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Research Article

Existence and Uniform Decay of Weak Solutions for Nonlinear Thermoelastic System with Memory

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A nonlinear thermoelastic system with memory is considered, which is derived from a physical model with vibration in temperature environment. By some skillful and technical arguments, results of existence, uniqueness, and uniform decay on this generalized system are obtained.

1. Introduction

In this work we consider the following initial boundary value problem:

$$\begin{gathered}
u'' - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t h(t-\tau)\Delta u(\tau)d\tau - \Delta u' + |u|^{\rho-2}u + \beta_1\theta = f \quad \text{in } Q, \\
\theta' - \Delta\theta + \beta_2 u' = g \quad \text{in } Q, \\
u = \theta = 0 \quad \text{on } \Sigma, \\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $Q := \Omega \times [0,T]$, $\Sigma := \partial\Omega \times [0,T]$, $\rho \geq 2$, Ω is a bounded domain in R^n with C^2 boundary, $u' = du/dt$, $u'' = d^2u/dt^2$, $M(s)$ is C^1 class function like $1 + s^\gamma$, $\gamma \geq 1$ and β_1, β_2 are positive constants:

$$\|\nabla u\|^2 = \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx, \qquad \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}, \tag{1.2}$$

f, g is a known function and the function $h(t)$ is positive and satisfies some conditions to be specified later.

However, (1.1) consists of a dynamical equation coupling a heat equation, which can be used to describe some physical process of thermoelastic material. Also, $u(x,t)$ and $\theta(x,t)$ represent the displacement and temperature, respectively, at position x and time t. The coupling of the heat equation in the model of vibrations presents important aspects because it represents better than the reality, that is, allowing to influence the vibrations in a more adequate way. $M(s)$ appearing in the dynamical part of system (1.1) is a nonlinear perturbation of Moeover, Kirchhoff-Carrier's model which describes small vibrations of a stretched string (dimension $n = 1$) when tension is assumed to have only a vertical component at each point of the string. Many researchers have investigated several types of problems involving the Kirchhoff equation among which we can cite the work in [1, 2]. Clark and Lima [3] studied the local existence for $0 < T_0 < T$ of solutions to the mixed problem:

$$u'' - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + |u|^\rho u + \theta = f \quad \text{in } Q,$$
$$\theta' - \Delta\theta + u' = g \quad \text{in } Q. \tag{1.3}$$

In this paper, we prove the global existence and uniqueness of weak solutions of (1.1) based on different definition of weak solution and estimate techniques from [3], we consider the Kirchhoff equation with the strong damping term $\Delta u'$ and so-called "memory" term $\int_0^t h(t-\tau)\Delta u(\tau)d\tau$. Here we consider the memory effect in (1.1) because physically some materials could produce the viscosity of memory type [4]. Hence under appropriate assumptions on $h(t)$, ρ, f, and g, and making use of Galerkin's approximations and compactness argument, we establish global existence and uniqueness. Meanwhile, by some suitable estimate techniques, we deal with the memory term and another nonlinear term appearing in the mixed problem of viscoelastic wave equation. In order to obtain the exponential decay of the energy, we make use of the perturbed energy method, see Komornik and Zuazua [5].

The rest of this paper is organized as follows: In Section 2 we give out assumptions and state the main result. In Section 3 we exploit Faedo-Galerkin's approximation, priori estimates, and compactness arguments to obtain the existence of solutions of a penalty problem. In Section 4, uniqueness is proved. In Section 5, the exponential decay of solution is obtained by using the perturbed energy method.

2. Assumptions and Main Results

Throughout this paper, we use the following notation:

$$(u,v) = \int_\Omega u(x)v(x)\ dx, \qquad \|u\|^2 = \int_\Omega |u(x)|^2\ dx. \tag{2.1}$$

Now we state the main hypotheses in this paper.

(A.1) Assumption on Kernel h

Let $h : R^+ \to R^+$ be a nonnegative and bounded C^2 function and suppose that there exist positive constants $\xi_1,\ \xi_2,\ \xi_3$ such that

$$-\xi_1 h(t) \le h'(t) \le -\xi_2 h(t) \quad \forall t \ge 0, \tag{2.2}$$

$$0 \le h''(t) \le \xi_3 h(t) \quad \forall t \ge 0. \tag{2.3}$$

Moreover, h verifies $l := 1 - \int_0^\infty h(s)ds > 0$.

(A.2) Assumption on ρ, μ

Let ρ satisfies that

$$\begin{aligned} &2 \le \rho \le \frac{2n-2}{n-2} \quad \text{if } n \ge 3, \\ &2 \le \rho < \infty \quad \text{if } n = 1, 2; \end{aligned} \tag{2.4}$$

μ is given by the Sobolev embedding inequality $\|u\|_2 \le \mu\|\nabla u\|$ for $u \in H_0^1(\Omega)$, in the general case, we denote $\|u\|_\rho \le C\|\nabla u\|$.

(A.3) Assumption on Initial Condition, f and g

Assume that $u_0, u_1, \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, and $f,\ g \in C^1_{loc}(0,\infty; L^2(\Omega))$. Next we define the energy $E(t)$ with

$$E(t) = \frac{1}{2}\left(\|u'(t)\|^2 + \|\theta(t)\|^2 + \|\nabla u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u(t)\|_\rho^\rho \right). \tag{2.5}$$

The main result is as follow.

