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Research Article

Oscillation Theorems for Second-Order Damped Nonlinear Differential Equations

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We present new oscillation criteria for the differential equation of the form $[r(t)U(t)]'+p(t)k_2(x(t), x'(t))|x(t)|^{\nu}U(t) + q(t)\phi(x(g_1(t)), x'(g_2(t)))f(x(t)) = 0$, where $U(t) = k_1(x(t), x'(t))|x'(t)|^{\alpha-1}x'(t)$, $\alpha \leq \beta$, $\nu = (\beta - \alpha)/(\alpha + 1)$. Our research is different from most known ones in the sense that H function is not employed in our results, though Riccati's substitution and its generalized forms are used. Our criteria which are established under quite general assumptions are an extension for previous results. In particular, by taking $\beta = \alpha$, the above-mentioned equation can be reduced into the various types of equations concerned by people currently.

1. Introduction

The existence of the oscillatory solutions of the nonlinear differential equation with damping,

$$[r(t)x'(t)]' + p(t)x'(t) + q(t)g(x(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

has received considerable attention from researchers for a long time.

People previously focused on the cases $r(t) > 0$, $p(t) \geq 0$, $q(t) > 0$. In recent years, people concerned that $r(t) > 0$, $p(t)$, $q(t)$ may change sign for $t \in [t_0, \infty)$, regarding work in this area can be seen in literature [1–4].

Recently, Li [5] has extended (1.1) to more general equations of the form

$$(r(t)x'(t))' + p(t)x'(t) + q(t)g(x(t))f(x'(t)) = 0, \quad t \geq t_0. \tag{1.2}$$

Yamaoka [6] has studied the following class of particular equation:

$$\left(|x'(t)|^{p-2}x'(t)\right)' + \frac{2p-1}{t}|x'(t)|^{p-2}x'(t) + q(t)g(x(t)) = 0, \quad t \geq t_0,\ p > 1. \tag{1.3}$$

Tiryaki and Zafer [1] and other authors [2, 7] have considered the following equation of the form

$$(r(t)h(x(t))x'(t))' + p(t)x'(t) + q(t)g(x(t)) = 0. \tag{1.4}$$

Zheng [8] has discussed the oscillation problem for the following equation:

$$(r(t)h(x(t))\psi(x'(t)))' + p(t)\psi(x'(t)) + q(t)g(x(t)) = 0. \tag{1.5}$$

It is worth noting that (1.1), (1.2), and (1.3) can transform into an undamping equation. For example, the equation

$$\left(r(t)|x'(t)|^{\alpha-1}x'(t)\right)' + p(t)|x'(t)|^{\alpha-1}x'(t) + q(t)g(x(t)) = 0 \tag{1.6}$$

can transform into the undamping equation

$$\left(\tilde{r}(t)|x'(t)|^{\alpha-1}x'(t)\right)' + \tilde{q}(t)g(x(t))f(x'(t)) = 0, \quad t \geq t_0, \tag{1.7}$$

where $\tilde{r}(t) = r(t)e^{\int_T^t (p(s)/r(s))ds}$, $\tilde{q}(t) = q(t)e^{\int_T^t (p(s)/r(s))ds}$. Although (1.4) and (1.5) can not be transformed into the undamping equation, but from the conditions $0 < c \leq h(x) \leq c_1 < \infty$ given by [1, 8, 9], if $h(t)$ is changed into c or c_1, the above-mentioned equations consistent with (1.6). This shows that under the above conditions, there is no essential difference between (1.4), (1.5), and the undamping equation. We note that the condition $\int_{t_0}^{\infty}(1/r(s))^{1/\alpha}ds = \infty$ must be used for (1.5); however, at this point the condition $\int_{t_0}^{\infty}(1/\tilde{r}(s))^{1/\alpha}ds = \infty$ cannot be guaranteed.

We have removed the condition $\int_{t_0}^{\infty}(1/r(s))^{1/\alpha}ds = \infty$, considered the oscillation problem for the following equation:

$$[r(t)\chi(x'(t))]' + q(t)\phi(x(g_1(t)), x'(g_2(t)))f(x(t)) = 0, \tag{1.8}$$

applied the results to the above-mentioned equation, and obtained a very good result.

In this paper, we consider the oscillatory behavior of the following differential equation of the form:

$$[r(t)U(t)]' + p(t)k_2(x(t), x'(t))|x(t)|^{(\beta-\alpha)/(\alpha+1)}U(t) + q(t)\phi(x(g_1(t)), x'(g_2(t)))f(x(t)) = 0, \tag{1.9}$$

where $U(t) = k_1(x(t), x'(t))|x'(t)|^{\alpha-1}x'(t)$.

Today, Riccati transformation, and its generalized forms are one of the most effective method in the oscillatory theory of nonlinear differential equations. Most obvious merits of Riccati's approach is that $q(t)$ may change sign in (1.8). For getting the more general results [10, 11], a lot of authors have introduced to a class of Y function

$$Y = \left\{\Phi \in C^1(E,R) \mid \Phi(t,t,l) = \Phi(t,l,l) = 0,\ \Phi(t,s,l) \neq 0,\ l < s < t\right\},$$
$$E = \{(t,s,l) \mid t_0 \leq l \leq s \leq t < \infty\}, \tag{1.10}$$

where $\partial\Phi/\partial s$ exists on E and is integral with respect to s. By using this method, peoples have obtained some general results, but its shortcoming is that the property of $q(t)$ can be weakened as $t \to \infty$. We use the method similar to [4], that is, replace the above-mentioned function $\Phi(t,s,l)$ with $\rho \in C^1([t_0,\infty), R^+)$. Perhaps the reason that people like to use this method is that integrating by parts with respect to s on $[l,t]$ can employ $\Phi(t,t,l) = \Phi(t,l,l) = 0$.

