Analytical Investigation of the Onset of Bifurcation Cascade in Two Logistic-like Maps

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Two modifications of the classical one-dimensional logistic map are proposed, which permitted analytical study of the onset of the bifurcation (period doubling) cascade. The modified maps are two-parametric ones. This introduces new features in their behaviour and makes them more flexible. The results can prove to be useful in the ecological (population dynamics) modeling, where the logistic map is a basic model and also in other fields of application.

Keywords: Logistic map; Fixed points; Stability; Bifurcation cascade; Chaos

1. INTRODUCTION

The now well-studied logistic map

$$X_{n+1} = F(X_n) = rX_n(1 - X_n)$$
$$X \in [0,1], 0 \le r \le 4 \tag{1}$$

is one of the most popular "icons" of the nonlinear discrete dynamics (Ott, 1993; Peitgen *et al.*, 1992). Various modifications and generalizations of (1) are known. For example, letting $2Z = r(1 - 2X)$ and $4C = r(2 - r)$, one obtains the equivalent version of (1)

$$Z_{n+1} = Z_n^2 + C \tag{2}$$

suitable for extension in the complex plane (Peitgen *et al.*, 1992). Another example is the cubic logistic map (Korsch and Jodle, 1994)

$$X_{n+1} = sX_n(1 - X_n^2),$$
$$X \in [0,1], 0 \le s \le 3\sqrt{3}/2 \cong 2.5981 \tag{3}$$

This map has one critical point $X_c = 1/\sqrt{3}$ and a nontrivial fixed point $\bar{X}_1 = \sqrt{(r-1)/r}$, which is stable in the interval $1 < s < 2$. Between 2 and 2.5981 the map develops a bifurcation cascade of period doubling, leading to chaos. Polynomials of higher degree have also been worked out (Sprott, 1993).

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Here we study analytically the map

$$X_{n+1} = F(X_n) = \frac{r}{1 + bX_n} X_n(1 - X_n) \qquad (4)$$

where $b > -1$, $0 \leq r \leq r_1(b)$ (r_1 to be specified later under the condition $0 \leq X \leq 1$). To our knowledge, such a map has not been studied before. In Section 2 we show how the properties of the original logistic map (1) change when the new parameter varies. The results can be of interest for some applications. The derivations and discussions go simultaneously. Section 3 deals with an alternative polynomial type version of (4).

2. ANALYTICAL RESULTS AND DISCUSSIONS

Simple calculations show that (4) has negative Schwarzian derivative $S_F(X, b, r) < 0$ and one critical point ($F'(X_c, b, r) = 0$)

$$X_c(b) = \frac{1}{b}(-1 + \sqrt{1 + b}) \qquad (5)$$

at which

$$F(X_c) = G(b)r, \quad G(b) = \frac{1}{b^2}(2 + b - 2\sqrt{1 + b}). \quad (6)$$

Obviously $G(0) = 1/4$ and $0 < X_c(b) \leq 1/2$ for $b \geq 0$, while $1/2 \leq X_c(b) < 1$ for $0 \geq b > -1$. Hence, the

value of r at which $F(X_c) = 1$ is given by

$$r_1(b) = 1/G(b) \qquad (7)$$

(See Tab. I for X_c and r_1 at $b > 0$ and Tab. II at $b < 0$)

The new map (4) has two fixed points: $\bar{X}_0 = 0$ for all r and b, and

$$\bar{X}_1 = \frac{r - 1}{r + b} \text{ for } r > 1, b > -1. \qquad (8)$$

Since $F'(0) = r, \bar{X}_0$ is stable (unstable) for $r < 1$ ($r > 1$). For the other fixed point

$$F'(\bar{X}_1) = \frac{1}{b + 1}\left(2 - r + \frac{b}{r}\right), b > -1. \qquad (9)$$

It is stable if $|F'(\bar{X}_1)| < 1$. Hence

$$1 < r < r_s(b) = \frac{b + 3}{2} + \left[\frac{(b + 3)^2}{4} + b\right]^{1/2}, \quad (10)$$

where r_s is the positive root of the equation

$$r^2 - (b + 3)r - b = 0 \qquad (11)$$

(See Tabs. I and II for the values of r_s).