Theorem 2.1. *If assumptions (1)–(3) hold, then there exists a unique weak solution $\{u,\theta\}$ with $u \in L^\infty(0,T;H_0^1(\Omega))$, $u' \in L^\infty(0,T;H_0^1(\Omega))$, $u'' \in L^\infty(0,T;L^2(\Omega))$, $\theta \in L^\infty(0,T;H_0^1(\Omega))$, and $\theta' \in L^\infty(0,T;L^2(\Omega))$ such that*

$$\begin{aligned} &(u'', w) + (\nabla u, \nabla w) + \|\nabla u\|^{2\gamma}(\nabla u, \nabla w) - \int_0^t h(t-\tau)(\nabla u(\tau), \nabla w)d\tau \\ &\quad + (\nabla u', \nabla w) + \beta_1(\theta, w) + \left(|u|^{\rho-2}u, w\right) - (f, w) = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &(\theta', w) + (\nabla\theta, \nabla w) + \beta_2(u', w) - (g, w) = 0 \quad \forall w \in H_0^1(\Omega), \\ &u(0) = u_0, \qquad u'(0) = u_1, \qquad \theta(0) = \theta_0. \end{aligned} \tag{2.7}$$

Furthermore, if $f = g = 0$, β_1, β_2 *satisfy that* $2/\mu^2 \geq \beta_1 + \beta_2$ *and* β_1 *small enough, we have the following decay estimate:*

$$E(t) \leq C\exp(-\xi t), \quad \forall t \geq t_0, \tag{2.8}$$

where C *and* ξ *are positive constants.*

3. Existence of Solutions

Proof of Theorem 2.1. We use Galerkin's approximation. Let $w_1, \ldots, w_m$ be a basis in $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$, and V_m the subspace of $H_0^1(\Omega)$ generated by the first m of $\{w_j\}$. For each $m \in N$, we seek the approximate solution:

$$u_m(t,x) = \sum_{j=1}^{m} g_m(t) w_j(x), \qquad \theta_m(t,x) = \sum_{j=1}^{m} \widetilde{g}_m(t) w_j(x), \tag{3.1}$$

of the following Cauchy problem:

$$\begin{gathered}(u_m'', w) + (\nabla u_m, \nabla w) + \|\nabla u_m\|^{2\gamma}(\nabla u_m, \nabla w) - \int_0^t h(t-\tau)(\nabla u_m(\tau), \nabla w)d\tau \\ +(\nabla u_m', \nabla w) + \beta_1(\theta_m, w) + \left(|u_m|^{\rho-2}u_m, w\right) - (f, w) = 0 \quad \forall w \in V_m,\end{gathered} \tag{3.2}$$

$$(\theta_m', w) + (\nabla\theta_m, \nabla w) + \beta_2(u_m', w) - (g, w) = 0 \quad \forall w \in V_m \tag{3.3}$$

satisfying the initial conditions

$$\begin{gathered}u_m(0) = u_{0m} = \sum_{j=1}^{m}(u_0, w_j)w_j \longrightarrow u_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega), \\ u_m'(0) = u_{1m} = \sum_{j=1}^{m}(u_1, w_j)w_j \longrightarrow u_1 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega), \\ \theta_m(0) = \theta_{0m} = \sum_{j=1}^{m}(\theta_0, w_j)w_j \longrightarrow \theta_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega).\end{gathered} \tag{3.4}$$

According to the ODE theory, we can solve the system (3.2)-(3.3) by Picard's iteration. Hence, this system has unique solution on interval $[0, T_m]$ for each m. The following estimates allow us to extend the solution to the closed interval $[0, T]$.

In the following proof, we will use c_i, $i = 0, 1, 2, \ldots$, to denote various positive constants which may be different in different places and may be dependent on T in some cases.

The First Estimate

Taking $w = u_m'(t)$ in (3.2) and $w = \theta_m(t)$ in (3.3), respectively, then adding the results and using assumption (1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|u_m'(t)\right\|^2 + \|\theta_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u_m(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u_m(t)\|_\rho^\rho\right) \\
&\quad + \left\|\nabla u_m'(t)\right\|^2 + \|\nabla\theta_m(t)\|^2 \\
&\quad = \frac{d}{dt}\left[\int_0^t h(t-\tau)(\nabla u_m(\tau), \nabla u_m(t))d\tau\right] - (\beta_1+\beta_2)\left(u_m', \theta_m\right) + \left(f, u_m'(t)\right) \\
&\quad + (g, \theta_m) - \int_0^t h'(t-\tau)(\nabla u_m(\tau), \nabla u_m(t))d\tau - h(0)\|\nabla u_m(t)\|^2 \\
&\quad \le \frac{(\beta_1+\beta_2)^2+1}{2}\left\|u_m'(t)\right\|^2 + \|\theta_m\|^2 + \frac{d}{dt}\left[\int_0^t h(t-\tau)(\nabla u_m(\tau), \nabla u_m(t))d\tau\right] \\
&\quad + \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 + c_1\|\nabla u_m(t)\|^2 + c_2\int_0^t \|\nabla u_m(\tau)\|^2 d\tau.
\end{aligned} \tag{3.5}$$

Now integrating (3.5) over $(0, t)$ for $t < T$, we have

$$\begin{aligned}
&\frac{1}{2}\left(\left\|u_m'(t)\right\|^2 + \|\theta_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u_m(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u_m(t)\|_\rho^\rho\right) \\
&\quad + \int_0^t \left\|\nabla u_m'(\tau)\right\|^2 d\tau + \int_0^t \|\nabla\theta_m(\tau)\|^2 d\tau \\
&\quad \le \frac{(\beta_1+\beta_2)^2+1}{2}\int_0^t \left\|u_m'(\tau)\right\|^2 d\tau + \int_0^t h(t-\tau)(\nabla u_m(\tau), \nabla u_m(t))d\tau \\
&\quad + \int_0^t \|\theta_m(\tau)\|^2 d\tau + c_3\int_0^t \|\nabla u_m(\tau)\|^2 d\tau + c_4.
\end{aligned} \tag{3.6}$$

Moreover, from assumption (1), we have

$$\int_0^t h(t-\tau)(\nabla u_m(\tau), \nabla u_m(t))d\tau \le c_5(\eta)\int_0^t \|\nabla u_m(\tau)\|^2 d\tau + \eta\|\nabla u_m(t)\|^2, \tag{3.7}$$

where $\eta > 0$ is arbitrary.

