\left(fbd-1\right)}{-1+\left(d/\left(fbd-1\right)\right)fb}=d,$$

$$
\begin{aligned}
y_6 &= \frac{x_3}{-1 + x_3 y_4 x_5} = \frac{a/(cea-1)}{-1 + (a/(cea-1))ce} = a, \\
z_6 &= \frac{x_3 + y_3}{-1 + x_3 y_4 x_5} \\
&= \frac{(a/(cea-1)) + (d/(fbd-1))}{-1 + (a/(cea-1))ce} \\
&= \frac{a(fbd-1) + d(cea-1)}{fbd-1}.
\end{aligned} \tag{7}
$$

For $n = 6, 7, 8, 9, 10, 11$, assume that

$$
\begin{aligned}
x_7 &= \frac{y_4}{-1 + y_4 x_5 y_6} = \frac{c}{-1 + cea} = x_1, \\
y_7 &= \frac{x_4}{-1 + x_4 y_5 x_6} = \frac{f}{-1 + fbd} = y_1, \\
z_7 &= \frac{x_4 + y_4}{-1 + x_4 y_5 x_6} = \frac{f + c}{-1 + fbd} = z_1, \\
x_8 &= \frac{y_5}{-1 + y_5 x_6 y_7} = \frac{b}{-1 + bd(f/(-1 + fbd))} \\
&= b(fbd-1) = x_2, \\
y_8 &= \frac{x_5}{-1 + x_5 y_6 x_7} = \frac{e}{-1 + ea(c/(-1 + cea))} \\
&= e(cea-1) = y_2, \\
z_8 &= \frac{x_5 + y_5}{-1 + x_5 y_6 x_7} = \frac{e + b}{-1 + ea(c/(-1 + cea))} \\
&= (e+b)(cea-1) = z_2, \\
x_9 &= \frac{y_6}{-1 + y_6 x_7 y_8} = \frac{a}{-1 + a(c/(-1 + cea))e(cea-1)} \\
&= \frac{a}{cea-1} = x_3, \\
y_9 &= \frac{x_6}{-1 + x_6 y_7 x_8} \\
&= \frac{d}{-1 + d(f/(-1 + fbd))b(fbd-1)} \\
&= \frac{d}{fbd-1} = y_3, \\
z_9 &= \frac{x_6 + y_6}{-1 + x_6 y_7 x_8} \\
&= \frac{d + a}{-1 + d(f/(-1 + fbd))b(fbd-1)} \\
&= \frac{d + a}{fbd-1} = z_3, \\
x_{10} &= \frac{y_7}{-1 + y_7 x_8 y_9} \\
&= \frac{f/(-1 + fbd)}{-1 + (f/(-1 + fbd))b(fbd-1)(d/(fbd-1))} \\
&= \frac{f/(-1 + fbd)}{-1 + (fbd/(-1 + fbd))} = f = x_4, \\
y_{10} &= \frac{x_7}{-1 + x_7 y_8 x_9} \\
&= \frac{c/(-1 + cea)}{-1 + (c/(-1 + cea))e(cea-1)(a/(cea-1))} \\
&= \frac{c/(-1 + cea)}{-1 + (cea/(-1 + cea))} = c = y_4, \\
z_{10} &= \frac{x_7 + y_7}{-1 + x_7 y_8 x_9} \\
&= \frac{(c/(-1 + cea)) + (f/(-1 + fbd))}{-1 + (c/(-1 + cea))e(cea-1)(a/(cea-1))} \\
&= \frac{c(fbd-1) + f(cea-1)}{fbd-1} = z_4, \\
x_{11} &= \frac{y_8}{-1 + y_8 x_9 y_{10}} \\
&= \frac{e(cea-1)}{-1 + e(cea-1)(a/(cea-1))c} = e = x_5, \\
y_{11} &= \frac{x_8}{-1 + x_8 y_9 x_{10}} \\
&= \frac{b(fbd-1)}{-1 + b(fbd-1)(d/(fbd-1))f} = b = y_5, \\
z_{11} &= \frac{x_8 + y_8}{-1 + x_8 y_9 x_{10}} \\
&= \frac{b(fbd-1) + e(cea-1)}{-1 + b(fbd-1)(d/(fbd-1))f} \\
&= \frac{b(fbd-1) + e(cea-1)}{fbd-1} = z_5, \\
x_{12} &= \frac{y_9}{-1 + y_9 x_{10} y_{11}} = \frac{d/(fbd-1)}{-1 + (d/(fbd-1))fb} = d = x_6, \\
y_{12} &= \frac{x_9}{-1 + x_9 y_{10} x_{11}} = \frac{a/(cea-1)}{-1 + (a/(cea-1))ce} = a = y_6, \\
z_{12} &= \frac{x_9 + y_9}{-1 + x_9 y_{10} x_{11}} = \frac{(a/(cea-1)) + (d/(fbd-1))}{-1 + (a/(cea-1))ce} \\
&= \frac{a(fbd-1) + d(cea-1)}{fbd-1} = z_6
\end{aligned} \tag{8}
$$