For (1.9), we make the following assumptions:

(A) $xf(x) > 0$, $x \neq 0$, $f'(x)|f(x)|^{(1-\beta)/\beta} \geq C_1$, $\alpha, \beta > 0$, $\phi \in C(R^2, R^+)$, $0 < n_\phi \leq \phi(x,y) \leq N_\phi$;

(B) $0 < C_0 \leq k_1(u,v) \leq C_2$, $0 \leq C_3 \leq k_2(u,v) \leq C_4$;

(C) $r \in C^1(I, R^+)$, $g_i \in C(I, R^+)$, $0 \leq g_i'(t)$, $i = 1,2$, $I = [t_0, \infty)$, $p, q \in C([t_0,\infty), R)$.

In the paper, a solution of (1.9) is called oscillatory if it has zeros unbounded set. If the solutions are oscillatory, (1.9) is called to be oscillatory equation.

2. Main Theorem

We establish some lemmas which are useful in our discussions.

Lemma 2.1. *Let $a, b, \gamma, \lambda > 0$, then*

$$at - bt^{1+1/\gamma} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{a^{\gamma+1}}{b^\gamma}, \tag{2.1}$$

$$at^{-\lambda} + bt^\gamma \geq (\lambda+\gamma)\lambda^{-\lambda/(\lambda+\gamma)}\gamma^{-\gamma/(\lambda+\gamma)} a^{\gamma/(\lambda+\gamma)} b^{\lambda/(\lambda+\gamma)}. \tag{2.2}$$

Lemma 2.1 can easily be proved by using the extremum of one variable function. For the sake of convenience, we denote

$$q_-(t) = \min\{q(t), 0\}, \quad q_+(t) = \max\{q(t), 0\}, \quad \tilde{q}(t) = n_\phi q_+(s) + N_\phi q_-(s),$$
$$p_1(t) = C_4 p_+(t) + C_3 p_-(t), \qquad p_2(t) = C_3 p_+(t) + C_4 p_-(t). \tag{2.3}$$

Theorem 2.2. *Assume that* $\alpha = \beta$ *holds and there exists* $\rho \in C^1([t_0,\infty),R^+)$, $\forall T \in [t_0,\infty)$, *such that*

$$\int_T^t \left(\tilde{q}(s) - \frac{C_2\alpha^\alpha r(s)}{C_1^\alpha(\alpha+1)^{\alpha+1}}\left|\frac{\rho'(s)}{\rho(s)}\right|^{\alpha+1}\right) e^{\int_{t_0}^s (p_i(\tau)/r(\tau))d\tau}\rho(s)ds \longrightarrow \infty, \quad t \longrightarrow \infty,\ i = 1,2, \tag{2.4}$$

$$\lim_{t\to\infty} Q(t) = \int_T^t \tilde{q}(s)e^{\int_{t_0}^s (p_1(\tau)/r(\tau))d\tau}ds > 0. \tag{2.5}$$

If any one of the following two conditions holds, then the solution $x = x(t)$ *of* (1.9) *is oscillatory.*

(1)

$$R(t) = \int_{t_0}^t r^{-1/\alpha}(s)e^{-1/\alpha\int_{t_0}^s (p_1(\tau)/r(\tau))d\tau}ds \longrightarrow \infty, \quad t \longrightarrow \infty. \tag{2.6}$$

(2) $R(\infty) < \infty$, *and*

$$\begin{aligned}
&\int_T^t \left(Q(s)r^{-1}(s)e^{-\int_{t_0}^s (p_1(\tau)/r(\tau))d\tau}\right)^{1/\alpha} ds \\
&\quad = \int_T^t \left(r^{-1}(s)\int_T^s \tilde{q}(\varsigma)e^{-\int_\varsigma^s (p_1(\tau)/r(\tau))d\tau}d\varsigma\right)^{1/\alpha} ds \longrightarrow \infty, \quad t \longrightarrow \infty,
\end{aligned} \tag{2.7}$$

$$\lim_{t\to\infty} Q^{1/\alpha}(t)(R(\infty) - R(t)) \geq \frac{\alpha}{C_1}C_2^{1/\alpha}. \tag{2.8}$$

Proof. Let $x = x(t)$ be a nonoscillatory solution of (1.9). Then, there exists $T \geq t_0$ such that $x = x(t) \neq 0$, $t \in (T,\infty)$. Without loss of generality, we may assume that $x = x(t) \geq 0$, $t \in (T,\infty)$.

Define the Riccati Transformation by

$$W(t) = \frac{\rho(t)r(t)k_1(x(t),x'(t))|x'(t)|^{\alpha-1}x'(t)}{f(x(t))}, \quad t \geq T. \tag{2.9}$$

From conditions (A) and (B), we have

$$\frac{|W(t)|^{1/\alpha}|x(t)|}{C_0^{1/\alpha}(\rho(t)r(t))^{1/\alpha}} \geq |x'(t)| \geq \frac{|W(t)|^{1/\alpha}|x(t)|}{C_2^{1/\alpha}(\rho(t)r(t))^{1/\alpha}}, \quad |f(x)| \geq \left(\frac{C_1}{\alpha}\right)^\alpha |x|^\alpha, \quad t \geq T. \tag{2.10}$$