Hence letting η small enough and using Gronwall's inequality we obtain the first estimate:

$$\begin{aligned}&\left\|u_m'(t)\right\|^2+\|\theta_m(t)\|^2+\|\nabla u_m(t)\|^2+\|\nabla u_m(t)\|^{2(\gamma+1)}+\|u_m(t)\|_\rho^\rho\\&\quad+\int_0^t\left\|\nabla u_m'(\tau)\right\|^2d\tau+\int_0^t\|\nabla\theta_m(\tau)\|^2d\tau\le L_1,\end{aligned}\tag{3.8}$$

where L_1 is independent of m.

The Second Estimate

First we estimate the initial data $u_m''(0)$ in the L^2-norm. Taking $t=0$ and $w=u_m''(0)$ in (3.2) we have

$$\begin{aligned}\left\|u_m''(0)\right\|^2\le{}&\left(1+\|\nabla u_0\|^{2\gamma}\right)\left|\left(\Delta u_0,u_m''(0)\right)\right|+\beta_1\left(\theta_0,u''(0)\right)+\left(\Delta u_1,u''(0)\right)\\&+\|u_0\|_{\rho-1}^{\rho-1}\left\|u_m''(0)\right\|+\|f\|\left\|u_m''(0)\right\|.\end{aligned}\tag{3.9}$$

Hence, noticing the assumption on u_0, u_1, and θ_0, we deduce

$$\|u_m''(0)\|\le L_2,\tag{3.10}$$

where L_2 is independent of m.

Similarly, taking $t=0$ and $w=\theta_m'(0)$ in (3.3), we also deduce

$$\|\theta_m'(0)\|\le L_3,\tag{3.11}$$

where L_3 is independent of m.

Differentiating (3.2) and (3.3), replacing w by $u_m''(t)$ and $\theta_m'(t)$ respectively, and then adding the results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left\|u_m''(t)\right\|^2+\left\|\theta_m'(t)\right\|^2+\left\|\nabla u_m'(t)\right\|^2\right]\\&\quad+\left\|\nabla u_m''(t)\right\|^2+\left\|\nabla\theta_m'(t)\right\|^2+h(0)\left\|\nabla u_m'(t)\right\|^2\\&\le-2\gamma\|\nabla u_m(t)\|^{2\gamma-2}\left(\nabla u_m(t),\nabla u_m'(t)\right)\left(\nabla u_m(t),\nabla u_m''(t)\right)\\&\quad-\left(\nabla u_m'(t),\nabla u_m''(t)\right)\|\nabla u_m(t)\|^{2\gamma}-\left((\rho-1)|u_m(t)|^{\rho-2}u_m'(t),u_m''(t)\right)\\&\quad+(\beta_1+\beta_2)\left|\left(\theta_m'(t),u_m''(t)\right)\right|+\left(f',u_m''(t)\right)+\left(g',\theta_m'(t)\right)\\&\quad+\frac{d}{dt}\left[\int_0^t h'(t-\tau)\left(\nabla u_m(\tau),\nabla u_m'(t)\right)d\tau\right]-h'(0)\left(\nabla u_m(t),\nabla u_m'(t)\right)\\&\quad-\int_0^t h''(t-\tau)\left(\nabla u_m(\tau),\nabla u_m'(t)\right)d\tau+h(0)\frac{d}{dt}\left(\nabla u_m(t),\nabla u_m'(t)\right).\end{aligned}\tag{3.12}$$

From the first estimate and Young's inequality, we have

$$\begin{aligned} &2\gamma\|\nabla u_m(t)\|^{2\gamma-2}\left(\nabla u_m(t), \nabla u_m'(t)\right)\left(\nabla u_m(t), \nabla u_m''(t)\right) \\ &\quad + \left(\nabla u_m'(t), \nabla u_m''(t)\right)\|\nabla u_m(t)\|^{2\gamma} \\ &\quad \le c_1(\eta)\left\|\nabla u_m'(t)\right\|^2 + \eta\left\|\nabla u_m''(t)\right\|^2, \end{aligned} \tag{3.13}$$

where $\eta > 0$ is arbitrary.

Noticing $1/n + (n-2)/2n + 1/2 = 1$, assumption (2), and the first estimate, we have

$$\begin{aligned} &\left((\rho-1)u_m(t)|^{\rho-2}u_m'(t), u_m''(t)\right) \\ &\quad \le (\rho-1)\|u_m(t)\|_{n(\rho-2)}^{\rho-2}\left\|u_m'(t)\right\|_{2n/(n-2)}\left\|u_m''(t)\right\| \\ &\quad \le c_2\|\nabla u_m(t)\|^{\rho-2}\left\|\nabla u_m'(t)\right\|\left\|u_m''(t)\right\| \\ &\quad \le c_3\left\|\nabla u_m'(t)\right\|^2 + c_3\left\|u_m''(t)\right\|^2, \end{aligned} \tag{3.14}$$

$$h'(0)\left(\nabla u_m(t), \nabla u'{}_m(t)\right) \le \frac{h'(0)^2}{2}\|\nabla u_m(t)\|^2 + \frac{1}{2}\left\|\nabla u_m'(t)\right\|^2, \tag{3.15}$$

and by assumption (1), we have

$$\int_0^t h''(t-\tau)\left(\nabla u_m(\tau), \nabla u_m'(t)\right)d\tau \le c_4\int_0^t \|\nabla u_m(\tau)\|^2 d\tau + \frac{1}{2}\left\|\nabla u_m'(t)\right\|^2. \tag{3.16}$$