are true. Also, we have

$$
\begin{aligned}
x_1 &= \frac{c}{cea-1} = x_7 = x_{13} = \cdots = x_{6n+1}, \quad n \in \mathbb{N}_0, \\
x_2 &= b(fbd-1) = x_8 = x_{14} = \cdots = x_{6n+2}, \quad n \in \mathbb{N}_0,
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \frac{a}{cea-1} = x_9 = x_{15} = \cdots = x_{6n+3}, \quad n \in \mathbb{N}_0,\\
x_4 &= f = x_{10} = x_{16} = \cdots = x_{6n+4}, \quad n \in \mathbb{N}_0,\\
x_5 &= e = x_{11} = x_{17} = \cdots = x_{6n+5}, \quad n \in \mathbb{N}_0,\\
x_6 &= d = x_{12} = x_{18} = \cdots = x_{6n+6}, \quad n \in \mathbb{N}_0,\\
y_1 &= \frac{f}{fbd-1} = y_7 = y_{13} = \cdots = y_{6n+1}, \quad n \in \mathbb{N}_0,\\
y_2 &= e(cea-1) = y_8 = y_{14} = \cdots = y_{6n+2}, \quad n \in \mathbb{N}_0,\\
y_3 &= \frac{d}{fbd-1} = y_9 = y_{15} = \cdots = y_{6n+3}, \quad n \in \mathbb{N}_0,\\
y_4 &= c = y_{10} = y_{16} = \cdots = y_{6n+4}, \quad n \in \mathbb{N}_0,\\
y_5 &= b = y_{11} = y_{17} = \cdots = y_{6n+5}, \quad n \in \mathbb{N}_0,\\
y_6 &= a = y_{12} = y_{18} = \cdots = y_{6n+6}, \quad n \in \mathbb{N}_0,\\
z_1 &= \frac{f+c}{fbd-1} = z_7 = z_{13} = \cdots = z_{6n+1}, \quad n \in \mathbb{N}_0,\\
z_2 &= (e+b)(cea-1) = z_8 = z_{14} = \cdots = z_{6n+2}, \quad n \in \mathbb{N}_0,\\
z_3 &= \frac{d+a}{fbd-1} = z_9 = z_{15} = \cdots = z_{6n+3}, \quad n \in \mathbb{N}_0,\\
z_4 &= \frac{c(fbd-1)+f(cea-1)}{fbd-1} = z_{10}\\
&= z_{16} = \cdots = z_{6n+4}, \quad n \in \mathbb{N}_0,\\
z_5 &= \frac{b(fbd-1)+e(cea-1)}{fbd-1} = z_{11}\\
&= z_{17} = \cdots = z_{6n+5}, \quad n \in \mathbb{N}_0,\\
z_6 &= \frac{a(fbd-1)+d(cea-1)}{fbd-1} = z_{12}\\
&= z_{18} = \cdots = z_{6n+6}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{9}
$$

□

Theorem 2. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, and $z_{-2} = q$ be arbitrary real numbers, and let $\{x_n, y_n, z_n\}$ be a solution of the system*

$$
x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \qquad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n},
$$
$$
z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{10}
$$

