Hindawi Publishing Corporation
Differential Equations and Nonlinear Mechanics
Volume 2007, Article ID 87696, 16 pages
doi:10.1155/2007/87696

Research Article

Uniform Blow-Up Rates and Asymptotic Estimates of Solutions for Diffusion Systems with Nonlocal Sources

Zhoujin Cui and Zuodong Yang

Received 13 April 2006; Revised 27 November 2006; Accepted 19 December 2006

Recommended by Nicola Bellomo

This paper investigates the local existence of the nonnegative solution and the finite time blow-up of solutions and boundary layer profiles of diffusion equations with nonlocal reaction sources; we also study the global existence and that the rate of blow-up is uniform in all compact subsets of the domain, the blow-up rate of $|u(t)|_\infty$ is precisely determined.

1. Introduction

In this paper, we study the following reaction-diffusion system with nonlocal nonlinear source:

$$\begin{gathered} u_t = \triangle u + a(x)\,|\,v(t)\,|_r^p, \quad x \in \Omega,\ t > 0, \\ v_t = \triangle v + b(x)\,|\,u(t)\,|_r^q, \quad x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is an open ball of $\mathbb{R}^N$ centered at the origin of radius R, $|u(t)|_r = (\int_\Omega |u(x,t)|^r dx)^{1/r}$, $1 \le r < \infty$ and $p,q \ge r$. A nonnegative solution of (1.1) is a pair of nonnegative functions $(u(x,t),v(x,t))$ such that $(u(x,t),v(x,t)) \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T))$ and satisfies (1.1). For a solution $(u(x,t),v(x,t))$ of (1.1), we define

$$T^* = T^*(u,v) = \sup\{T > 0 : (u,v) \text{ are bounded and satisfy (1.1)}\}. \tag{1.2}$$

Note that if $T^* < +\infty$, then (u,v) blows up in L^∞ norm, in the sense that $\lim_{t\to T^*} |u(t)|_{L^\infty} = +\infty$ or $\lim_{t\to T^*} |v(t)|_{L^\infty} = +\infty$; in this case, we say that the solution blows up in finite time. If $T^* = \infty$, then (u,v) is a global solution of (1.1).

In the past several decades, many physical phenomena were formulated into nonlocal mathematical models (see [1–6]). It has also been suggested that nonlocal growth terms present a more realistic model of population dynamics (see [7]). System (1.1) is related to some ignition models in physics for compressible reactive gases.

A lot of effort has been devoted in the past few years to the study of blow-up rates and profiles for local semilinear parabolic equations of the type

$$u_t - \Delta u = u^p; \tag{1.3}$$

see [8–11] and the references therein. Several interesting blow-up results which concern the blow-up condition, blow-up set, and blow-up rate are presented; see [12–16] and references therein.

The blow-up property of the solution to a single equation of the form

$$\begin{gathered} u_t = \triangle u + a(x)\,|\,u(t)\,|_r^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{1.4}$$

has been discussed by many authors; see [1, 4] and the references therein. In [1], Souplet introduced a new method for investigating the rate and profile of blow-up of solutions to problem (1.4) with $a(x) = \text{constant} = 1$. He proved that if $p > 1$, then uniformly on compact subsets of Ω holds

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(x,t) = \lim_{t\to T}(T-t)^{1/(p-1)}\,|\,u(t)\,|_\infty = \left[(p-1)|\Omega|^{p/r}\right]^{-1/(p-1)}. \tag{1.5}$$

Very recently, Liu et al. [4] proved the global blow-up and determined the blow-up rate for problem (1.4) with $a(x) \neq$ constant.

Our present work is inspired by [1, 4], mentioned before, and [3, 5, 6, 12–19]. In [17], Escobedo and Herrero studied the system

$$u_t = \triangle u + v^p, \qquad v_t = \triangle v + u^q \tag{1.6}$$

with homogeneous Dirichlet boundary conditions. They showed that if $pq \le 1$, every solution of (1.6) is global, while for $pq > 1$, there are solutions that blow-up and others that are global according to the size of initial data. The blow-up rates of solutions to (1.6) were considered in [3, 5, 6].

In [12], Wang discussed the finite time blow-up of the positive solution to the problem

$$\begin{gathered} u_t = \triangle u + u^m v^n, \quad x \in \Omega,\ t > 0, \\ v_t = \triangle v + u^p v^q, \quad x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \ge 0, \quad v(x,0) = v_0(x) \ge 0, \quad x \in \Omega. \end{gathered} \tag{1.7}$$

Let λ_1 be the first eigenvalue of $-\Delta$ in Ω with null Dirichlet boundary condition. His results are the following.

(i) Assume that

$$m>1,\ n>0,\ p=0,\ q=1,\ \lambda_1<1,\quad m\le 1+\frac{n(1-\lambda_1)}{\lambda_1}, \tag{1.8}$$

or

$$q>1,\ p>0,\ n=0,\ m=1,\ \lambda_1<1,\quad q\le 1+\frac{p(1-\lambda_1)}{\lambda_1}. \tag{1.9}$$

Furthermore, if $m=1+n(1-\lambda_1)/\lambda_1$ in (1.8) or $q=1+p(1-\lambda_1)/\lambda_1$ in (1.9), it is assumed that $\lambda_1<2/3$. Then, for any nontrivial initial data, that is, $u_0(x)\not\equiv 0$, $v_0(x)\not\equiv 0$, the solution of (1.7) blows up in finite time.

(ii) If (1.8), (1.9), and the conditions that $m\le 1$, $q\le 1$, and $np\le(1-m)(1-q)$ do not hold, then the solution of (1.7) blows up in finite time for large initial data.

In [13], Wang evaluated the blow-up rate of the solution to (1.7) with $\Omega=B_R(0)$. Under some suitable conditions, he obtained that

$$\begin{aligned} c(T-t)^{-\theta}&\le \max_{0\le|x|\le R}u(\cdot,t)=u(0,t)\le C(T-t)^{-\theta},\quad t\in[0,T),\\ c(T-t)^{-\sigma}&\le \max_{0\le|x|\le R}v(\cdot,t)=v(0,t)\le C(T-t)^{-\sigma},\quad t\in[0,T), \end{aligned} \tag{1.10}$$

for some positive constants c and C, here $\theta=(1+n-q)/(np-(1-m)(1-q))$ and $\sigma=(1+p-m)/(np-(1-m)(1-q))$, and T is the blow-up time of (u,v).