Therefore, combining (3.14)–(3.16), (3.10), (3.11) and integrating (3.12) over $(0, t)$ we have

$$\begin{aligned} &\frac{1}{2}\left[\left\|u_m''(t)\right\|^2 + \left\|\theta_m'(t)\right\|^2 + \left\|\nabla u_m'(t)\right\|^2\right] \\ &\quad + (1-\eta)\int_0^t \left\|\nabla u_m''(\tau)\right\|^2 d\tau + \int_0^t \left\|\nabla\theta_m'(\tau)\right\|^2 d\tau + h(0)\int_0^t \left\|\nabla u_m'(\tau)\right\|^2 d\tau \\ &\le \left(c_1(\eta) + c_3 + 1\right)\int_0^t \left\|\nabla u_m'(\tau)\right\|^2 d\tau + \frac{(\beta_1+\beta_2)^2 + 1}{2}\int_0^t \left\|\theta_m'(\tau)\right\|^2 d\tau \\ &\quad + (c_3 + 1)\int_0^t \left\|u_m''(\tau)\right\|^2 d\tau + c_5\int_0^t \|\nabla u_m(\tau)\|^2 d\tau \\ &\quad + \int_0^t h'(t-\tau)\left(\nabla u_m(\tau), \nabla u_m'(t)\right)d\tau + h(0)\left(\nabla u_m(t), \nabla u_m'(t)\right) + c_6. \end{aligned} \tag{3.17}$$

Moreover, consider that

$$\begin{aligned}\int_0^t h'(t-\tau)\left(\nabla u_m(\tau),\nabla u_m'(t)\right)d\tau &\le c_7(\eta)\int_0^t \|\nabla u_m(\tau)\|^2 d\tau+\eta\left\|\nabla u_m'(t)\right\|^2,\\ h(0)\left(\nabla u_m(t),\nabla u_m'(t)\right)&\le c_8(\eta)\|\nabla u_m(t)\|^2+\eta\left\|\nabla u_m'(t)\right\|^2.\end{aligned}\tag{3.18}$$

Hence, from (3.17), (3.18), the first estimate, letting η small enough and using Gronwall's inequality, we get the second estimate:

$$\begin{gathered}\left\|u_m''(t)\right\|^2+\left\|\theta_m'(t)\right\|^2+\left\|\nabla u_m'(t)\right\|^2+\int_0^T\left\|\nabla u_m''(\tau)\right\|^2 d\tau+\int_0^T\left\|\nabla\theta_m'(\tau)\right\|^2 d\tau\le L_4,\\ \forall 0\le t\le T,\end{gathered}\tag{3.19}$$

where L_4 is independent of m.

The Third Estimate

Taking $w=\theta_m'(t)$ in (3.3), we have

$$\left\|\theta_m'\right\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta_m\|^2\le\beta_2\left|\left(u_m'(t),\theta_m'(t)\right)\right|+\left|\left(g(t),\theta_m'(t)\right)\right|.\tag{3.20}$$

Hence we easily get $\|\nabla\theta\|^2\le L_5,\ \forall 0\le t\le T$, and L_5 is independent of m.

The Fourth Estimate

Let $m_1\ge m_2$ be two natural numbers and consider $y_m:=u_{m_1}-u_{m_2}$, $z_m:=\theta_{m_1}-\theta_{m_2}$. From the system (3.2), we have

$$\begin{aligned}&\left(y_m'',w\right)+\left(\nabla y_m,\nabla w\right)+\left(\|\nabla u_{m_1}\|^{2\gamma}\nabla y_m,\nabla w\right)\\&\quad+\left(\left(\|\nabla u_{m_1}\|^{2\gamma}-\|\nabla u_{m_2}\|^{2\gamma}\right)\nabla u_{m_2},\nabla w\right)\\&\quad-\int_0^t h(t-\tau)\left(\nabla y_m(\tau),\nabla w\right)d\tau+\beta_1\left(z_m,w\right)+\left(\nabla y_m',\nabla w\right)\\&\quad+\left(|u_{m_1}(t)|^{\rho-2}u_{m_1}(t)-|u_{m_2}(t)|^{\rho-2}u_{m_2}(t),w\right)=0,\end{aligned}\tag{3.21}$$

Taking $w = y'_m$ in (3.21), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|y'_m(t)\right\|^2 + \left\|\nabla y_m(t)\right\|^2 + \|\nabla u_{m_1}(t)\|^{2\gamma}\left\|\nabla y_m(t)\right\|^2\right] + \left\|\nabla y'_m(t)\right\|^2 \\
&\quad\le \left(\left(\|\nabla u_{m_2}(t)\|^{2\gamma} - \|\nabla u_{m_1}(t)\|^{2\gamma}\right)\nabla u_{m_2}(t), \nabla y'_m(t)\right) \\
&\quad - \left(|u_{m_1}(t)|^{\rho-2}u_{m_1}(t) - |u_{m_2}(t)|^{\rho-2}u_{m_2}(t), y'_m(t)\right) \\
&\quad + \frac{1}{2}\left\|\nabla y_m(t)\right\|^2\frac{d}{dt}\|\nabla u_{m_1}(t)\|^{2\gamma} + \beta_1\left|\left(z_m(t), y'_m(t)\right)\right| \\
&\quad + \int_0^t h(t-\tau)\left(\nabla y_m(\tau), \nabla y'_m(t)\right)d\tau.
\end{aligned} \tag{3.22}$$