Also, assume that $b \neq 0$, $e \neq 0$, $fbd \neq -1$, and $cea \neq -1$. Then, all six-period solutions of (10) *are as follows:*

$$
\begin{aligned}
x_{6n+1} &= -\frac{c}{1+cea}, \qquad y_{6n+1} = -\frac{f}{fbd+1},\\
z_{6n+1} &= -\frac{-f(1+fbd)+c}{fbd+1},\\
x_{6n+1} &= -b(fbd+1), \qquad y_{6n+2} = -e(cea+1),\\
z_{6n+2} &= e - b(1+cea),\\
x_{6n+1} &= -\frac{a}{1+cea}, \qquad y_{6n+3} = -\frac{d}{fbd+1},\\
z_{6n+3} &= -\frac{-d(fbd+1)+a}{fbd+1},\\
x_{6n+1} &= f, \qquad y_{6n+4} = c,\\
z_{6n+4} &= \frac{-c(fbd+1)+f(1+cea+c^2e^2a^2)}{(fbd+1)(cea+1)},\\
x_{6n+1} &= e, \qquad y_{6n+5} = b,\\
z_{6n+5} &= \frac{-b(1+2fbd+f^2b^2d^2)+e(1+cea)}{fbd+1},\\
x_{6n+1} &= d, \qquad y_{6n+6} = a,\\
z_{6n+6} &= \frac{-a(fbd+1)+d(1+cea+c^2e^2a^2)}{(1+cea)(fbd+1)},\\
&\qquad n \in \mathbb{N}_0.
\end{aligned} \tag{11}
$$