In [14, 15], Galaktionov et al. considered the system

$$u_t=\triangle u^{\gamma+1}+v^p,\qquad v_t=\triangle v^{\mu+1}+u^q\quad \text{for } (x,t)\in\Omega\times(0,T) \tag{1.11}$$

with homogeneous Dirichlet boundary conditions, where $p>1$, $q>1$, $\gamma>0$, $\mu>0$. Several interesting results are established. Their results show that $P_c=pq-(1+\gamma)(1+\mu)$ is the critical exponent of (1.11), namely, if $P_c<0$ solutions are global for all initial data, and if $P_c>0$ solutions blow-up in finite time for sufficiently large initial data.

In this paper, we will prove that $P_c=pq-1$ is also the critical exponent of system (1.1).

The purpose of this paper is to determine the critical exponents as well as the estimates for blow-up rates and boundary layer profiles of the reaction-diffusion system (1.1). As for the function $a(x)$, $b(x)$, $u_0(x)$, $v_0(x)$, we assume that

(A_1) $a(x),b(x)\in C^2(\Omega)$, $u_0(x),v_0(x)\in C^{2+\alpha}(\Omega)$, $\alpha\in(0,1)$; $a(x),b(x),u_0(x),v_0(x)>0$ in Ω, and $a(x)=b(x)=u_0(x)=v_0(x)=0$ on $\partial\Omega$.

(A_2) $a(x)$, $b(x)$, $u_0(x)$, and $v_0(x)$ are radially symmetric, that is, $a(x)=a(r)$, $b(x)=b(r)$, $u_0(x)=u_0(r)$, and $v_0(x)=v_0(r)$ with $r=|x|$. $a(r)$, $b(r)$, $u_0(r)$, and $v_0(r)$ are nonincreasing for $r\in[0,R]$.

This paper is organized as follows. In Section 2, we investigate the global existence and finite time blow-up of system (1.1). Section 3 is devoted to the blow-up set and blow-up rate of solutions to (1.1). In Section 4, we give the boundary layer estimates.

2. Global existence and finite time blow-up

In this section, we start with the definition of super- and sub-solution of system (1.1).

Definition 2.1. A pair of nonnegative functions $(\overline{u}(x,t),\overline{v}(x,t))$ is called a supersolution of (1.1) if $(\overline{u}(x,t),\overline{v}(x,t)) \in C(\overline{\Omega}\times[0,T])\cap C^{2,1}(\Omega\times(0,T))$ and satisfy

$$\begin{gathered}
\overline{u}_t \geq \triangle\overline{u} + a(x)\,|\,\overline{v}(t)\,|_r^p, \quad (x,t)\in\Omega\times(0,T),\\
\overline{v}_t \geq \triangle\overline{v} + b(x)\,|\,\overline{u}(t)\,|_r^q, \quad (x,t)\in\Omega\times(0,T),\\
\overline{u}(x,t) \geq \overline{v}(x,t) \geq 0, \quad x\in\partial\Omega,\ t>0,\\
\overline{u}(x,0)\geq u_0(x), \quad \overline{v}(x,0)\geq v_0(x), \quad x\in\overline{\Omega}.
\end{gathered} \tag{2.1}$$

A pair of nonnegative functions $(\underline{u}(x,t),\underline{v}(x,t))$ is called a subsolution of (1.1) if $(\underline{u}(x,t),\underline{v}(x,t)) \in C(\overline{\Omega}\times[0,T])\cap C^{2,1}(\Omega\times(0,T))$ and satisfy

$$\begin{gathered}
\underline{u}_t \leq \triangle\underline{u} + a(x)\,|\,\underline{v}(t)\,|_r^p, \quad (x,t)\in\Omega\times(0,T),\\
\underline{v}_t \leq \triangle\underline{v} + b(x)\,|\,\underline{u}(t)\,|_r^q, \quad (x,t)\in\Omega\times(0,T),\\
\underline{u}(x,t) = \underline{v}(x,t) = 0, \quad x\in\partial\Omega,\ t>0,\\
\underline{u}(x,0)\leq u_0(x), \quad \underline{v}(x,0)\leq v_0(x), \quad x\in\overline{\Omega},
\end{gathered} \tag{2.2}$$

where $1\leq r<+\infty$, $p,q\geq r$.

We set $Q_T=\Omega\times(0,T]$ and $S_T=\partial\Omega\times(0,T]$. A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1). The following comparison lemma plays a crucial role in our proof which can be obtained by similar arguments as in [16].

Lemma 2.2. *Assume (A_1)-(A_2) hold, $w(x,t),z(x,t)\in C(\overline{Q}_T)\cap C^{2,1}(Q_T)$ and satisfy*

$$\begin{gathered}
w_t - \triangle w \geq a(x)d_1(t)\int_\Omega c_1(x,t)z(x,t)dx, \quad (x,t)\in Q_T,\\
z_t - \triangle z \geq b(x)d_2(t)\int_\Omega c_2(x,t)w(x,t)dx, \quad (x,t)\in Q_T,\\
w(x,t),z(x,t)\geq 0, \quad (x,t)\in S_T,\\
w(x,0),z(x,0)\geq 0, \quad x\in\Omega,
\end{gathered} \tag{2.3}$$

where $d_i(t),c_i(x,t)\geq 0$ $(i=1,2)$ in Q_T, and are bounded continuous functions. Then $w(x,t)$, $z(x,t)\geq 0$ on $\overline{Q}_T$.

Proof. Let $K=\max\{K_1,K_2\}+1$, where

$$\begin{gathered}
K_1 = \sup_{t\in(0,T]} a(0)d_1(t)\int_\Omega c_1(x,t)dx,\\
K_2 = \sup_{t\in(0,T]} b(0)d_2(t)\int_\Omega c_2(x,t)dx.
\end{gathered} \tag{2.4}$$

Since $c_i(x,t)$, $d_i(t)$ are bounded and continuous in Q_T, we know that $K<+\infty$. Let $w_1=e^{-Kt}w$, $z_1=e^{-Kt}z$, then we can deduce that $w_1(x,t),z_1(x,t)\geq 0$ on $\overline{Q}_T$. In fact, since $w_1(x,t),z_1(x,t)\geq 0$ for $(x,t)\in S_T$ or $x\in\Omega$, $t=0$, if $\min\{w_1(x,t),z_1(x,t)\}<0$ for some $(x,t)\in\overline{Q}_T$, then (w_1,z_1) has a negative minimum in Q_T. Without loss of generality, we can assume that $\min\{w_1(x,t),z_1(x,t)\}$ is taken at $(x_1,t_1)\in Q_T$ and $w_1(x_1,t_1)\leq w_1(x,t)$, $w_1(x_1,t_1)\leq z_1(x,t)$ for all $(x,t)\in\overline{Q}_T$. Using the first inequality in (2.3), we find that

$$w_{1t}-\triangle w_1\geq -Kw_1(x,t)+a(x)d_1(t)\int_\Omega c_1(x,t)z_1(x,t)dx,\quad (x,t)\in Q_T, \tag{2.5}$$

and then it follows from $c_1(x,t)\geq 0$ in Q_T and (A_2) that

$$w_{1t}(x_1,t_1)-\triangle w_1(x_1,t_1)\geq\left(-K+a(x_1)d_1(t_1)\int_\Omega c_1(x,t_1)dx\right)w_1(x_1,t_1)\geq -w_1(x_1,t_1)>0. \tag{2.6}$$