Noticing that

$$\begin{aligned}
&\int_0^t h(t-\tau)\left(\nabla y_m(\tau), \nabla y'_m(t)\right)d\tau \\
&\qquad = -h(0)\left\|\nabla y_m(t)\right\|^2 - \int_0^t h'(t-\tau)\left(\nabla y_m(\tau), \nabla y_m(t)\right)d\tau \\
&\qquad + \frac{d}{dt}\left(\int_0^t h(t-\tau)\left(\nabla y_m(\tau), \nabla y_m(t)\right)d\tau\right),
\end{aligned} \tag{3.23}$$

hence, using assumption (2.2) and integrating (3.22) over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\left[\left\|y'_m(t)\right\|^2 + \left\|\nabla y_m(t)\right\|^2 + \|\nabla u_{m_1}(t)\|^{2\gamma}\left\|\nabla y_m(t)\right\|^2\right] \\
&\quad + h(0)\int_0^t \left\|\nabla y_m(\tau)\right\|^2 d\tau + \int_0^t \left\|\nabla y'_m(\tau)\right\|^2 d\tau \\
&\le \int_0^t \left|\|\nabla u_{m_1}(\tau)\|^{2\gamma} - \|\nabla u_{m_2}(\tau)\|^{2\gamma}\right| \|\nabla u_{m_2}(\tau)\| \left\|\nabla y'_m(\tau)\right\| d\tau \\
&\quad + \int_0^t \left\||u_{m_1}(\tau)|^{\rho-2}u_{m_1}(\tau) - |u_{m_2}(\tau)|^{\rho-2}u_{m_2}(\tau)\right\| \left\|y'_m(\tau)\right\| d\tau \\
&\quad + \int_0^t \beta_1\left|\left(z_m(\tau), y'_m(\tau)\right)\right| d\tau + \frac{1}{2}\int_0^t \left\|\nabla y_m(\tau)\right\|^2 \frac{d}{d\tau}\|\nabla u_{m_1}(\tau)\|^{2\gamma} d\tau \\
&\quad + c_1\int_0^t h(t-\tau)\left|\left(\nabla y_m(\tau), \nabla y_m(t)\right)\right| d\tau + c_2\left(\left\|y_{1m}\right\|^2 + \left\|\nabla y_{0m}\right\|^2\right).
\end{aligned} \tag{3.24}$$

Notice that

$$c_1\int_0^t h(t-\tau)\left|\left(\nabla y_m(\tau), \nabla y_m(t)\right)\right| d\tau \le c_3(\eta)\int_0^t \left\|\nabla y_m(\tau)\right\|^2 d\tau + \eta\left\|\nabla y_m(t)\right\|^2. \tag{3.25}$$

where $\eta > 0$ is arbitrary:

$$\left| \|\nabla u_{m_1}(\tau)\|^{2\gamma} - \|\nabla u_{m_2}(\tau)\|^{2\gamma} \right| \le c_4 \left(\|\nabla u_{m_1}(\tau)\|^{2\gamma-1} + \|\nabla u_{m_2}(\tau)\|^{2\gamma-1} \right) \|\nabla y_m(\tau)\|,$$
$$\frac{d}{d\tau} \|\nabla u_{m_1}(\tau)\|^{2\gamma} \le c_5 \|\nabla u_{m_1}(\tau)\|^{2\gamma-1} \|\nabla u'_{m_1}(\tau)\|. \tag{3.26}$$

Moreover, by mean value theorem and assumption (2), we have

$$\left\| |u_{m_1}(\tau)|^{\rho-2} u_{m_1}(\tau) - |u_{m_2}(\tau)|^{\rho-2} u_{m_2}(\tau) \right\|$$
$$\le c_6 \left(\|\nabla u_{m_1}(\tau)\|^{\rho-2} + \|\nabla u_{m_2}(\tau)\|^{\rho-2} \right) \|\nabla y_m(\tau)\|. \tag{3.27}$$

Therefore, by (3.25)–(3.27), letting $\eta > 0$ small enough, by the first estimate, and using the Gronwall's lemma of integral form (see [6]) in (3.24) we obtain that

$$\|y'_m(t)\|^2 + \|\nabla y_m(t)\|^2 + \int_0^T \|\nabla y'_m(\tau)\|^2 d\tau$$
$$\le c_7(T) \left(\|y_{1m}\|^2 + \|\nabla y_{0m}\|^2 + \int_0^T \|z_m(\tau)\|^2 d\tau \right). \tag{3.28}$$

Passage to the Limit

From above estimates, we deduce that there exist functions $u,\ \theta$ and subsequences of $\{u_m\}$, $\{\theta_m\}$ which we still denote by $\{u_m\}$, $\{\theta_m\}$ satisfying

$$\begin{aligned}
u_m &\longrightarrow u \quad \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right) \text{ weak}*, \\
u'_m &\longrightarrow u' \quad \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right) \text{ weak}*, \\
u''_m &\longrightarrow u'' \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right) \text{ weak}*, \\
u''_m &\longrightarrow u'' \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ weakly}, \\
\theta_m &\longrightarrow \theta \quad \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right) \text{ weak}*, \\
\theta'_m &\longrightarrow \theta' \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ weakly}, \\
\theta'_m &\longrightarrow \theta' \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right) \text{ weak}*.
\end{aligned} \tag{3.29}$$

Moreover, according to the compactness of Aubin-Lions, we have

$$u_m \longrightarrow u \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{3.30}$$

$$\theta_m \longrightarrow \theta \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{3.31}$$

Hence combing (3.31) and the fourth estimate (3.28), we deduce that

$$u_m \longrightarrow u \text{ strongly in } C^0\left(0,T;H_0^1(\Omega)\right). \tag{3.32}$$

Thus we can pass the limit in system (3.2)-(3.3). Let $m \to \infty$, we prove that $\{u,\ \theta\}$ is a weak solution of the system (1.1). □

4. Uniqueness of the Solution

The proof of uniqueness of solution is similar to the fourth estimate, but for integrity, we still give the detailed proof.