Proof. For $n = 0, 1, 2, 3, 4, 5$, we have

$$
\begin{aligned}
x_1 &= \frac{y_{-2}}{-1 - y_{-2}x_{-1}y_0} = \frac{c}{-1-cea} = -\frac{c}{1+cea},\\
y_1 &= \frac{x_{-2}}{-1 - x_{-2}y_{-1}x_0} = \frac{f}{-1-fbd} = -\frac{f}{1+fbd},\\
z_1 &= \frac{x_{-2}+y_{-2}}{-1 - y_{-2}x_{-1}y_0} = \frac{f+c}{-1-fbd} = -\frac{f+c}{1+fbd},\\
x_2 &= \frac{y_{-1}}{-1 - y_{-1}x_0y_1} = \frac{b}{-1-bd(-(f/(fbd+1)))}\\
&= \frac{b}{-(1/(fbd+1))} = -b(1+fbd),\\
y_2 &= \frac{x_{-1}}{-1 - x_{-1}y_0x_1} = \frac{e}{-1-ea(-(c/(1+cea)))}\\
&= \frac{e}{-(1/(1+cea))} = -e(1+cea),\\
z_2 &= \frac{x_{-1}+y_{-1}}{-1 - x_{-1}y_0x_1} = \frac{e+b}{-1-ea(-(c/(1+cea)))}\\
&= \frac{e+b}{-(1/(1+cea))} = -(e+b)(1+cea),\\
x_3 &= \frac{y_0}{-1 - y_0x_1y_2}\\
&= \frac{a}{-1-a(-(c/(1+cea)))(-e(1+cea))}\\
&= -\frac{a}{1+cea},
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= \frac{x_0}{-1 - x_0 y_1 x_2} \\
&= \frac{d}{-1 - d\left(-\left(f/\left(1+fbd\right)\right)\right)\left(-b\left(1+fbd\right)\right)} \\
&= -\frac{d}{1+fbd}, \\
z_3 &= \frac{x_0 + y_0}{-1 - x_0 y_1 x_2} \\
&= \frac{d+a}{-1 - d\left(-\left(f/\left(1+fbd\right)\right)\right)\left(-b\left(1+fbd\right)\right)} \\
&= -\frac{d+a}{1+fbd}, \\
x_4 &= \frac{y_1}{-1 - y_1 x_2 y_3} \\
&= \frac{-\left(f/\left(1+fbd\right)\right)}{-1 + \left(f/\left(1+fbd\right)\right) b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right)} \\
&= \frac{-\left(f/\left(1+fbd\right)\right)}{-1 + \left(fbd/\left(1+fbd\right)\right)} = \frac{-\left(f/\left(1+fbd\right)\right)}{-\left(1/\left(1+fbd\right)\right)} = f, \\
y_4 &= \frac{x_1}{-1 - x_1 y_2 x_3} \\
&= \frac{-\left(c/\left(1+cea\right)\right)}{-1 + \left(c/\left(1+cea\right)\right) e\left(1+cea\right)\left(a/\left(1+cea\right)\right)} \\
&= \frac{-\left(c/\left(1+cea\right)\right)}{-1 + \left(cea/\left(1+cea\right)\right)} = \frac{-\left(c/\left(1+cea\right)\right)}{-\left(1/\left(1+cea\right)\right)} = c, \\
z_4 &= \frac{x_1 + y_1}{-1 - x_1 y_2 x_3} \\
&= \frac{-\left(c/\left(1+cea\right)\right) - \left(f/\left(1+fbd\right)\right)}{-1 + \left(c/\left(1+cea\right)\right) e\left(1+cea\right)\left(a/\left(1+cea\right)\right)} \\
&= \frac{c\left(1+fbd\right) + f\left(1+cea\right)}{1+fbd}, \\
x_5 &= \frac{y_2}{-1 - y_2 x_3 y_4} \\
&= \frac{-e\left(1+cea\right)}{-1 - e\left(1+cea\right)\left(a/\left(1+cea\right)\right) c} = e, \\
y_5 &= \frac{x_2}{-1 - x_2 y_3 x_4} \\
&= \frac{-b\left(1+fbd\right)}{-1 - b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right) f} = b, \\
z_5 &= \frac{x_2 + y_2}{-1 - x_2 y_3 x_4} \\
&= \frac{-b\left(1+fbd\right) - e\left(1+cea\right)}{-1 - b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right) f} \\
&= \frac{b\left(1+fbd\right) + e\left(1+cea\right)}{1+fbd}, \\
x_6 &= \frac{y_3}{-1 - y_3 x_4 y_5} \\
&= \frac{-\left(d/\left(1+fbd\right)\right)}{-1 + \left(d/\left(1+fbd\right)\right) fb} = d, \\
y_6 &= \frac{x_3}{-1 - x_3 y_4 x_5} \\
&= \frac{-\left(a/\left(1+cea\right)\right)}{-1 + \left(a/\left(1+cea\right)\right) ce} = a, \\
z_6 &= \frac{x_3 + y_3}{-1 - x_3 y_4 x_5} \\
&= \frac{-\left(a/\left(1+cea\right)\right) - \left(d/\left(1+fbd\right)\right)}{-1 + \left(a/\left(1+cea\right)\right) ce} \\
&= \frac{a\left(1+fbd\right) + d\left(1+cea\right)}{1+fbd}.
\end{aligned}
\tag{12}
$$

For $n = 6, 7, 8, 9, 10, 11$, assume that

$$
\begin{aligned}
x_7 &= \frac{y_4}{-1 - y_4 x_5 y_6} = \frac{c}{-1 - cea} = -\frac{c}{1+cea} = x_1, \\
y_7 &= \frac{x_4}{-1 - x_4 y_5 x_6} = \frac{f}{-1 - fbd} = -\frac{f}{1+fbd} = y_1, \\
z_7 &= \frac{x_4 + y_4}{-1 - x_4 y_5 x_6} = \frac{f+c}{-1 - fbd} = -\frac{f+c}{1+fbd} = z_1, \\
x_8 &= \frac{y_5}{-1 - y_5 x_6 y_7} = \frac{b}{-1 + bd\left(f/\left(1+fbd\right)\right)} \\
&= -b\left(1+fbd\right) = x_2, \\
y_8 &= \frac{x_5}{-1 - x_5 y_6 x_7} = \frac{e}{-1 + ea\left(c/\left(1+cea\right)\right)} \\
&= -e\left(1+cea\right) = y_2, \\
z_8 &= \frac{x_5 + y_5}{-1 - x_5 y_6 x_7} = \frac{e+b}{-1 + ea\left(c/\left(1+cea\right)\right)} \\
&= -\left(e+b\right)\left(1+cea\right) = z_2, \\
x_9 &= \frac{y_6}{-1 - y_6 x_7 y_8} = \frac{a}{-1 - a\left(c/\left(1+cea\right)\right) e\left(1+cea\right)} \\
&= -\frac{a}{1+cea} = x_3, \\
y_9 &= \frac{x_6}{-1 - x_6 y_7 x_8} = \frac{d}{-1 - d\left(f/\left(1+fbd\right)\right) b\left(1+fbd\right)} \\
&= -\frac{d}{1+fbd} = y_3,
\end{aligned}
$$