On the contrary, if $w_1(x,t)$ attains negative minimum at (x_1,t_1), then,

$$w_1(x,t)\leq 0,\qquad \triangle w_1(x_1,t_1)\geq 0,\qquad w_{1t}(x_1,t_1)\leq 0, \tag{2.7}$$

and hence

$$w_{1t}(x_1,t_1)-\triangle w_1(x_1,t_1)\geq 0, \tag{2.8}$$

which leads to a contradiction to inequality (2.6). Thus $\min\{w_1(x,t),z_1(x,t)\}\geq 0$ on $\overline{Q}_T$, and therefore $w(x,t),z(x,t)\geq 0$ on $\overline{Q}_T$. □

In order to get global existence and blow-up results, we need the following comparison principle which is a direct consequence of Lemma 2.2.

Corollary 2.3. *Let (u,v) be the unique nonnegative solution of (1.1). Assume that a pair of nonnegative functions $(w,z)\in C(\overline{Q}_T)\cap C^{2,1}(Q_T)$ and satisfy*

$$\begin{gathered}\omega_t\geq(\leq)\triangle\omega+a(x)\,|\,z(t)\,|_r^p,\quad (x,t)\in\Omega\times(0,T),\\ z_t\geq(\leq)\triangle z+b(x)\,|\,\omega(t)\,|_r^q,\quad (x,t)\in\Omega\times(0,T),\\ \omega(x,t)\geq(=)z(x,t)\geq(=)0,\quad x\in\partial\Omega,\ t>0,\\ \omega(x,0)\geq(\leq)u_0(x),\quad z(x,0)\geq(\leq)v_0(x),\quad x\in\overline{\Omega}.\end{gathered} \tag{2.9}$$

Then $(w(x,t),z(x,t))\geq(\leq)(u(x,t),v(x,t))$ on $\overline{Q}_T$.

Proof. We only prove $(w(x,t),z(x,t))\geq(u(x,t),v(x,t))\geq(0,0)$. A similar argument can be proved in other case. Let $\varphi_1(x,t)=w(x,t)-u(x,t)$, $\varphi_2(x,t)=z(x,t)-v(x,t)$. By the mean value theorem,

$$
\begin{aligned}
|z(t)|_r^p - |v(t)|_r^p &= \left(\int_\Omega |z(x,t)|^r dx\right)^{p/r} - \left(\int_\Omega |v(x,t)|^r dx\right)^{p/r} \\
&= \frac{p}{r}(\eta_1(t))^{(p-r)/r}\left[\int_\Omega (z^r(x,t) - v^r(x,t))dx\right] \\
&= p(\eta_1(t))^{(p-r)/r}\left[\int_\Omega (\eta_2(x,t))^{r-1}(z(x,t) - v(x,t))dx\right] \\
&= p(\eta_1(t))^{(p-r)/r}\left[\int_\Omega (\eta_2(x,t))^{r-1}\varphi_2(x,t)dx\right],
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
|w(t)|_r^q - |u(t)|_r^q &= \frac{q}{r}(\eta_3(t))^{(q-r)/r}\left[\int_\Omega (w^r(x,t) - u^r(x,t))dx\right] \\
&= q(\eta_3(t))^{(q-r)/r}\left[\int_\Omega (\eta_4(x,t))^{r-1}(w(x,t) - u(x,t))dx\right] \\
&= q(\eta_3(t))^{(q-r)/r}\left[\int_\Omega (\eta_4(x,t))^{r-1}\varphi_1(x,t)dx\right],
\end{aligned}
$$

where $\eta_1, \eta_3 \geq 0$ are some intermediate values between $|z(t)|_r^r = \int_\Omega |z|^r dx$ and $|v(t)|_r^r = \int_\Omega |v|^r dx$, $|w(t)|_r^r = \int_\Omega |w|^r dx$, and $|u(t)|_r^r = \int_\Omega |v|^r dx$, respectively, $\eta_2, \eta_4 \geq 0$ are some intermediate values between $z(x,t)$ and $v(x,t)$, $w(x,t)$ and $u(x,t)$, respectively. Then by (2.9)-(2.10), the functions φ_1, φ_2 satisfies the relation

$$
\begin{gathered}
\varphi_{1t} \geq \triangle\varphi_1 + a(x)p(\eta_1(t))^{(p-r)/r}\left[\int_\Omega (\eta_2(x,t))^{r-1}\varphi_2(x,t)dx\right], \quad (x,t) \in \Omega\times(0,T), \\
\varphi_{2t} \geq \triangle\varphi_2 + b(x)q(\eta_3(t))^{(q-r)/r}\left[\int_\Omega (\eta_4(x,t))^{r-1}\varphi_1(x,t)dx\right], \quad (x,t) \in \Omega\times(0,T), \\
\varphi_1(x,t), \varphi_2(x,t) \geq 0, \quad x \in \partial\Omega,\ t > 0, \\
\varphi_1(x,0), \varphi_2(x,0) \geq 0, \quad x \in \overline{\Omega}.
\end{gathered} \tag{2.11}
$$

Lemma 2.2 implies that $\varphi_1, \varphi_2 \geq 0$, that is, $(w(x,t), z(x,t)) \geq (u(x,t), v(x,t))$. □

From Corollary 2.3, we have the following lemma.

Lemma 2.4. *Let (u,v) be the unique nonnegative solution of (1.1), and suppose that $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ are supersolution and subsolution of problem (1.1), respectively, then $(\overline{u},\overline{v}) \geq (u,v) \geq (\underline{u},\underline{v})$ on $\overline{Q}_T$.*

Theorem 2.5. *Assume (A_1)-(A_2) hold, and $pq < 1$, then every nonnegative solution of system (1.1) exists globally.*

Proof. Let $\varphi(x)$ be the unique positive solution of the linear elliptic problem

$$
-\Delta\varphi(x) = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{2.12}
$$

Denote $C = \max_{x\in\Omega}\varphi(x)$. Then, $0 \le \varphi(x) \le C$. We define the functions $\overline{u}(x,t)$ and $\overline{v}(x,t)$ as

$$\overline{u} = \big(K(\varphi+1)\big)^{l_1}, \qquad \overline{v} = \big(K(\varphi+1)\big)^{l_2}, \tag{2.13}$$

where $l_1, l_2 < 1$ and $K > 0$ will be fixed later. Clearly, $(\overline{u},\overline{v})$ is bounded for any $T > 0$ and $\overline{u} \ge K^{l_1}$, $\overline{v} \ge K^{l_2}$.