Let (u_1,θ_1) and (u_2,θ_2) be two solutions of couple system (1.1) under the conditions of Theorem 2.1, then we have $(u,\theta) := (u_1 - u_2, \theta_1 - \theta_2)$ verifying

$$\begin{aligned}
&(u'',w) + (\nabla u,\nabla w) + \left(\|\nabla u_1\|^{2\gamma}\nabla u,\nabla w\right) + \left(\left(\|\nabla u_1\|^{2\gamma} - \|\nabla u_2\|^{2\gamma}\right)\nabla u_2,\nabla w\right)\\
&\quad - \int_0^t h(t-\tau)(\nabla u(\tau),\nabla w)d\tau + \beta_1(\theta,w) + (\nabla u',\nabla w)\\
&\quad + \left(|u_1(t)|^{\rho-2}u_1(t) - |u_2(t)|^{\rho-2}u_2(t),w\right) = 0,
\end{aligned} \tag{4.1}$$

$$\begin{gathered}
(\theta',w) + (\nabla\theta,\nabla w) + \beta_2(u',w) = 0 \quad \forall w \in H_0^1(\Omega),\\
u(0) = u'(0) = \theta(0) = 0.
\end{gathered} \tag{4.2}$$

Taking $w = u'$ in (4.1) and $w = \theta$ in (4.2), respectively, and adding the results, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u_1(t)\|^{2\gamma}\|\nabla u(t)\|^2\right] + \left\|\nabla u'(t)\right\|^2 + \|\nabla\theta(t)\|^2\\
&\quad\le \left(\left(\|\nabla u_2(t)\|^{2\gamma} - \|\nabla u_1(t)\|^{2\gamma}\right)\nabla u_2(t),\nabla u'(t)\right)\\
&\qquad - \left(|u_1(t)|^{\rho-2}u_1(t) - |u_2(t)|^{\rho-2}u_2(t),u'(t)\right) + \frac{1}{2}\|\nabla u(t)\|^2\frac{d}{dt}\|\nabla u_1(t)\|^{2\gamma}\\
&\qquad + (\beta_1+\beta_2)\left|(\theta(t),u'(t))\right| + \int_0^t h(t-\tau)(\nabla u(\tau),\nabla u'(t))d\tau.
\end{aligned} \tag{4.3}$$

Noticing that

$$\begin{aligned}
&\int_0^t h(t-\tau)\left(\nabla u(\tau),\nabla u'(t)\right)d\tau \\
&\quad= -h(0)\|\nabla u(t)\|^2 - \int_0^t h'(t-\tau)(\nabla u(\tau),\nabla u(t))d\tau + \frac{d}{dt}\left(\int_0^t h(t-\tau)(\nabla u(\tau),\nabla u(t))d\tau\right),
\end{aligned} \tag{4.4}$$

hence, using assumption (2.2) and integrating (4.3) over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\left[\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u_1(t)\|^{2\gamma}\|\nabla u(t)\|^2\right] \\
&\quad + h(0)\int_0^t \|\nabla u(\tau)\|^2 d\tau + \int_0^t \left\|\nabla u'(\tau)\right\|^2 d\tau + \int_0^t \|\nabla\theta(\tau)\|^2 d\tau \\
&\leq \int_0^t \left|\|\nabla u_1(\tau)\|^{2\gamma} - \|\nabla u_2(\tau)\|^{2\gamma}\right| \|\nabla u_2(\tau)\| \left\|\nabla u'(\tau)\right\| d\tau \\
&\quad + \int_0^t \left\||u_1(\tau)|^{\rho-2}u_1(\tau) - |u_2(\tau)|^{\rho-2}u_2(\tau)\right\| \left\|u'(\tau)\right\| d\tau \\
&\quad + \int_0^t (\beta_1+\beta_2)\left|\left(\theta(\tau),u'(\tau)\right)\right| d\tau + \frac{1}{2}\int_0^t \|\nabla u(\tau)\|^2 \frac{d}{d\tau}\|\nabla u_1(\tau)\|^{2\gamma} d\tau \\
&\quad + c_1\int_0^t h(t-\tau)|(\nabla u(\tau),\nabla u(t))| d\tau.
\end{aligned} \tag{4.5}$$

Notice that

$$c_1\int_0^t h(t-\tau)|(\nabla u(\tau),\nabla u(t))|d\tau \leq c_2(\eta)\int_0^t \|\nabla u(\tau)\|^2 d\tau + \eta\|\nabla u(t)\|^2, \tag{4.6}$$

where $\eta > 0$ is arbitrary:

$$\begin{aligned}
\left|\|\nabla u_1(\tau)\|^{2\gamma} - \|\nabla u_2(\tau)\|^{2\gamma}\right| &\leq c_3\left(\|\nabla u_1(\tau)\|^{2\gamma-1} + \|\nabla u_2(\tau)\|^{2\gamma-1}\right)\|\nabla u(\tau)\|, \\
\frac{d}{d\tau}\|\nabla u_1(\tau)\|^{2\gamma} &\leq c_4\|\nabla u_1(\tau)\|^{2\gamma-1}\left\|\nabla u'_1(\tau)\right\|.
\end{aligned} \tag{4.7}$$

Moreover, by mean value theorem and assumption (2), we have

$$\left\||u_1(\tau)|^{\rho-2}u_1(\tau) - |u_2(\tau)|^{\rho-2}u_2(\tau)\right\| \leq c_5\left(\|\nabla u_1(\tau)\|^{\rho-2} + \|\nabla u_2(\tau)\|^{\rho-2}\right)\|\nabla u(\tau)\|. \tag{4.8}$$

Therefore, by (4.6)–(4.8), Cauchy inequality, Young's inequality, and using Gronwall's lemma in (4.5), we get

$$\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \|\nabla u(t)\|^2 + \int_0^T \left\|\nabla u'(\tau)\right\|^2 d\tau + \int_0^T \|\nabla\theta(\tau)\|^2 d\tau = 0. \tag{4.9}$$

Thus, we have proved the uniqueness consequence.