$$
\begin{aligned}
z_9 &= \frac{x_6 + y_6}{-1 - x_6 y_7 x_8} = \frac{d + a}{-1 - d\left(f/\left(1 + fbd\right)\right) b\left(1 + fbd\right)} \\
&= -\frac{d + a}{1 + fbd} = z_3, \\
x_{10} &= \frac{y_7}{-1 - y_7 x_8 y_9} \\
&= \frac{-\left(f/\left(1 + fbd\right)\right)}{-1 + \left(f/\left(1 + fbd\right)\right) b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right)} \\
&= \frac{-\left(f/\left(1 + fbd\right)\right)}{-1/\left(1 + fbd\right)} = f = x_4, \\
y_{10} &= \frac{x_7}{-1 - x_7 y_8 x_9} \\
&= \frac{-\left(c/\left(1 + cea\right)\right)}{-1 + \left(c/\left(1 + cea\right)\right) e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right)} \\
&= \frac{-\left(c/\left(1 + cea\right)\right)}{-1/\left(1 + cea\right)} = c = y_4, \\
z_{10} &= \frac{x_7 + y_7}{-1 - x_7 y_8 x_9} \\
&= \frac{-\left(c/\left(1 + cea\right)\right) - \left(f/\left(1 + fbd\right)\right)}{-1 + \left(c/\left(1 + cea\right)\right) e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right)} \\
&= \frac{c\left(1 + fbd\right) + f\left(1 + cea\right)}{1 + fbd} = z_4, \\
x_{11} &= \frac{y_8}{-1 - y_8 x_9 y_{10}} \\
&= \frac{-e\left(1 + cea\right)}{-1 - e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right) c} = e = x_5, \\
y_{11} &= \frac{x_8}{-1 - x_8 y_9 x_{10}} \\
&= \frac{-b\left(1 + fbd\right)}{-1 - b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right) f} = b = y_5, \\
z_{11} &= \frac{x_8 + y_8}{-1 - x_8 y_9 x_{10}} \\
&= \frac{-b\left(1 + fbd\right) - e\left(1 + cea\right)}{-1 - b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right) f} \\
&= \frac{b\left(1 + fbd\right) + e\left(1 + cea\right)}{1 + fbd} = z_5, \\
x_{12} &= \frac{y_9}{-1 - y_9 x_{10} y_{11}} \\
&= \frac{-\left(d/\left(1 + fbd\right)\right)}{-1 + \left(d/\left(1 + fbd\right)\right) fb} = d = x_6, \\
y_{12} &= \frac{x_9}{-1 - x_9 y_{10} x_{11}} \\
&= \frac{-\left(a/\left(1 + cea\right)\right)}{-1 + \left(a/\left(1 + cea\right)\right) ce} = a = y_6, \\
z_{12} &= \frac{x_9 + y_9}{-1 - x_9 y_{10} x_{11}} \\
&= \frac{-\left(a/\left(1 + cea\right)\right) - \left(d/\left(1 + fbd\right)\right)}{-1 + \left(a/\left(1 + cea\right)\right) ce} \\
&= \frac{a\left(1 + fbd\right) + d\left(1 + cea\right)}{1 + fbd} = z_6
\end{aligned} \tag{13}
$$