Differentiating $W(t)$, and applying (1.9) and (2.10), we have

$$
\begin{aligned}
W'(t) &= \frac{\rho'(t)}{\rho(t)}W(t) - \frac{p(t)k_2(x(t),x'(t))}{r(t)}W(t) \\
&\quad - \frac{\rho(t)q(t)\left|f(x(t))\right|}{|x(t)|^{\alpha}}\phi\left(x(g_1(t)),x'(g_2(t))\right) - \frac{\alpha W(t)x'(t)}{x(t)} \\
&\le \begin{cases} \left(\dfrac{\rho'(t)}{\rho(t)} - \dfrac{p_1(t)}{r(t)}\right)W(t) - \left(\dfrac{C_1}{\alpha}\right)^{\alpha}\rho(t)\tilde{q}(t) - \dfrac{\alpha|W(t)|^{(\alpha+1)/\alpha}}{C_2^{1/\alpha}(\rho(t)r(t))^{1/\alpha}}, & W(t)<0, \\ \left(\dfrac{\rho'(t)}{\rho(t)} - \dfrac{p_2(t)}{r(t)}\right)W(t) - \left(\dfrac{C_1}{\alpha}\right)^{\alpha}\rho(t)\tilde{q}(t) - \dfrac{\alpha|W(t)|^{(\alpha+1)/\alpha}}{C_2^{1/\alpha}(\rho(t)r(t))^{1/\alpha}}, & W(t)>0. \end{cases}
\end{aligned}
\tag{2.11}
$$

By (2.1), we have

$$
W'(t) \le \begin{cases} -\left(\left(\dfrac{C_1}{\alpha}\right)^{\alpha}\rho(t)\tilde{q}(t) - \dfrac{C_2 r(t)}{(\alpha+1)^{\alpha+1}}\left(\dfrac{\rho'(t)}{\rho(t)} - \dfrac{p_1(t)}{r(t)}\right)^{\alpha+1}\right)\rho(t), & W(t)<0, \\ -\left(\left(\dfrac{C_1}{\alpha}\right)^{\alpha}\rho(t)\tilde{q}(t) - \dfrac{C_2 r(t)}{(\alpha+1)^{\alpha+1}}\left(\dfrac{\rho'(t)}{\rho(t)} - \dfrac{p_2(t)}{r(t)}\right)^{\alpha+1}\right)\rho(t), & W(t)>0. \end{cases}
\tag{2.12}
$$

Integrating the above inequality from T to $t \ge T$, we have

$$
W(t) \le W(T) - \int_T^t \left(\left(\frac{C_1}{\alpha}\right)^{\alpha}\tilde{q}(s) - \frac{C_2 r(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(s)}{\rho(s)} - \frac{p_i(s)}{r(s)}\right)^{\alpha+1}\right)\rho(s), \quad i=1,2. \tag{2.13}
$$

Condition (2.4) shows that $\lim_{t\to\infty}W(t) = -\infty$. Without loss of generality, we may assume that $W(t)<0$, $t \ge T$, by applying (2.11), we have

$$
\begin{gathered}
\tilde{q}(t)e^{\int_{t_0}^{t}(p_1(s)/r(s))ds} + C_1C_2^{-1/\alpha}r^{-1/\alpha}(t)e^{-(1/\alpha)\int_{t_0}^{t}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(t) \le Z'(t), \\
Z(t) = -\frac{W(t)}{\rho(t)}e^{\int_{t_0}^{t}(p_1(s)/r(s))ds} > 0, \qquad t \ge T.
\end{gathered}
\tag{2.14}
$$

Integrating the above inequality from T to $t \ge T$, we obtain

$$
Z(T) + Q(t) + C_1C_2^{-1/\alpha}\int_T^t r^{-1/\alpha}(\tau)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(\tau)d\tau \le Z(t). \tag{2.15}
$$

Let

$$F(t)=\int_T^t r^{-1/\alpha}(\tau)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(s)ds>0,$$
$$F'(t)=r^{-1/\alpha}(t)e^{-(1/\alpha)\int_{t_0}^{t}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(t),$$
$$t>T. \tag{2.16}$$

We will discuss in the following two cases.

(1) By (2.1) and (2.5), we see that

$$C_1^{(\alpha+1)/\alpha}C_2^{-(\alpha+1)/\alpha^2}r^{-1/\alpha}(t)e^{-(1/\alpha)\int_{t_0}^{t}(p_1(s)/r(s))ds}\le F'(t)F^{-(\alpha+1)/\alpha}(t),\quad t\ge T_1>T. \tag{2.17}$$

Integrating the above inequality from T_1 to $t\ge T_1$, we have

$$\alpha F^{-1/\alpha}(T_1)>\alpha F^{-1/\alpha}(T_1)-\alpha F^{-1/\alpha}(t)$$
$$>C_1^{(\alpha+1)/\alpha}C_2^{-(\alpha+1)/\alpha^2}\int_{T_1}^t r^{-1/\alpha}(s)e^{-(1/\alpha)\int_{t_0}^{s}(p_1(\tau)/r(\tau))d\tau}ds\longrightarrow\infty,\quad t\longrightarrow\infty. \tag{2.18}$$

But, it is impossible that the above inequality holds.

(2) Observe that $W(t)<0$ and by (2.8), we have $x'(t)<0$, so that $x=x(t)$ is monotonic decreasing function for $t\ge T$, and if $\lim_{t\to\infty}x(t)=c$, then $c=0$. Otherwise, if $c>0$, by (2.15) and (2.9), we have

$$Q(t)f(c)\le Z(t)f(c)\le Z(t)f(x(t))=-e^{\int_{t_0}^{\tau}(p_1(s)/r(s))ds}r(t)k_1(x(t),x'(t))\left|x'(t)\right|^{\alpha},\quad t\ge T. \tag{2.19}$$

By condition (B), we have

$$-x'(t)\ge C_2^{-1/\alpha}f^{1/\alpha}(c)Q^{1/\alpha}(t)r^{-1/\alpha}(t)e^{-(1/\alpha)\int_{t_0}^{t}(p_1(s)/r(s))ds},\quad t\ge T. \tag{2.20}$$

Integrating the above inequality from T_1 to $t\ge T_1$ leads to

$$x(T)-c\ge x(T)-x(t)\ge C_2^{-1/\alpha}f^{1/\alpha}(c)\int_T^t Q^{1/\alpha}(s)r^{-1/\alpha}(s)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}ds\longrightarrow\infty,\quad t\longrightarrow\infty. \tag{2.21}$$