5. Asymptotic Behavior of the Solution

In this section, we follow the additional assumptions appeared in Theorem 2.1. We introduce the energy

$$\begin{aligned} e(t) := \frac{1}{2}\Bigg(&\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \left(1 - \int_0^t h(s)ds\right)\|\nabla u(t)\|^2 \\ &+ (h\Box\nabla u)(t) + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u(t)\|_\rho^\rho\Bigg), \end{aligned} \tag{5.1}$$

where we define

$$(h\Box y)(t) = \int_0^t h(t-\tau)\left\|y(t) - y(\tau)\right\|_2^2 d\tau. \tag{5.2}$$

Remark 5.1. Taking $w = u'(t)$ in (2.6) and $w = \theta(t)$ in (2.7), respectively, then adding the results we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \|\nabla u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u(t)\|_\rho^\rho\right) \\ &\quad + \left\|\nabla u'(t)\right\|^2 + \|\nabla\theta(t)\|^2 + (\beta_1+\beta_2)\left(\theta(t), u'(t)\right) \\ &= \int_0^t h(t-\tau)\left(\nabla u(\tau), \nabla u'(t)\right)d\tau. \end{aligned} \tag{5.3}$$

Noticing

$$\begin{aligned} &\int_0^t h(t-\tau)\left(\nabla u(\tau), \nabla u'(t)\right)d\tau \\ &\quad = \frac{1}{2}(h'\Box\nabla u)(t) - \frac{1}{2}(h\Box\nabla u)'(t) + \frac{1}{2}\left(\int_0^t h(s)ds\|\nabla u(t)\|^2\right)' - \frac{1}{2}h(t)\|\nabla u(t)\|^2, \end{aligned} \tag{5.4}$$

and combining the assumptions on $\beta_1,\ \beta_2$ appeared in Theorem 2.1, we deduce

$$\begin{aligned}e'(t) &\leq -M_1\left\|\nabla u'(t)\right\|^2 - M_1\|\nabla\theta(t)\|^2 + \frac{1}{2}(h'\Box\nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(t)\|^2 \\ &\leq -M_1\left\|\nabla u'(t)\right\|^2 - M_1\|\nabla\theta(t)\|^2 - \frac{\xi_2}{2}(h\Box\nabla u)(t) \\ &\leq 0,\end{aligned} \tag{5.5}$$

where we denote $M_1 = 1-(\beta_1+\beta_2)(\mu^2/2) \geq 0$. Thus, we have the energy $e(t)$ is uniformly bounded (by $e(0)$) and is decreasing in t.

Remark 5.2. Furthermore, from the assumption (1), we have

$$\begin{aligned}E(t) &\leq \frac{1}{2}\left(\left\|u'(t)\right\|^2 + \|\theta(t)\|^2 + \frac{1}{l}\left(1-\int_0^t h(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{2}{\rho}\|u(t)\|_\rho^\rho\right) \\ &\leq l^{-1}e(t).\end{aligned} \tag{5.6}$$

For every $\varepsilon > 0$, we define the perturbed energy by setting

$$e_\varepsilon(t) = e(t) + \varepsilon\psi(t), \qquad \text{where } \psi(t) = (u'(t), u(t)). \tag{5.7}$$

Lemma 5.3. *There exists $M_2 > 0$ such that*

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon M_2 e(t), \quad \forall t \geq 0. \tag{5.8}$$

Proof. From (5.7), we obtain

$$\begin{aligned}\left|\psi(t)\right| &\leq \mu\left\|u'(t)\right\|\|\nabla u(t)\| \\ &\leq \frac{\mu}{2}\left\|u'(t)\right\|^2 + \frac{\mu}{2}\|\nabla u(t)\|^2 \\ &\leq \frac{\mu}{l}e(t),\end{aligned} \tag{5.9}$$

hence we have

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon M_2 e(t), \quad \forall t \geq 0, \tag{5.10}$$

where $M_2 = \mu/l$. □

Lemma 5.4. *There exists* $M_3 > 0$ *and* $\overline{\varepsilon}$ *such that for* $\varepsilon \in (0, \overline{\varepsilon}]$,

$$e'_{\varepsilon}(t) \leq -\varepsilon M_3 e(t). \tag{5.11}$$

Proof. By using the problem (1.1), we obtain

$$\begin{aligned}
\varphi'(t) &= \left\|u'(t)\right\|^2 + \left(u''(t), u(t)\right) \\
&= \left\|u'(t)\right\|^2 - \|\nabla u(t)\|^2 - \|\nabla u(t)\|^{2\gamma+2} + \int_0^t h(t-\tau)(\nabla u(\tau), \nabla u(t))d\tau \\
&\quad - \left(\nabla u'(t), \nabla u(t)\right) - \beta_1(\theta(t), u(t)) - \left(|u|^{\rho-2}u, u\right).
\end{aligned} \tag{5.12}$$

Notice that

$$\begin{aligned}
&\int_0^t h(t-\tau)(\nabla u(\tau), \nabla u(t))d\tau \\
&\quad = \int_0^t h(t-\tau)(\nabla u(\tau) - \nabla u(t), \nabla u(t))d\tau + \|\nabla u(t)\|^2 \int_0^t h(t-\tau)d\tau \\
&\quad \leq \frac{1}{4\eta}\int_0^t h(t-\tau)\|\nabla u(\tau) - \nabla u(t)\|^2 d\tau + \left(1+\eta\right)\|\nabla u(t)\|^2 \int_0^t h(t-\tau)d\tau \\
&\quad = \frac{1}{4\eta}(h\Box\nabla u)(t) + \left(1+\eta\right)\|\nabla u(t)\|^2 \int_0^t h(t-\tau)d\tau \\
&\quad \leq \frac{1}{4\eta}(h\nabla u)(t) + \left(1+\eta\right)(1-l)\|\nabla u(t)\|^2,
\end{aligned} \tag{5.13}$$

$$\beta_1(\theta(t), u(t)) \leq \frac{\beta_1\mu^2}{2}\|\nabla\theta(t)\|^2 + \frac{\beta_1\mu^2}{2}\|\nabla u(t)\|^2, \tag{5.14}$$

$$(\nabla u'(t), \nabla u(t)) \leq \eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\|\nabla u'(t)\|^2, \tag{5.15}$$

where $\eta > 0$ is arbitrary.