For $r > r_s$ two additional fixed points $\bar{X}_2, \bar{X}_3$ appear as roots of the equation $\bar{X} = F(F(\bar{X}))$ with $F(X)$ from (4). However, $\bar{X}_0$ and $\bar{X}_1$ (already unstable) satisfy this equation too. After factoring

TABLE I   Numerical values of X_c, r_1 and r_s for $b \geq 0$

b	0	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1.00
$X_c(b)$	0.5	0.4772	0.4580	0.4415	0.4271	0.4204	0.4173	0.4148	0.4142
$r_1(b)$	4	4.3909	4.7664	5.1298	5.4833	5.6568	5.7411	5.8130	5.8284
$r_s(b)$	3	3.2613	3.5138	3.7596	4	4.1185	4.1660	4.2244	4.2361
$r_1 - r_s = \Delta_r$	1	1.1296	1.2526	1.3702	1.4833	1.5383	1.5751	1.5886	1.5923

TABLE II   Numerical values of X_c, r_1 and r_s for $-1 < b \leq 0$

b	0	-0.2	-0.4	-0.6	-0.8	-0.9	-0.95	-0.99
$X_c(b)$	0.5	0.5279	0.5635	0.6126	0.6910	0.7597	0.8173	0.9091
$r_1(b)$	4	3.6036	3.1492	2.6647	2.0944	1.7325	1.4972	1.21
$r_s(b)$	3	2.7266	2.4358	2.1165	1.7403	1.5	1.3422	1.1465
$r_1 - r_s = \Delta_r$	1	0.877	0.7134	0.5482	0.3541	0.2325	0.155	0.0635

out them and solving the resulting quadratic equation

$$r(b - r)\bar{X}^2 - (r + 1)(b - r)\bar{X} - (r + 1) = 0$$

one obtains

$$\bar{X}_{2,3}(r,b) = \frac{r + 1}{2r} \pm \left[\frac{(r + 1)^2}{4r^2} - \frac{r + 1}{r(r - b)} \right]^{1/2}.$$
$$(12)$$

The 2-cycle (12) is stable for $r_s(b) < r < r_{s1}(b)$, where r_{s1} is the smallest positive root of the equation $F'(\bar{X}_2) \cdot F'(\bar{X}_3) = -1$. However, explicit solution for r_{s1} (b) cannot be obtained. But $r_{s1}(0) = 1 + \sqrt{6}$ as it must be (Hilborn, 1994). Due to the rapidly increasing complexity, further analytical treatment is futureless.

Looking on Table I we see that for $b > 0$ the range $\Delta_r(b) = r_1(b) - r_s(b)$ inside which all known bifurcation events (period-doubling cascade, bands of regularity, full chaos) are expected to be observed is wider than $\Delta_r(0) = 4 - 3 = 1$ for the original map (1). In other words, through the new parameter $b(> 0)$ one can stretch this range to a desired width. Moreover $r_1(b) > 4$ and $r_s(b) > 3$. For various applications where the logistic map serves as a model (ecology, economics, *etc.*) this can help for better fitting of the model to the data.

An alternative approach to (4) is to fix r and to consider b as a free parameter. With a view to (1), two particular values of r are of special interest — $r = 3$ and $r = 4$. In the first case $(r = 3)$

$$\bar{X}_1 = \frac{2}{3 + b}, \quad F'(\bar{X}_1) = \frac{b - 3}{3(b + 1)} \qquad (13)$$

Hence, $|F'| > 1$ for $-1 < b < 0$ and $|F'| < 1$ for $b > 0$. In the second case $(r = 4)$

$$\bar{X}_1 = \frac{3}{4 + b}, \quad F'(\bar{X}_1) = \frac{b - 8}{4(b + 1)} \qquad (14)$$

Hence, $|F'| > 1$ (instability) for $-1 < b < 4/5$ and $|F'| < 1$ (stability) for $b > 4/5$.

In the general case, solving (11) with respect to b yields

$$b_s = \frac{r_s(r_s - 3)}{r_s + 1}, \quad r_s \geq 3 \qquad (15)$$

Particularly, for sequence of values $r'_s = 3, r''_s$, $r'''_s, \ldots, r^\infty_s \cong 3.57$ where 2^n-cycle $(n = 1, 2, \ldots)$ first appears in (1), the corresponding sequence $b'_s = 0, b''_s, b'''_s, \ldots, b^\infty_s \cong 0.4453$ can be calculated. At $n \to \infty$ the b-sequence obeys the Feigenbaum limit $\delta \cong 4.669\ldots$, as the r-sequence does.

Finally, if $b \equiv r$, *i.e.*,

$$X_{n+1} = \frac{rX_n(1 - X_n)}{1 + rX_n}, \quad r > 1, \qquad (16)$$

then

$$\bar{X}_1 = \frac{r - 1}{2r}, \quad F'(\bar{X}_1) = \frac{3 - r}{1 + r}$$

Hence, $|F'| < 1$ for all $r > 1$ and $\bar{X}_1$ remains a stable point. These are qualitatively new features of the map (4). In this connection, an intriguing question arises related to the modification of (2) on the analogy of (4):

$$Z_{n+1} = \frac{Z_n^2 + C}{1 + bZ_n} \qquad (17)$$

where $Z = X + iY, C = C_R + iC_I, i = \sqrt{-1}$, b-real parameter. In fact, (17) is a two-dimensional map. At $b = 0$ it generates the classical sets of Julia and Mandelbrot (Peitgen *et al.*, 1992). There are three possibilities for the new parameter: b-independent, $b \equiv C_R, b \equiv C_I$. The impact of each one on these sets remains to be investigated numerically.