Then we have

$$\begin{gathered}
\overline{u}_t - \Delta\overline{u} = -K^{l_1}\big(l_1(l_1-1)(\varphi+1)^{l_1-2}|\nabla\varphi|^2 + l_1(\varphi+1)^{l_1-1}\Delta\varphi\big) \ge l_1(C+1)^{l_1-1}K^{l_1}, \\
a(x)|\overline{v}|_r^p = a(x)K^{pl_2}\,\big|(\varphi+1)^{l_2}\big|_r^p \le a(0)|\Omega|^{p/r}(C+1)^{pl_2}K^{pl_2}, \\
\overline{v}_t - \Delta\overline{v} \ge l_2(C+1)^{l_2-1}K^{l_2}, \qquad b(x)|\overline{u}|_r^q \le b(0)|\Omega|^{q/r}(C+1)^{ql_1}K^{ql_1}.
\end{gathered} \tag{2.14}$$

Denote

$$K_1 = \left(\frac{a(0)|\Omega|^{p/r}}{l_1}(C+1)^{pl_2-l_1+1}\right)^{1/(l_1-pl_2)}, \qquad K_2 = \left(\frac{b(0)|\Omega|^{q/r}}{l_2}(C+1)^{ql_1-l_2+1}\right)^{1/(l_2-ql_1)}. \tag{2.15}$$

Now, since $pq < 1$, we can choose two positive constants $l_1, l_2 < 1$ such that

$$p < \frac{l_1}{l_2} < \frac{1}{q}, \tag{2.16}$$

hence $pl_2 < l_1$, $ql_1 < l_2$. We can choose K sufficiently large such that

$$K > \max\{K_1, K_2\}, \tag{2.17}$$

$$\big(K(\varphi+1)\big)^{l_1} \ge u_0(x), \qquad \big(K(\varphi+1)\big)^{l_2} \ge v_0(x). \tag{2.18}$$

Now, it follows from (2.14)–(2.18) that $(\overline{u},\overline{v})$ is a positive supersolution of (1.1). Hence by Lemma 2.4, $(u,v) \le (\overline{u},\overline{v})$, which implies that (u,v) exists globally. This completes the proof. □

Theorem 2.6. *Assume (A_1)-(A_2) hold, and $pq > 1$, then the nonnegative solution of system (1.1) exists globally for "small" initial data.*

Proof. Clearly, there exist positive constants $l_1, l_2 < 1$ such that

$$p > \frac{l_1}{l_2} > \frac{1}{q}, \tag{2.19}$$

hence $pl_2 > l_1$, $ql_1 > l_2$. We can choose K sufficiently small such that

$$K < \min\{K_1, K_2\}. \tag{2.20}$$

Furthermore, assume that u_0, v_0 are small enough to satisfy (2.18). Then it follows from (2.14), (2.18)–(2.20) that $(\overline{u},\overline{v})$ is a positive supersolution of (1.1). We can also see that the solution is bounded from below. This completes the proof. □

Remark 2.7. Furthermore, denote by $\psi(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\psi(x) = 1, \quad x \in \Omega_1; \qquad \psi(x) = 0, \quad x \in \partial\Omega_1, \tag{2.21}$$

here $\Omega_1 \subset\subset \Omega$. It is obvious that $\psi(x)$ depends on Ω_1 continuously. By the comparison principle for elliptic equation, we have $\psi < \varphi$ on Ω_1.

THEOREM 2.8. *Assume (A_1)-(A_2) hold, if $pq = 1$, then the nonnegative solution of (1.1) is global if the domain ($|\Omega|$) is sufficiently small.*

Proof. If $pq = 1$, there exist positive constants $l_1, l_2 < 1$ such that

$$p = \frac{l_1}{l_2} = \frac{1}{q}, \tag{2.22}$$

hence $pl_2 = l_1$, $ql_1 = l_2$. Without loss of generality, we may assume that every domain under consideration is in a sufficiently large ball B. Denote by $\varphi_B(x)$ the unique positive solution of the following linear elliptic problem:

$$-\Delta\varphi(x) = 1, \quad x \in B; \qquad \varphi(x) = 0, \quad x \in \partial B. \tag{2.23}$$

Let $C_0 = \max_{x\in B}\varphi_B(x)$. From Remark 2.7, we have $C \le C_0$. Then we may assume that $|\Omega|$ is sufficiently small such that

$$|\Omega| < \min\left\{\left(\frac{l_1}{a(0)(C_0+1)}\right)^{r/p}, \left(\frac{l_2}{b(0)(C_0+1)}\right)^{r/q}\right\}. \tag{2.24}$$

Furthermore, choose K large enough to satisfy (2.18). Then, it follows from (2.14), (2.18), and (2.24) that $(\overline{u}, \overline{v})$ is a positive supersolution of (1.1). By Lemma 2.4, we achieve the desired result. □

THEOREM 2.9. *Assume (A_1)-(A_2) hold, and $pq > 1$, then the nonnegative solution of system (1.1) blows up if initial data is sufficiently large.*

Proof. Let $\varphi(x)$ be the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and let λ_1 be the corresponding eigenvalue. We choose $\varphi(x)$ such that $\varphi(x) > 0$ in Ω and $\max_{x\in\overline{\Omega}}\varphi(x) = 1$.

Since $pq > 1$, there exist two positive constants m, n such that $p > m/n$, $q > n/m$. Set $\gamma = \min\{np - m + 1, mq - n + 1\}$, $L = \min\{a(x)m^{-1}(\int_\Omega |\varphi|^{nr}dy)^{p/r}, b(x)n^{-1}(\int_\Omega |\varphi|^{mr}dy)^{q/r}\}$. Let $s(t)$ be the solution of the Cauchy problem: $s' = -\lambda_1 s + Ls^\gamma$, $s(0) = s_0 > 0$. Since $\gamma > 1$, then $s(t)$ blows up in finite time for sufficiently large datum s_0.