But, this is impossible. We choose $\rho(t)=1$, thus (2.15) has the following form:

$$Z(T)+Q(t)+C_1C_2^{-1/\alpha}\int_T^t r^{-1/\alpha}(\tau)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(\tau)d\tau\le Z(t),$$
$$Z(t)=-e^{\int_{t_0}^{t}(p_1(s)/r(s))ds}W(t). \tag{2.22}$$

When $\int_T^t r^{-1/\alpha}(\tau)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(\tau)d\tau < M^{\alpha+1}$, by considering Hölder's inequality and (2.10), we have

$$\begin{aligned}-\ln\frac{x(t)}{x(T)} &= \int_T^t - x'(s)x^{-1}(s)ds\\ &\le\left(\int_T^t r(s)e^{-\int_{t_0}^{\tau}(p_1(s)/r(s))ds}\frac{|x'(s)|^{\alpha+1}}{x^{\alpha+1}(s)}ds\right)^{1/(\alpha+1)}\left(\int_T^t r^{-1/\alpha}(s)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}ds\right)^{\alpha/(\alpha+1)}\\ &\le C_0^{-(\alpha+1)/\alpha}\left(\int_{t_N}^t r^{-1/\alpha}(s)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(s)ds\right)^{1/(\alpha+1)}(R(t)-R(T))^{\alpha/(\alpha+1)}\\ &\le MC_0^{-(\alpha+1)/\alpha}R^{\alpha/(\alpha+1)}(t),\end{aligned} \tag{2.23}$$

such that $MC_0^{-(\alpha+1)/\alpha}R^{\alpha/(\alpha+1)}(\infty) \ge MC_0^{-(\alpha+1)/\alpha}R^{\alpha/(\alpha+1)}(t) \ge -\ln(x(t)/x(T)) \to \infty$, $t \to \infty$. However, this is also impossible.

If $\int_T^t r^{-1/\alpha}(\tau)e^{-(1/\alpha)\int_{t_0}^{\tau}(p_1(s)/r(s))ds}Z^{(\alpha+1)/\alpha}(\tau)d\tau \to \infty$, $t \to \infty$, then by (2.8), we see that $\lim_{t\to\infty}Q(t) = \infty$. From (2.15), we obtain

$$\begin{aligned}&Z(t) \ge Z(T) + Q(T_2) + C_1C_2^{-1/\alpha}F(t), \quad t \ge T_2 \ge T,\\ &C_1C_2^{-1/\alpha}F'(t)\left(Z(T) + Q(T_2) + C_1C_2^{-1/\alpha}F(t)\right)^{-(\alpha+1)/\alpha}\\ &\qquad \ge C_1C_2^{-1/\alpha}r^{-1/\alpha}(t)e^{-(1/\alpha)\int_{t_0}^{t}(p_1(s)/r(s))ds}, \quad t \ge T_2.\end{aligned} \tag{2.24}$$

Integrating the above inequality from T_2 to $t \ge T_2$, we have

$$\alpha Q^{-1/\alpha}(T_2) > \alpha\left(Z(T) + Q(T_2) + C_1C_2^{-1/\alpha}F(T_2)\right)^{-1/\alpha} \ge C_1C_2^{-1/\alpha}(R(\infty) - R(T_2)), \quad T_2 \ge T. \tag{2.25}$$

Let $T_2 \to \infty$; the above inequality contradicts (2.8); this completes the proof. □

Theorem 2.3. *Suppose that* $\alpha \le \beta$ *and*

$$\int_{t_0}^{\infty}\left(\left(\frac{C_1}{\beta}\right)^{\beta}\widetilde{q}(s) - \frac{C_2\alpha^{\alpha}|p_1(s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}r^{\alpha}(s)}\right)ds = \infty, \quad i = 1,2. \tag{2.26}$$

If there exists $0 < \varepsilon < ((\alpha+1)\beta/\alpha)C_2^{-1/\alpha}$, $t_0 \le t_1 < t_2 < \cdots < t_n < \cdots \to \infty$, *such that*

$$\int_{t_n}^t \theta(s)ds > 0, \quad t > t_n, \quad \lim_{t\to\infty}\left(\int_{t_n}^t \theta(s)ds\right)^{1/\alpha}(R(\infty) - R(t))^{(\beta-\alpha)/\lambda(\alpha+1)+1} = \infty, \quad n = 1,2,\ldots, \tag{2.27}$$

where

$$\theta(t)=\left(\frac{C_1}{\beta}\right)^{\beta}\tilde{q}(t)-\frac{1}{\alpha+1}\frac{|p_1(t)|^{\alpha+1}}{\varepsilon^{\alpha}r^{\alpha}(t)},\qquad R(t)=\int_{t_0}^{t}r^{-1/\alpha}(s)ds,$$
$$C_\varepsilon=\left(\beta C_2^{-1/\alpha}-\frac{\varepsilon\alpha}{\alpha+1}\right),\qquad \delta=\left(\frac{C_0}{C_2}\right)^{(\alpha+1)/\alpha},\qquad \lambda=\frac{\beta(\delta-1)}{\alpha}+1-\frac{\varepsilon\delta}{\alpha+1},\tag{2.28}$$

then every solution of (1.9) *is oscillatory.*

Note

From (2.27), it is easy to obtain the following equation:

$$\lim_{t\to\infty}\left(\int_{t_n}^{t}\theta(sds)\right)^{1/\alpha}(R(\infty)-R(t))=\infty,\quad \lim_{t\to\infty}\left(\int_{t_n}^{t}\theta(s)ds\right)^{1/\alpha}\ln\frac{R(\infty)}{R(t)}=\infty,\quad n=1,2,\ldots.\tag{2.29}$$

Proof. Let $x=x(t)$ be a nonoscillatory solution of (1.9). Then, there exists $T\geq t_0$ such that $x=x(t)\neq 0$, $t>T$. We may assume that $x=x(t)>0$, $t>T$.