3. AN ALTERNATIVE MAP TO (4)

We now consider the cubic map

$$X_{n+1} = F(X_n) = rX_n(1 + aX_n)(1 - X_n)$$
$$X \in [0, 1], \quad -1 \leq a \leq 1, \quad 0 < r \leq r_1(a), \quad (18)$$

with $r_1(a)$ to be defined below (*cf.* $r_1(b)$, (7)). Since $(1 + bX)^{-1} \approx 1 - bX$, (18) is closely related to (4),

TABLE III   Selected characteristics of the cubic map (18)

a	-1	$-1/2$	0	$1/2$	1
$X_c(a)$	$1/3$	$(\sqrt{3}-1)/\sqrt{3} \approx 0.4226$	$1/2$	$(\sqrt{7}-1)/3 \approx 0.5486$	$1/\sqrt{3} \approx 0.5774$
$\bar{X}(a,r)(r>1)$	$(\sqrt{r}-1)/\sqrt{r}$	$(3/2)-\sqrt{(8+r)/4r}$	$(r-1)/r$	$(-1/2)+\sqrt{(9r-8)/4r}$	$\sqrt{(r-1)/r}$
$r_1(a)$	$27/4 = 6.75$	6	4	3.1689	$(3\sqrt{3}/2) \approx 2.5981$
$r_s(a)$	4	3.8379	3	2.4194	2
$r_1 - r_s = \Delta_r(a)$	2.75	2.1621	1	0.7495	0.5981

but it may represent independent interest too. Obviously, at $a = 1$ one obtains (3), while

$$X_{n+1} = rX_n(1 - X_n)^2 \text{ at } a = -1. \quad (19)$$

This relationship implies a similarity in some respects in the behaviour of both maps. Actually, let us calculate and compare the same characteristics (5)–(10) for the map (18):

– the critical point and the nonzero fixed point

$$X_c(a) = \frac{1}{3a}\left(a - 1 \pm \sqrt{a^2 + a + 1}\right), \quad (20)$$

$$\bar{X}(a,r) = \frac{a-1}{2a} \pm \sqrt{\frac{(a-1)^2}{4a^2} + \frac{r-1}{ar}}, \quad r > 1 \quad (21)$$

with $(+)$ at $a > 0$ and $(-)$ at $a < 0$;
– the stability multiplier

$$F'(\bar{X}) = 3 - 2r - r(a - 1)\bar{X}(a,r); \quad (22)$$

(Unlike (10), the inequality $|F'| < 1$ cannot be solved explicitly to obtain $r_s(a)$).
– the upper bound $r_1(a)$ at which $F(X_c) = 1$

$$r_1(a) = 3/X_c(a)(2 + (a - 1)X_c(a)). \quad (23)$$

Some particular cases and numerical values of the above characteristics are summarized in Table III. Comparing to Table I and Table II, we observe the same effect of stretching $(a < 0)$ or shrinking $(a > 0)$ of the first bifurcation range $\Delta_r = r_1 - r_s$. Numerically the differences are not essential. However, from technical point of view, the map (4) is easier to work out than (18). In the same

time, the extension of (18) in the complex plane

$$Z_{n+1} = (Z_n^2 + C)(1 + aZ_n) \quad (24)$$

is simpler than (17) when represented as two-dimensional map. Like before, the new parameter a can be free, or $a \equiv C_R$, or $a \equiv C_I$. The impact of the particular choice on the Julia and Mandelbrot sets remains to be investigated numerically and compared to (17).

4. CONCLUSION

In conclusion, we have shown that the proposed modification (4) of the classical logistic map (1) can change the properties of the latter quantitatively (Tabs. I and II) and qualitatively as well. An appropriate choice of r and b can prevent development of bifurcation cascade and chaos. This makes (4) more flexible as a model in ecology (population dynamics). In such a context, (4) can be interpreted as a logistic map with a strengthened $(b > 0)$ or weakened $(b < 0)$ feedback $(1 - X_n)$. Accordingly, $\Delta_r(b < 0) < \Delta_r(0) < \Delta_r(b > 0)$, where $\Delta_r(0) = 1$. The same interpretation is valid about (18) and Table III. When compared to each other, the map (4) shows some advantages (deeper and simpler analytical study of the onset of bifurcation cascade). However, (24) is simpler for numerical realization than (17).

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