Set $\underline{u}(x,t) = s^m(t)\varphi^m(x)$, $\underline{v}(x,t) = s^n(t)\varphi^n(x)$. We can assert that $(\underline{u},\underline{v})$ is a subsolution of system (1.1). A direct computation yields

$$\begin{aligned}
\Delta\underline{u}+a(x)\Big(\int_\Omega|\underline{v}|^r dy\Big)^{p/r} &= s^m\big(m\varphi^{m-1}\Delta\varphi + m(m-1)\varphi^{m-2}|\nabla\varphi|^2\big) + a(x)s^{np}\Big(\int_\Omega|\varphi|^{nr}dy\Big)^{p/r} \\
&\geq ms^m\varphi^m\Big(-\lambda_1 + a(x)s^{np-m}m^{-1}\Big(\int_\Omega|\varphi|^{nr}dy\Big)^{p/r}\Big) \\
&\geq ms^{m-1}\varphi^m s' = \underline{u}_t, \\
\Delta\underline{v}+b(x)\Big(\int_\Omega|\underline{u}|^r dy\Big)^{q/r} &= s^n\big(n\varphi^{n-1}\Delta\varphi + n(n-1)\varphi^{n-2}|\nabla\varphi|^2\big) + b(x)s^{mq}\Big(\int_\Omega|\varphi|^{mr}dy\Big)^{q/r} \\
&\geq ns^n\varphi^n\Big(-\lambda_1 + b(x)s^{mq-n}n^{-1}\Big(\int_\Omega|\varphi|^{mr}dy\Big)^{q/r}\Big) \\
&\geq ns^{n-1}\varphi^n s' = \underline{v}_t.
\end{aligned} \tag{2.25}$$

Therefore, $(\underline{u},\underline{v})$ is a subsolution of problem (1.1) provided that the initial data are sufficiently large such that $u_0 \geq \underline{u}(x,0)$, $v_0 \geq \underline{v}(x,0)$. By Lemma 2.4, we get that $(\underline{u},\underline{v}) \leq (u,v)$ and (u,v) blows up in finite time. □

From Theorems 2.5-2.6 and Theorems 2.8-2.9, we see that the critical exponent of the system is $pq = 1$.

Remark 2.10. If $a(x) =$ constant, $b(x) =$ constant, then the conclusions of Theorems 2.5-2.6 and Theorems 2.8-2.9 still hold for $\Omega \subset \mathbb{R}^N$ being a bounded domain with smooth boundary.

3. Uniform blow-up profiles

In this section, we assume that the nonnegative solution (u,v) of (1.1) blows up in finite time, we denote the blow-up time of the solution (u,v) by T^*. Throughout this section, we investigate the blow-up profile of the system (1.1). At first, we cite an important result which belongs to Liu et al. for uncouple diffusion equations with nonlocal nonlinear source (see [4]) as the basic lemma of our discussion. In the proof, the authors make use of the maximum principle (see [20, 21]) and sub-supersolution method (see [16]).

From [4, Theorem 3.1], we give the following lemma.

Lemma 3.1. *Let $u \in C^{2,1}(\overline{\Omega}\times(0,T))$ be the solution of the problem*

$$\begin{aligned}
u_t &= \triangle u + a(x)g(t), \quad x\in\Omega,\ t>0, \\
u(x,t) &= 0, \quad x\in\partial\Omega,\ t>0, \\
u(x,0) &= u_0(x), \quad x\in\Omega,
\end{aligned} \tag{3.1}$$

where the function $g(t) \geq 0$ will depend on the solution u, and $G(t) = \int_0^t g(s)ds$. Assume *that* (A_1), (A_2) *hold, and* $g(t)$ *is nonnegative, continuous, and nondecreasing on* $(0,T^*)$, $\lim_{t\to T^*} G(t) = +\infty$, *then*

$$\lim_{t\to T^*} \frac{u(x,t)}{G(t)} = a(x), \tag{3.2}$$

uniformly in all compact subsets of Ω.

In this section, we sometimes use the notation $u \sim v$ for $\lim_{t\to T^*} u(t)/v(t) = 1$. Denote

$$\begin{aligned} g_1(t) &= |v(t)|_r^p, \quad g_2(t) = |u(t)|_r^q, \\ G_1(t) &= \int_0^t g_1(s)ds, \quad G_2(t) = \int_0^t g_2(s)ds, \end{aligned} \qquad t \in (0,T^*), \tag{3.3}$$

and set

$$\begin{aligned} U(t) &= \max_{x\in\overline{\Omega}} u(x,t), \quad V(t) = \max_{x\in\overline{\Omega}} v(x,t), \quad t \in [0,T^*), \\ a_0 &= \max_{x\in\overline{\Omega}} a(x), \qquad b_0 = \max_{x\in\overline{\Omega}} b(x), \end{aligned} \tag{3.4}$$

then we have the following lemma.

Lemma 3.2. *Let* (u,v) *be a nonnegative solution of (1.1). Assume that the initial data* u_0 *and* v_0 *satisfy* (A_1)*-*(A_2)*, and*
(i) (u,v) *has blow-up time* $T^* < \infty$,
(ii) $u_t, v_t \geq 0$ *for* $(x,t) \in \Omega \times (0,T^*)$.

Then, we have

$$\lim_{t\to T^*} G_1(t) = \lim_{t\to T^*} G_2(t) = +\infty, \tag{3.5}$$

and there exist two positive constants C_1 and C_2 such that

$$u(x,t) \leq a_0 G_1(t) + C_1, \quad v(x,t) \leq b_0 G_2(t) + C_2, \quad (x,t) \in \overline{\Omega} \times [0,T^*). \tag{3.6}$$

Proof. Rewrite system (1.1) as follows:

$$\begin{aligned} u_t &= \Delta u(x,t) + a(x) g_1(t), \quad (x,t) \in \Omega \times (0,T^*), \\ v_t &= \Delta v(x,t) + b(x) g_2(t), \quad (x,t) \in \Omega \times (0,T^*). \end{aligned} \tag{3.7}$$

Using similar arguments as in [22], we give the proof of this lemma. Let

$$U(t) = \max_{x\in\overline{\Omega}} u(x,t) = u(x_0,t), \qquad V(t) = \max_{x\in\overline{\Omega}} v(x,t) = v(x_1,t). \tag{3.8}$$

Then functions $U(t)$, $V(t)$ satisfy

$$U'(t) = u_t(x_0,t) = \triangle u(x_0,t) + a(x_0) g_1(t), \qquad V'(t) = v_t(x_1,t) = \triangle v(x_1,t) + b(x_1) g_2(t) \tag{3.9}$$

since $\triangle u(x_0,t) \le 0$, $\triangle v(x_1,t) \le 0$, we get

$$0 \le U'(t) \le a_0 g_1(t), \quad 0 \le V'(t) \le b_0 g_2(t), \quad \text{a.e. } (0,T^*). \tag{3.10}$$

Integrating the above inequalities over $(0,t)$ for $t \in (0,T^*)$, we get

$$0 \le U(t) \le U(0) + a_0 G_1(t), \qquad 0 \le V(t) \le V(0) + b_0 G_2(t). \tag{3.11}$$