Introduce the Riccati transformation $W(t)=r(t)U(t)/x(t)^{\beta}$, $t\geq T$. From conditions (A) and (B), we have

$$\left(\frac{W(t)x^{\beta}(t)}{C_0 r(t)}\right)^{1/\alpha}\geq|x'(t)|\geq\left(\frac{W(t)x^{\beta}(t)}{C_2 r(t)}\right)^{1/\alpha},\quad |f(x)|\geq\left(\frac{C_1}{\beta}\right)^{\beta}|x|^{\beta},\quad t\geq T.\tag{2.30}$$

Differentiating $W(t)$, and applying (1.9) and the above inequality, leads to

$$W'(t)=-\frac{p(t)k_2(x(t),x'(t))x^{(\beta-\alpha)/(\alpha+1)}(t)W(t)}{r(t)}-\frac{q(t)\phi(x(g_1(t)),x'(g_2(t)))}{x^{\beta}(t)}-\frac{\beta W(t)x'(t)}{x(t)}$$
$$\leq\begin{cases}-\dfrac{p_1(t)}{r(t)}x^{(\beta-\alpha)/(\alpha+1)}(t)W(t)-\left(\dfrac{C_1}{\beta}\right)^{\beta}\tilde{q}(t)\\ \quad-\beta C_2^{-1/\alpha}r^{-1/\alpha}(t)x^{(\beta-\alpha)/\alpha}(t)|W(t)|^{(\alpha+1)/\alpha},\quad W(t)<0\\ -\dfrac{p_2(t)}{r(t)}x^{(\beta-\alpha)/(\alpha+1)}(t)W(t)-\left(\dfrac{C_1}{\beta}\right)^{\beta}\tilde{q}(t)\\ \quad-\beta C_2^{-1/\alpha}r^{-1/\alpha}(t)x^{(\beta-\alpha)/\alpha}(t)W^{(\alpha+1)/\alpha}(t),\quad W(t)>0.\end{cases}\tag{2.31}$$

By (2.1), we see that

$$W'(t) \leq \begin{cases} -\left(\dfrac{C_1}{\beta}\right)^{\beta}\tilde{q}(t) + \dfrac{C_2\alpha^{\alpha}|p_1(t)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}r^{\alpha}(t)}, & Z'(t) < 0 \\ -\left(\dfrac{C_1}{\beta}\right)^{\beta}\tilde{q}(t) + \dfrac{C_2\alpha^{\alpha}|p_2(t)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}r^{\alpha}(t)}, & Z'(t) > 0. \end{cases} \tag{2.32}$$

The following proof is similar to that in Theorem 2.2, using (2.26), we find that $\lim_{t\to\infty}W(t) = -\infty$. Thus there exists $T_1 \geq T$, such that $W(t) < 0$, $x'(t) < 0$, $t \geq T_1$. Because $x = x(t)$ is monotonic decreasing function on $[T_1, \infty)$; hence $\lim_{t\to\infty}x(t) = c \geq 0$. By (2.31), we have

$$\begin{aligned} W'(t) \leq{}& \frac{|p_1(t)|}{r(t)}x^{(\beta-\alpha)/(\alpha+1)}(t)|W(t)| - \left(\frac{C_1}{\beta}\right)^{\beta}\tilde{q}(t) \\ &- \beta C_2^{-1/\alpha}r^{-1/\alpha}(t)x^{(\beta-\alpha)/(\alpha+1)}(t)|W(t)|^{(\alpha+1)/\alpha}, \quad W(t) < 0,\ t \geq T_1. \end{aligned} \tag{2.33}$$

By using of weighted mean inequality, we can transform the above inequality into

$$W'(t) \leq -\theta(t) - C_{\varepsilon}r^{-1/\alpha}(t)\left|x^{(\beta-\alpha)/(\alpha+1)}(t)W(t)\right|^{(\alpha+1)/\alpha}, \quad W(t) < 0. \tag{2.34}$$

We need to show that $c = 0$. Otherwise, if $c > 0$, by the above inequality, we have

$$W'(t) \leq -\theta(t) - C_{\varepsilon}c^{(\beta-\alpha)/(\alpha+1)}r^{-1/\alpha}(t)|W(t)|^{(\alpha+1)/\alpha}, \quad W(t) < 0. \tag{2.35}$$

By choosing $N \geq 1$, such that $t_N \geq T_1$, integrating the above inequality from T_N to $t \geq T_N$ and by (2.27), we can get

$$W(t) \leq W(t_N) - \int_{t_N}^{t_n}\theta(s)ds - C_{\varepsilon}c^{(\beta-\alpha)/(\alpha+1)}\int_{t_N}^{t}r^{1/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha}ds, \quad t \geq t_n,\ n > N,\ W(t) < 0, \tag{2.36}$$

or

$$\begin{aligned} &\frac{C_{\varepsilon}c^{(\beta-\alpha)/(\alpha+1)}r^{-1/\alpha}(t)|W(t)|^{(\alpha+1)/\alpha}}{\left(|W(t_N)| + \int_{t_N}^{t_n}\theta(s)ds + C_{\varepsilon}c^{(\beta-\alpha)/(\alpha+1)}\int_{t_N}^{t}r^{-1/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha}ds\right)^{(\alpha+1)/\alpha}} \\ &\quad \geq C_{\varepsilon}c^{(\beta-\alpha)/(\alpha+1)}r^{-1/\alpha}(t), \quad t \geq t_n,\ n > N. \end{aligned} \tag{2.37}$$