Hence, from (5.12)–(5.15), we have

$$\begin{aligned}
\varphi'(t) \leq \left\|u'(t)\right\|^2 + \left(-l + \eta(2-l) + \frac{\beta_1\mu^2}{2}\right)\|\nabla u(t)\|^2 - \|\nabla u(t)\|^{2\gamma+2} \\
+ \frac{1}{4\eta}(h\Box\nabla u)(t) + \frac{1}{4\eta}\left\|\nabla u'(t)\right\|^2 + \frac{\beta_1\mu^2}{2}\|\nabla\theta(t)\|^2 - \|u(t)\|_\rho^\rho.
\end{aligned} \tag{5.16}$$

Therefore, from (5.5) and (5.16), we get

$$\begin{aligned}
{e'}_{\varepsilon}(t) &= e'(t) + \varepsilon\psi'(t) \\
&\leq -M_1\left\|\nabla u'(t)\right\|^2 - M_1\|\nabla\theta(t)\|^2 - \frac{\xi_2}{2}(h\Box\nabla u)(t) \\
&\quad + \varepsilon\mu^2\left\|\nabla u'(t)\right\|^2 + \varepsilon\left(-l + \eta(2-l) + \frac{\beta_1\mu^2}{2}\right)\|\nabla u(t)\|^2 - \varepsilon\|\nabla u(t)\|^{2\gamma+2} \\
&\quad + \frac{\varepsilon}{4\eta}(h\Box\nabla u)(t) + \frac{\varepsilon}{4\eta}\left\|\nabla u'(t)\right\|^2 + \frac{\varepsilon\beta_1\mu^2}{2}\|\nabla\theta(t)\|^2 - \varepsilon\|u(t)\|_{\rho}^{\rho} \\
&\leq -\left(M_1 - \varepsilon\mu^2 - \frac{\varepsilon}{4\eta}\right)\left\|\nabla u'(t)\right\|^2 - \left(M_1 - \frac{\varepsilon\beta_1\mu^2}{2}\right)\|\nabla\theta(t)\|^2 \\
&\quad - \left(\frac{\xi_2}{2} - \frac{\varepsilon}{4\eta}\right)(h\Box\nabla u)(t) - \varepsilon\left(l - \eta(2-l) - \frac{\beta_1\mu^2}{2}\right)\|\nabla u(t)\|^2 \\
&\quad - \varepsilon\|\nabla u(t)\|^{2\gamma+2} - \varepsilon\|u(t)\|_{\rho}^{\rho}.
\end{aligned} \tag{5.17}$$

Taking β_1 and η small enough, we have $l - \eta(2-l) - \beta_1\mu^2/2 \geq 0$. Moreover if we denote

$$\widetilde{\varepsilon} = \min\left\{\frac{M_1}{\mu^2 + 1/4\eta}, \frac{2M_1}{\beta_1\mu^2}, 2\eta\xi_2\right\}, \tag{5.18}$$

and choosing $\varepsilon \in (0, \widetilde{\varepsilon}]$, we obtain

$${e'}_{\varepsilon}(t) \leq -\varepsilon M_3 e(t) \tag{5.19}$$

for some constant $M_3 > 0$. □

Proof of Decay

Let us define $\widehat{\varepsilon} = \min\{1/2M_2, \widetilde{\varepsilon}\}$ and consider $\varepsilon \in (0, \widehat{\varepsilon}\ ]$. From Lemma 5.3, we have

$$(1 - M_2\varepsilon)e(t) \leq e_{\varepsilon}(t) \leq (1 + M_2\varepsilon)e(t)\ , \tag{5.20}$$

and so

$$\frac{1}{2}e(t) \leq e_{\varepsilon}(t) \leq \frac{3}{2}e(t). \tag{5.21}$$

From (5.21), we get

$$-\varepsilon M_3 e(t) \leq -\varepsilon\frac{2}{3}M_3 e_{\varepsilon}(t). \tag{5.22}$$

Hence from (5.22) and Lemma 5.4, we obtain

$$e'_{\varepsilon}(t) \leq -\varepsilon \frac{2}{3} M_3 e_{\varepsilon}(t). \tag{5.23}$$

that is,

$$\frac{d}{dt}\left(e_{\varepsilon}(t)\ \exp\left\{\frac{2\varepsilon}{3} M_3 t\right\}\right) \leq 0. \tag{5.24}$$

Integrating last inequality over $[0,t]$, we get

$$e_{\varepsilon}(t) \leq e_{\varepsilon}(0)\ \exp\left\{-\frac{2\varepsilon}{3} M_3 t\right\}. \tag{5.25}$$

From (5.21) and (5.25), we have

$$e(t) \leq 3e(0)\ \exp\left\{-\frac{2\varepsilon}{3} M_3 t\right\}. \tag{5.26}$$

Hence, from (5.6) and (5.26), we obtain

$$E(t) \leq l^{-1} e(t) \leq 3e(0) l^{-1}\ \exp\left\{-\frac{2\varepsilon}{3} M_3 t\right\}, \quad t \geq t_0,\ \forall \varepsilon \in (0, \hat{\varepsilon}], \tag{5.27}$$

that is,

$$E(t) \leq C\ \exp(-\xi t), \quad \forall t \geq t_0, \tag{5.28}$$

where $C = 3e(0)l^{-1}$ and $\xi = (2\varepsilon/3)M_3$.

Therefore, we have proved the exponential decay of solution.

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