Since the nonnegative solution (u,v) of (1.1) blows up in finite time T^*, we know that

$$\lim_{t\to T^*} U(t) = \lim_{t\to T^*} \max_{x\in\overline{\Omega}} u(x,t) = +\infty, \qquad \lim_{t\to T^*} V(t) = \lim_{t\to T^*} \max_{x\in\overline{\Omega}} v(x,t) = +\infty. \tag{3.12}$$

Then (3.5) follows from (3.11), (3.12), and the facts that $U(0) = \max_{x\in\overline{\Omega}} u_0 < +\infty$ and $V(0) = \max_{x\in\overline{\Omega}} v_0 < +\infty$. Moreover, inequality (3.6) follows from (3.11), (3.12), and nonnegativity of $U(t)$ and $V(t)$, where $C_1 = U(0) = \max_{x\in\overline{\Omega}} u_0(x)$ and $C_2 = V(0) = \max_{x\in\overline{\Omega}} v_0(x)$. □

Remark 3.3. Lemma 3.2 implies that if u and v have a finite blow-up time T^*, then $G_1(t)$ and $G_2(t)$ blow-up in the same time T^* also.

From Lemmas 3.1 and 3.2, we get the following theorem immediately.

Theorem 3.4. *Let (u,v) be a classical solution of (1.1) with blow-up time T^*, then*

$$\lim_{t\to T^*} \frac{u(x,t)}{G_1(t)} = a(x), \qquad \lim_{t\to T^*} \frac{v(x,t)}{G_2(t)} = b(x), \tag{3.13}$$

uniformly in all compact subsets of Ω.

As a straightforward result of Theorem 3.4, we have the following theorem on the blow-up set.

Theorem 3.5. *Let (u,v) be blow-up solution of (1.1), then the blow-up set of (1.1) is the whole domain Ω, that is to say, the blow-up solution (u,v) has a global blow-up.*

Theorem 3.6. *Assume $pq > 1$, let (u,v) be a solution of (1.1) with blow-up time T^*, then*

$$\lim_{t\to T^*} (T^* - t)^{\alpha} u(x,t) = a(x) C_1 \left(\int_\Omega a^r(x)dx \right)^{pq/r(1-pq)} \left(\int_\Omega b^r(x)dx \right)^{p/r(1-pq)}, \tag{3.14}$$

$$\lim_{t\to T^*} (T^* - t)^{\beta} v(x,t) = b(x) C_2 \left(\int_\Omega a^r(x)dx \right)^{q/r(1-pq)} \left(\int_\Omega b^r(x)dx \right)^{pq/r(1-pq)}, \tag{3.15}$$

in which $\alpha = (p+1)/(pq-1)$, $\beta = (q+1)/(pq-1)$, $C_1 = ((p+1)/(pq-1))^{\alpha}((q+1)/(p+1))^{\alpha p/(p+1)}$, $C_2 = ((p+1)/(q+1))^{\beta q/(q+1)}((q+1)/(pq-1))^{\beta}$.

Proof. By (3.13) in Theorem 3.4, it follows that

$$\forall x \in \Omega, \quad \lim_{t\to T^*} \frac{|u(x,t)^r|}{G_1^r(t)} = a^r(x), \quad \lim_{t\to T^*} \frac{|v(x,t)^r|}{G_2^r(t)} = b^r(x). \tag{3.16}$$

Moreover, (3.6) in Lemma 3.2 implies that for all $\varepsilon > 0$, $0 \le |u(x,t)^r|/G_1^r(t) \le a^r(x) + \varepsilon$, $0 \le |v(x,t)^r|/G_2^r(t) \le b^r(x) + \varepsilon$ in Ω for t close enough to T^*. By the Lebesgue's dominated convergence theorem, we infer that $\int_\Omega |u(y,t)|^r dy \sim \int_\Omega a^r(x)dx G_1^r(t)$, $\int_\Omega |v(y,t)|^r dy \sim \int_\Omega b^r(x)dx G_2^r(t)$ as $t \to T^*$, then we have

$$\begin{aligned} G_1'(t) &= g_1(t) = |v(t)|_r^p = \left(\int_\Omega |v(y,t)|^r dy\right)^{p/r} \sim \left(\int_\Omega b^r(x)dx\right)^{p/r} G_2^p(t), \\ G_2'(t) &= g_2(t) = |u(t)|_r^q = \left(\int_\Omega |u(y,t)|^r dy\right)^{q/r} \sim \left(\int_\Omega a^r(x)dx\right)^{q/r} G_1^q(t), \end{aligned} \tag{3.17}$$

which implies that

$$\left(\int_\Omega a^r(x)dx\right)^{q/r} G_1^q G_1' \sim \left(\int_\Omega b^r(x)dx\right)^{p/r} G_2^p G_2' \quad \text{as } t \longrightarrow T^*. \tag{3.18}$$

Because $G_1(t), G_2(t) \to \infty$ as $t \to T^*$, it follows from (3.18) that

$$\left(\int_\Omega a^r(x)dx\right)^{q/r} \frac{G_1^{q+1}(t)}{q+1} \sim \left(\int_\Omega b^r(x)dx\right)^{p/r} \frac{G_2^{p+1}(t)}{p+1}. \tag{3.19}$$

From (3.17) and (3.19), we have

$$\begin{aligned} G_1'(t) &\sim \left(\int_\Omega b^r(x)dx\right)^{p/r} G_2^p(t) \sim \left(\frac{p+1}{q+1}\right)^{p/(p+1)} \left(\int_\Omega a^r(x)dx\right)^{pq/r(p+1)} \\ &\quad \times \left(\int_\Omega b^r(x)dx\right)^{p/r(p+1)} G_1^{p(q+1)/(p+1)}, \end{aligned} \tag{3.20}$$

it follows that

$$\frac{p+1}{1-pq}\left(G_1^{(1-pq)/(p+1)}\right)' \sim \left(\frac{p+1}{q+1}\right)^{p/(p+1)} \left(\int_\Omega a^r(x)dx\right)^{pq/r(p+1)} \left(\int_\Omega b^r(x)dx\right)^{p/r(p+1)}, \tag{3.21}$$

that is,

$$\frac{p+1}{1-pq}\left(G_1^{(1-pq)/(p+1)}\right)' = \left(\frac{p+1}{q+1}\right)^{p/(p+1)} \left(\int_\Omega a^r(x)dx\right)^{pq/r(p+1)} \left(\int_\Omega b^r(x)dx\right)^{p/r(p+1)} + \alpha(t), \tag{3.22}$$

where $\alpha(t) \to 0$ as $t \to T^*$. Integrating over (t, T^*), we have

$$G_1(T^* - t)^\alpha \sim C_1 \left(\int_\Omega a^r(x)dx\right)^{pq/r(1-pq)} \left(\int_\Omega b^r(x)dx\right)^{p/r(1-pq)}. \tag{3.23}$$