Differentiating the above inequality on the interval $[t_n, \infty]$, we have

$$\alpha\left(|W(t_N)| + \int_{t_N}^{t_n} \theta(s)ds + C_\varepsilon c^{(\beta-\alpha)/(\alpha+1)} \int_{t_N}^{t_n} r^{-1/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha} ds\right)^{-1/\alpha} \geq C_\varepsilon c^{(\beta-\alpha)/(\alpha+1)}(R(\infty) - R(t_n)), \quad t > t_n,\ n > N. \tag{2.38}$$

This is a contradiction to (2.29); hence, we have $\lim_{t\to\infty} x(t) = 0$. According to the above discussion, we have

$$|W(t)| \geq |W(t_N)| + \int_{t_N}^{t_n} \theta(s)ds + C_\varepsilon \int_{t_N}^{t} r^{-1/\alpha}(s)\left|x^{(\beta-\alpha)/(\alpha+1)}(t)W(s)\right|^{(\alpha+1)/\alpha} ds, \quad t \geq t_n,\ n > N,\ W(t) < 0, \tag{2.39}$$

$$\left(|W(t_N)| + \int_{t_N}^{t_n} \theta(s)ds + C_\varepsilon \int_{t_N}^{t_n} r^{-1/\alpha}(s)\left|x^{(\beta-\alpha)/(\alpha+1)}(t)W(s)\right|^{(\alpha+1)/\alpha} ds\right)^{-1/\alpha} \geq C_\varepsilon \int_{t_n}^{\infty} r^{-1/\alpha}(s)x^{(\beta-\alpha)/(\alpha+1)}(s)ds, \quad n > N. \tag{2.40}$$

We will discuss in the following two cases.

If there exists $M > 0$, such that $\int_{t_N}^{t} r^{-1/\alpha}(s)x^{(\beta-\alpha)/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha} ds \leq M^{\alpha+1}$, $t \geq t_N$, by considering Hölder's inequality and (2.39), we have

$$\begin{aligned}
&\frac{\alpha+1}{\beta-\alpha}\left(x^{(\alpha-\beta)/(\alpha+1)}(t) - x^{(\alpha-\beta)/(\alpha+1)}(t_N)\right) \\
&\quad= \int_{t_N}^{t} - x'(s)x^{-(\beta+1)/(\alpha+1)}(s)ds \\
&\quad\leq \left(\int_{t_N}^{t} r(s)\frac{|x'(s)|^{\alpha+1}}{x^{\beta+1}(s)} ds\right)^{1/(\alpha+1)} \left(\int_{t_N}^{t} r^{-1/\alpha}(s)ds\right)^{\alpha/(\alpha+1)} \\
&\quad\leq C_0^{-(\alpha+1)/\alpha}\left(\int_{t_N}^{t} r^{-1/\alpha}(s)x^{(\beta-\alpha)/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha} ds\right)^{1/(\alpha+1)} (R(t) - R(t_N))^{\alpha/(\alpha+1)} \\
&\quad\leq MC_0^{-(\alpha+1)/\alpha} R^{\alpha/(\alpha+1)}(t).
\end{aligned} \tag{2.41}$$

By choosing $n > N$, such that $2^{(\alpha+1)/(\beta-\alpha)}x(t) < x(t_N)$, $t \geq t_n$, $n > N$, from the above inequality, we have $x^{(\beta-\alpha)/(\alpha+1)}(t) \geq ((\alpha+1)C_0^{(\alpha+1)/\alpha}/2(\beta-\alpha)M)R^{-\alpha/(\alpha+1)}(t)$, $t \geq t_n$, $n > N$. Inserting it in (2.40), we can get

$$\begin{aligned}\left(\int_{t_N}^{t_n}\theta(s)ds\right)^{-1/\alpha} &\geq \left(|W(t_N)| + \int_{t_N}^{t_n}\theta(s)ds + \int_{t_N}^{t_n}\frac{\widetilde{C}_\varepsilon|W(s)|^{(\alpha+1)/\alpha}}{R(s)r^{1/\alpha}(s)}ds\right)^{-1/\alpha}\\ &\geq \frac{\widetilde{C}_\varepsilon}{\alpha}\ln\frac{R(\infty)}{R(t_n)}, \quad \widetilde{C}_\varepsilon = C_\varepsilon\left(\frac{(\alpha+1)C_0^{(\alpha+1)/\alpha}}{2(\beta-\alpha)M}\right)^{(\alpha+1)/\alpha}.\end{aligned} \tag{2.42}$$

This is contradiction to (2.29).

If $\int_{t_N}^{\infty}(x^{(\beta-\alpha)/\alpha}(s)|W(s)|^{(\alpha+1)/\alpha}/(\rho(s)r(s))^{1/\alpha})ds = \infty$ for $t \geq t_n$, $n > N$, along with (2.39) and (2.30), we have

$$\frac{C_2 r(t)|x'(t)|^\alpha}{|x(t)|^\beta} \geq |W(t)| \geq |W(t_N)| + C_\varepsilon C_0^{(\alpha+1)/\alpha}\int_{t_N}^{t} r(s)\frac{|x'(s)|^{\alpha+1}}{|x(s)|^{\beta+1}}ds, \quad t \geq t_n,\ n > N, \tag{2.43}$$

leading to

$$C_\varepsilon C_0^{(\alpha+1)/\alpha}\frac{r(t)|x'(t)|^{\alpha+1}}{|x(t)|^{\beta+1}}\left(|W(t_N)| + C_\varepsilon C_0^{(\alpha+1)/\alpha}\int_{t_N}^{t} r(s)\frac{|x'(s)|^{\alpha+1}}{|x(s)|^{\beta+1}}ds\right)^{-1} \geq -\frac{C_\varepsilon}{C_2}C_0^{(\alpha+1)/\alpha}\frac{x'(t)}{x(t)}. \tag{2.44}$$