are true. Also, we have

$$
\begin{aligned}
& x_1 = -\frac{c}{1 + cea} = x_7 = x_{13} = \cdots = x_{6n+1}, \quad n = 0, 1, 2, 3, \ldots, \\
& x_2 = -b\left(1 + fbd\right) = x_8 = x_{14} = \cdots = x_{6n+2}, \quad n = 0, 1, 2, 3, \ldots, \\
& x_3 = -\frac{a}{1 + cea} = x_9 = x_{15} = \cdots = x_{6n+3}, \quad n = 0, 1, 2, 3, \ldots, \\
& x_4 = f = x_{10} = x_{16} = \cdots = x_{6n+4}, \quad n = 0, 1, 2, 3, \ldots, \\
& x_5 = e = x_{11} = x_{17} = \cdots = x_{6n+5}, \quad n = 0, 1, 2, 3, \ldots, \\
& x_6 = d = x_{12} = x_{18} = \cdots = x_{6n+6}, \quad n = 0, 1, 2, 3, \ldots, \\
& y_1 = -\frac{f}{1 + fbd} = y_7 = y_{13} = \cdots = y_{6n+1}, \quad n \in \mathbb{N}_0, \\
& y_2 = -e\left(1 + cea\right) = y_8 = y_{14} = \cdots = y_{6n+2}, \quad n \in \mathbb{N}_0, \\
& y_3 = -\frac{d}{1 + fbd} = y_9 = y_{15} = \cdots = y_{6n+3}, \quad n \in \mathbb{N}_0, \\
& y_4 = c = y_{10} = y_{16} = \cdots = y_{6n+4}, \quad n \in \mathbb{N}_0, \\
& y_5 = b = y_{11} = y_{17} = \cdots = y_{6n+5}, \quad n \in \mathbb{N}_0, \\
& y_6 = a = y_{12} = y_{18} = \cdots = y_{6n+6}, \quad n \in \mathbb{N}_0, \\
& z_1 = -\frac{f + c}{1 + fbd} = z_7 = z_{13} = \cdots = z_{6n+1}, \quad n \in \mathbb{N}_0, \\
& z_3 = -\frac{d + a}{1 + fbd} = z_9 = z_{15} = \cdots = z_{6n+3}, \quad n \in \mathbb{N}_0, \\
& z_4 = \frac{c\left(1 + fbd\right) + f\left(1 + cea\right)}{1 + fbd} = z_{10} = z_{16} = \cdots = z_{6n+4}, \quad n \in \mathbb{N}_0, \\
& z_5 = \frac{b\left(1 + fbd\right) + e\left(1 + cea\right)}{1 + fbd} = z_{11} = z_{17} = \cdots = z_{6n+5}, \quad n \in \mathbb{N}_0, \\
& z_6 = \frac{a\left(1 + fbd\right) + d\left(1 + cea\right)}{1 + fbd} = z_{12} = z_{18} = \cdots = z_{6n+6}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{14}
$$

□

The following corollary follows from Theorem 1.

Corollary 3. *The following conclusions are valid for $n \in \mathbb{N}$:*

(i) $x_{6n+2}y_{6n+3} = x_{6n+6}y_{6n+5}$,

(ii) $x_{6n+1}y_{6n+2} = x_{6n+5}y_{6n+6}$,

(iii) $x_{6n+1}y_{6n+6} = x_{6n+3}y_{6n+4}$,

(iv) $x_{6n+4}y_{6n+3} = x_{6n+6}y_{6n+1}$.

The following corollary follows from Theorem 2.

Corollary 4. *The following conclusions are valid for $n \in \mathbb{N}$:*

(i) $x_{6n+1}y_{6n+6} = x_{6n+3}y_{6n+4}$,

(ii) $x_{6n+6}y_{6n+1} = x_{6n+4}y_{6n+3}$,

(iii) $x_{6n+3}y_{6n+2} = x_{6n+5}y_{6n+6}$,

(iv) $x_{6n+1}y_{6n+2} = x_{6n+5}y_{6n+4}$.

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