From (3.23) and Theorem 3.4, we have

$$\begin{aligned} (T^* - t)^\alpha u(x,t) &\sim G_1(t) a(x) (T^* - t)^\alpha \\ &\sim a(x) C_1 \left(\int_\Omega a^r(x)dx\right)^{pq/r(1-pq)} \left(\int_\Omega b^r(x)dx\right)^{p/r(1-pq)}. \end{aligned} \tag{3.24}$$

Then we get (3.14). The second equality (3.15) can be proved analogously. This completes the proof. □

Remark 3.7. From Theorem 3.6, we have

$$
\begin{aligned}
G_1(t) &\sim C_1\left(\int_\Omega a^r(x)dx\right)^{pq/r(1-pq)}\left(\int_\Omega b^r(x)dx\right)^{p/r(1-pq)}(T^*-t)^{-\alpha},\\
G_2(t) &\sim C_2\left(\int_\Omega a^r(x)dx\right)^{q/r(1-pq)}\left(\int_\Omega b^r(x)dx\right)^{pq/r(1-pq)}(T^*-t)^{-\beta}
\end{aligned}
\tag{3.25}
$$

as $t \to T^*$, C_1, C_2 defined as in Theorem 3.6.

Remark 3.8. If $a(x) =$ constant, $b(x) =$ constant, then the conclusions of Theorem 3.6 and Remark 3.7 still hold for $\Omega \subset \mathbb{R}^N$ being a bounded domain with smooth boundary.

4. Boundary layer estimates

Throughout this section, we deal with boundary layer estimate of (1.1) with $a(x) = a$, $b(x) = b$ in which a, b are constants. At first we cite some conclusions belonging to Souplet (see [1]) for the uncoupled equation (3.1) with $a(x) = 1$.

Definition 4.1. Say that g is standard if it satisfies the following power-like conditions

$$
k_1(T-t)^{-1} \le \frac{g(t)}{G(t)} \le k_2(T-t)^{-1} \quad \text{as } t \longrightarrow T \tag{4.1}
$$

for some constant $k_2 \ge k_1 \ge 0$.

Remark 4.2. According to the note after [1, Definition 4.1], we note that if g is standard, then $C_1(T-t)^{-(k_1+1)} \le g(t) \le C_2(T-t)^{-(k_1+1)}$ as $t \to T$. Conversely, g is standard whenever $c_1(T-t)^{-\gamma} \le g(t) \le c_2(T-t)^{-\gamma}$. Therefore, $g(t)$ is standard, if and only if $c_1'(T-t)^{-\gamma+1} \le G(t) \le c_2'(T-t)^{-\gamma+1}$ as $t \to T$ for some $\gamma > 1$ and $c_2 \ge c_1 > 0$, $c_2' \ge c_1' > 0$.

Lemma 4.3 [1, Theorem 4.5]. *Let $g(t)$ be standard and let $\omega(x,t)$ be a solution of (3.1) in which $a(x) = 1$ with blow-up time T. Denote by $d(x) = \operatorname{dist}(x,\partial\Omega)$. Then for all $K > 0$, there exist constants $m_k, m_k' > 0$, and some $t_0 \in (0,T)$ such that*

$$
m_k\frac{d(x)}{\sqrt{T-t}}G(t) \le \omega(x,t) \le m_k'\frac{d(x)}{\sqrt{T-t}}G(t) \tag{4.2}
$$

for $(x,t) \in \{(x,t) \in \Omega\times[t_0,T) : d(x) \le K\sqrt{T-t}\}$.

Lemma 4.4 [1, Theorem 4.6]. *Let $g(t)$ and $G(t)$ be standard, and let $\omega(x,t)$ be a solution of (3.1) in which $a(x) = 1$ with blow-up time T. Then $|\omega(x,t)|_\infty(1 - C(T-t)/d^2(x)) \le \omega(x,t)$ in $\Omega\times[t_0,T)$ for some $C > 0$ and some $t_0 \in (0,T)$.*

The above lemmas will be used to determine the boundary layer estimates of solutions to problem (1.1). By using the conclusions of blow-up rates for problem (1.1) in Section 3 together with Lemmas 4.3 and 4.4, we have the following results.

LEMMA 4.5. *For system (1.1) with $a(x) = a$, $b(x) = b$, the same conclusions of Lemmas 4.3 and 4.4 still hold.*

THEOREM 4.6. *Under the assumptions of Theorem 3.6, let (u,v) be a solution of (1.1) with blow-up time T. Then for all $K > 0$, there exist some constants $C_2 \geq C_1 > 0$, $C_4 \geq C_3 > 0$ and some $t_0 \in (0,T)$, such that (u,v) satisfies*

$$\begin{aligned} C_1 \frac{d(x)}{\sqrt{T-t}} |u(t)|_\infty \leq u(x,t) \leq C_2 \frac{d(x)}{\sqrt{T-t}} |u(t)|_\infty, \\ C_3 \frac{d(x)}{\sqrt{T-t}} |v(t)|_\infty \leq v(x,t) \leq C_4 \frac{d(x)}{\sqrt{T-t}} |v(t)|_\infty \end{aligned} \tag{4.3}$$

for $(x,t) \in \{(x,t) \in \Omega \times [t_0,T) : d(x) \leq K\sqrt{T-t}\}$.

Proof. From (3.25), we have $G_1(t) \sim d_1(T-t)^{-\alpha}$, $G_2(t) \sim d_2(T-t)^{-\beta}$ as $t \to T$, in which $d_1, d_2 > 0$, $\alpha, \beta > 0$. For some $t_0 \in [0,T)$, there exist four positive constants m_i $(1 \leq i \leq 4)$ such that

$$\begin{aligned} m_1(T-t)^{-\alpha} \leq G_1(t) \leq m_2(T-t)^{-\alpha}, \\ m_3(T-t)^{-\beta} \leq G_2(t) \leq m_4(T-t)^{-\beta} \quad \text{for } t \in [t_0,T). \end{aligned} \tag{4.4}$$

It follows that

$$\begin{aligned} m_1(T-t)^{-\delta_1+1} \leq G_1(t) \leq m_2(T-t)^{-\delta_1+1}, \\ m_3(T-t)^{-\delta_2+1} \leq G_2(t) \leq m_4(T-t)^{-\delta_2+1} \quad \text{for } t \in [t_0,T), \end{aligned} \tag{4.5}$$

where $\delta_1 = \alpha + 1 > 1$, $\delta_2 = \beta + 1 > 1$. Hence by Remark 4.2, it follows that $g_1(t)$, $g_2(t)$ are standard. By using Lemma 4.3 and (3.13), we get the results immediately. □

THEOREM 4.7. *Under the assumptions of Theorem 3.6, let (u,v) be a solution of (1.1) with blow-up time T. Then for all $K > 0$, there exist some constants $C_5, C_6 > 0$ and some $t_0 \in (0,T)$, such that (u,v)*

$$\begin{aligned} |u(t)|_\infty \left(1 - \frac{C_5(T-t)}{d^2(x)}\right) \leq u(x,t), \\ |v(t)|_\infty \left(1 - \frac{C_6(T-t)}{d^2(x)}\right) \leq v(x,t) \end{aligned} \tag{4.6}$$

for all $(x,t) \in \Omega \times [t_0,T)$.