Integrating the above inequality from t_n to $t > t_n$ and by (2.40), we can get

$$\ln\frac{C_2 r(t)|x'(t)|^\alpha}{-W(t_N)|x(t)|^\beta} \geq \frac{C_\varepsilon}{C_2}C_0^{(\alpha+1)/\alpha}\ln\frac{x(t_n)}{x(t)}, \tag{2.45}$$

or

$$-x'(t)x^{(\beta/\alpha-\varepsilon/(\alpha+1))\delta-\beta/\alpha}(t) \geq |W(t_N)|^{1/\alpha}C_2^{-1/\alpha}x^{(\beta/\alpha-\varepsilon/(\alpha+1))\delta}(t_n)(r(t))^{-1/\alpha}. \tag{2.46}$$

Integrating the above inequality on the interval $[t, \infty)$, we have

$$x(t) \geq C^{1/\lambda}(R(\infty) - R(t))^{1/\lambda}, \quad C = \lambda|W(t_N)|^{1/\alpha}C_2^{-1/\alpha}x^{(\beta/\alpha-\varepsilon/(\alpha+1))\delta}(t_n). \tag{2.47}$$

If $R_\rho(\infty) = \infty$, the above inequality cannot be satisfied; hence, $R_\rho(\infty) < \infty$. Inserting it in (2.40), we can get

$$\left(\frac{\beta-\alpha}{\lambda(\alpha+1)}+1\right) \geq C_\varepsilon C^{(\beta-\alpha)/\lambda(\alpha+1)}\left(\int_{t_N}^{t_n}\theta(sds)\right)^{1/\alpha}(R(\infty)-R(t_n))^{(\beta-\alpha)/\lambda(\alpha+1)+1}. \tag{2.48}$$

This is contradiction to (2.26).

Hence, we complete the proof of Theorem 2.3. □

3. Some Examples

Example 3.1. Let us consider the oscillatory behavior of the following differential equation:

$$\left(t^\lambda\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + pt^{\lambda_1}\left|x'(t)\right|^{\alpha-1}x'(t) + qt^{\lambda_2}|x(t)|^{\alpha-1}x(t)\phi\left(x(t),x'(t)\right) = 0, \quad t \geq t_0. \tag{3.1}$$

Comparing (3.1) with (1.9), we can find that

$$\begin{gathered} r(t) = t^\lambda, \quad C_2 = C_0 = 1, \quad p(t) = pt^{\lambda_1}, \quad p > 0, \quad C_1 = 1, \quad C_3 = C_4 = 1, \\ q(t) = qt^{\lambda_2}, \quad \beta = \alpha, \quad p_1(t) = pt^{\lambda_1}, \quad p_2(t) = 0, \quad \widetilde{q}(t) = qn_\phi t^{\lambda_2}. \end{gathered} \tag{3.2}$$

Let $\rho(t) = t^\mu$, (2.4)–(2.7) are transformed into the equations

$$\int_{t_0}^{t}\left(qn_\phi - \frac{\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}}s^{\lambda-\lambda_2-\alpha-1}\right)e^{p\int_{t_0}^{s}\tau^{\lambda_1-\lambda}d\tau}s^{\lambda_2+\mu}ds \longrightarrow \infty, \quad t \longrightarrow \infty, \tag{3.3}$$

$$\int_T^t s^{\lambda_2}e^{p\int_{t_0}^{s}\tau^{\lambda_1-\lambda}d\tau}ds > 0, \tag{3.4}$$

$$R(t) = \int_{t_0}^{t} s^{-\lambda/\alpha}e^{-(p/\alpha)\int_{t_0}^{s}\tau^{\lambda_1-\lambda}d\tau}ds \longrightarrow \infty, \quad t \longrightarrow \infty, \tag{3.5}$$

$$\int_T^t\left(s^{-\lambda}\int_T^s \varsigma^{\lambda_2}e^{-p\int_{\varsigma}^{s}\tau^{\lambda_1-\lambda}d\tau}d\varsigma\right)^{1/\alpha}ds \longrightarrow \infty, \quad t \longrightarrow \infty, \tag{3.6}$$

$$\lim_{t\to\infty}\left(\int_T^t s^{\lambda_2}e^{p\int_{t_0}^{s}\tau^{\lambda_1-\lambda}d\tau}ds\right)^{1/\alpha}(R(\infty)-R(t)) > \frac{\alpha C_2^{1/\alpha}}{qn_\phi C_1}. \tag{3.7}$$

We will discuss inthe following cases.

(1) $\lambda_1 + 1 < \lambda$, choosing $\mu = -\min\{\lambda_2+1, 0\}$, provided that $\lambda \leq \alpha$, $\lambda - \lambda_2 < \alpha + 1$, or $qn_\phi > \mu^{\alpha+1}/(\alpha+1)^{\alpha+1}$ is satisfied for $\alpha \geq \lambda = \lambda_2 + \alpha + 1$, then (3.3)–(3.5) hold, and the solution of (3.1) is oscillatory.

(2) $\lambda = \lambda_1 + 1$, $p + \lambda + \alpha \geq 0$, choosing $\mu = -\min\{\lambda_2 + 1, 0\}$, provided that $\lambda \leq \alpha$, and $\lambda - \lambda_2 < \alpha + 1$ or $qn_\phi > |\mu - p|^{\alpha+1}/(\alpha+1)^{\alpha+1}$, $\alpha \geq \lambda = \lambda_2 + \alpha + 1$, then (3.3)–(3.5) hold, the solution of (3.1) is oscillatory.