Proof of this theorem is similar to the above theorem, so we omit it here.

Remark 4.8. Theorem 4.6 implies some boundary layer estimates that

$$\lim_{t \to T} \frac{u(x,t)}{|u(t)|_\infty} = 0, \qquad \lim_{t \to T} \frac{v(x,t)}{|v(t)|_\infty} = 0 \tag{4.7}$$

for $x \in \{x \in \Omega : d(x) \le K\sqrt{T-t}\}$ satisfying $d(x)/\sqrt{T-t} \to 0$ as $t \to T$. Similarly, it follows from Theorem 4.7 that

$$\lim_{t\to T} \frac{u(x,t)}{|u(t)|_\infty} = 1, \qquad \lim_{t\to T} \frac{v(x,t)}{|v(t)|_\infty} = 1 \tag{4.8}$$

for $x \in \Omega$ satisfying $d(x)/\sqrt{T-t} \to \infty$ as $t \to T$.

Due to the above discussion, we know that the size of boundary layer of (1.1) decays like $\sqrt{T-t}$.

Acknowledgments

This project is supported by the National Natural Science Foundation of China (Grant no. 10571022). This project is supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant no. 04KJB110062; no. 06KJB110056) and the Science Foundation of Nanjing Normal University (Grant no. 2003SXXXGQ2B37).

References

[1] P. Souplet, "Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source," *Journal of Differential Equations*, vol. 153, no. 2, pp. 374–406, 1999.

[2] K. Bimpong-Bota, P. Ortoleva, and J. Ross, "Far-from-equilibrium phenomena at local sites of reaction," *Journal of Chemical Physics*, vol. 60, no. 8, pp. 3124–3133, 1974.

[3] H. Li and M. Wang, "Blow-up properties for parabolic systems with localized nonlinear sources," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 771–778, 2004.

[4] Q. Liu, Y. Li, and H. Gao, "Uniform blow-up rate for diffusion equations with nonlocal nonlinear source," to appear in *Nonlinear Analysis*.

[5] M. Wang, "Blow-up rate estimates for semilinear parabolic systems," *Journal of Differential Equations*, vol. 170, no. 2, pp. 317–324, 2001.

[6] G. Caristi and E. Mitidieri, "Blow-up estimates of positive solutions of a parabolic system," *Journal of Differential Equations*, vol. 113, no. 2, pp. 265–271, 1994.

[7] J. Furter and M. Grinfeld, "Local vs. nonlocal interactions in population dynamics," *Journal of Mathematical Biology*, vol. 27, no. 1, pp. 65–80, 1989.

[8] F. B. Weissler, "An L^∞ blow-up estimate for a nonlinear heat equation," *Communications on Pure and Applied Mathematics*, vol. 38, no. 3, pp. 291–295, 1985.

[9] Y. Giga and R. V. Kohn, "Asymptotically self-similar blow-up of semilinear heat equations," *Communications on Pure and Applied Mathematics*, vol. 38, no. 3, pp. 297–319, 1985.

[10] Z. Yang and Q. Lu, "Blow-up estimates for a non-Newtonian filtration equation," *Journal of Mathematical Research and Exposition*, vol. 23, no. 1, pp. 7–14, 2003.

[11] Z. Yang, "Nonexistence of positive solutions to a quasi-linear elliptic equation and blow-up estimates for a nonlinear heat equation," *Rocky Mountain Journal of Mathematics*, vol. 36, no. 4, pp. 1399–1414, 2006.

[12] M. Wang, "Global existence and finite time blow up for a reaction-diffusion system," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 51, no. 1, pp. 160–167, 2000.

[13] M. Wang, "Blowup estimates for a semilinear reaction diffusion system," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 46–51, 2001.

[14] V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "A parabolic system of quasilinear equations. I," *Differential Equations*, vol. 19, pp. 1558–1571, 1983.

[15] V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "A parabolic system of quasilinear equations. II," *Differential Equations*, vol. 21, pp. 1049–1062, 1985.
[16] Y. Chen, "Blow-up for a system of heat equations with nonlocal sources and absorptions," *Computers & Mathematics with Applications*, vol. 48, no. 3-4, pp. 361–372, 2004.
[17] M. Escobedo and M. A. Herrero, "A semilinear parabolic system in a bounded domain," *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 165, no. 1, pp. 315–336, 1993.
[18] Z. Yang and Q. Lu, "Nonexistence of positive solutions to a quasilinear elliptic system and blow-up estimates for a non-Newtonian filtration system," *Applied Mathematics Letters*, vol. 16, no. 4, pp. 581–587, 2003.
[19] Z. Yang and Q. Lu, "Blow-up estimates for a quasi-linear reaction-diffusion system," *Mathematical Methods in the Applied Sciences*, vol. 26, no. 12, pp. 1005–1023, 2003.
[20] F. Dickstein and M. Escobedo, "A maximum principle for semilinear parabolic systems and applications," *Nonlinear Analysis*, vol. 45, no. 7, pp. 825–837, 2001.
[21] P. Pucci and J. Serrin, "The strong maximum principle revisited," *Journal of Differential Equations*, vol. 196, no. 1, pp. 1–66, 2004.
[22] A. Friedman and B. McLeod, "Blow-up of positive solutions of semilinear heat equations," *Indiana University Mathematics Journal*, vol. 34, no. 2, pp. 425–447, 1985.

Zhoujin Cui: Institute of Mathematics, School of Mathematics and Computer Science,
Nanjing Normal University, Nanjing, Jiangsu 210097, China
Email address: czj1982@sina.com

Zuodong Yang: Institute of Mathematics, School of Mathematics and Computer Science,
Nanjing Normal University, Nanjing, Jiangsu 210097, China; College of Zhongbei,
Nanjing Normal University, Nanjing, Jiangsu 210046, China
Email address: zdyang_jin@263.net