(3) $\lambda = \lambda_1 + 1$, $p + \lambda + \alpha < 0$, we can see that $R(\infty) < \infty$, choosing $\mu = -\min\{\lambda_2 + 1, 0\}$, and therefore, (3.6) and (3.7) are transformed into the following equation:

$$\int_T^t \left(\frac{qn_\phi}{\lambda_2 + p + 1} \left(s^{-\lambda+\lambda_2+1} - s^{-\lambda-p} T^{\lambda_2+p+1} \right) \right)^{1/\alpha} ds \longrightarrow \infty, \quad t \longrightarrow \infty, \lambda_2 + p + 1 > 0,$$
$$\lim_{t\to\infty} t^{(\lambda_2-\lambda+\alpha+1)/\alpha} > \frac{(\lambda + p - \alpha) C_2^{1/\alpha}}{qn_\phi C_1} (\lambda_2 + p + 1)^{1/\alpha}, \quad \lambda_2 + p + 1 > 0, \tag{3.8}$$

provided that $\lambda_2 + p + 1 > 0$ and $\lambda < \lambda_2 + \alpha + 1$, or $\lambda = \lambda_2 + \alpha + 1$, $((\lambda + p - \alpha)C_2^{1/\alpha}/qn_\phi C_1)(\lambda_2 + p + 1)^{1/\alpha} < 1$, then (3.3)-(3.4) and (3.6)-(3.7) hold, the solution of (3.1) is oscillatory.

In particular, we chose $\alpha = 1$, $p = 0$, $\lambda = 0$, $\lambda_2 = -2$, $\phi(x(t), x'(t)) = 1 = n_\phi$, $\mu = 1$, $q > 1/4$, thus the conditions of the case (1) can be satisfied. This is the sufficient condition for all solutions of $x''(t) + (q/t^2)x(t) = 0$ to be oscillatory.

If we choose $\alpha = P - 1$, $p = 2(P-1)$, $\lambda = 0$, $\lambda_1 = -1$, $\lambda_2 = -P$, $\phi(x,y) \geq ((P-1)/P)^P + \tilde{\lambda}/\ln^2|x| > ((P-1)/P)^P = n_\phi$, $\tilde{\lambda} > 0$, $\mu = P - 1$, $q \geq 1$, the conditions of the case (2) can be satisfied. Compared with the conditions: $q = 1$, $\tilde{\lambda} > (1/2)((P-1)/P)^{P+1}$ in [6], our results are more general.

Example 3.2. Let us consider the oscillatory behavior of the following differential equation:

$$x''(t) - \sin t x'(t) + \frac{1 + \cos t}{1 + \sin^2 t} x(t) \left(1 + |x(t)|^2 \right), \quad t \geq 0. \tag{3.9}$$

Comparing (3.9) with (1.9), we can see that $\tilde{q}(t) = (1 + \cos t)/(1 + \sin^2 t)$, $\alpha = 1$, $r(t) = 1$, $f(x) = x(1 + x^2)$, $C_1 = 3/\sqrt[3]{2}$, $C_0 = C_2 = C_3 = C_4 = 1$, $p_1(t) = p_2(t) = -\sin t$, $\beta = 3$. Choosing $\varepsilon = 1$, clearly, the conditions (2.26) and (2.27) of Theorem 6 can be satisfied. Therefore, we may conclude that (3.9) is oscillatory. Example 3.2 is Example 2 of [5]. It is easy to verify that Example 1 of [5] also satisfies with Theorem 6.

Example 3.3. Let us consider the oscillatory behavior of the following differential equation:

$$\left[\frac{1}{1 + t^2} x'(t) \right]' - \frac{a + b \sin t}{t} x'(t) + q(t) |x(t)|^{\beta-1} x(t), \quad t \geq 0. \tag{3.10}$$

Comparing (3.10) with (1.9), we can see that $\tilde{q}(t) = q(t)$, $\alpha = 1$, $r(t) = 1/(1+t^2)$, $f(x) = |x|^{\beta-1}x$, $C_1 = \beta$, $C_0 = C_2 = C_3 = C_4 = 1$, and $p_1(t) = p_2(t) = (a + b \sin t)/t$. Thus, we have $R(t) = t - t_0 + (1/3)(t^3 - t_0^3) \to \infty$, $t \to \infty$, so the second condition of Theorem 2.3 in (2.27) is satisfied. Therefore, another condition is

$$\int_{t_0}^{\infty} \left(q(s) - \frac{(a + b \sin s)^2}{4\beta} \left(1 + \frac{1}{s^2} \right) \right) ds = \infty, \tag{3.11}$$

and there exists $0 < \varepsilon < 2\beta$, $t_0 \le t_1 < t_2 < \cdots < t_n < \cdots \to \infty$ such that

$$\int_{t_n}^{t} \left(q(s) - \frac{(a + b\sin s)^2}{2\varepsilon} \left(1 + \frac{1}{s^2} \right) \right) ds > 0, \quad t > t_n,\ n = 1, 2, \ldots. \tag{3.12}$$

If only $\int_{t_0}^{\infty}(q(s) - (2a^2 + b^2)/8\beta)ds = \infty$, $\int_{t_0}^{\infty}(q(s) - (2a^2 + b^2)/4\varepsilon)ds > M$, where M is a sufficiently large constant, then the conditions (3.11) and (3.12) can be satisfied, the solution of (3.10) is oscillatory.

By taking $q(t) = \sin^2 t$, provided that $2a^2 + b^2 < 4\beta$, $\varepsilon = (1/2\beta + 1/(2a^2 + b^2))^{-1}$, (3.11) and (3.12) hold. By the way, we also note that for (3.10), the example given in [3],

$$q(t) = \begin{cases} \dfrac{3 + t^2}{4} \left[\dfrac{2}{t - (6n - 4)\pi} + \dfrac{1 + t^2}{t} \right]^2, & (6n - 4)\pi \le t \le \left(6n - \dfrac{7}{2} \right)\pi, \\[2ex] \dfrac{3 + t^2}{4} \left[\dfrac{2}{(6n - 3)\pi - t} - \dfrac{1 + t^2}{t} \right]^2, & \left(6n - \dfrac{7}{2} \right)\pi \le t \le (6n - 3)\pi, \end{cases} \quad n = \{1, 2, \ldots\} \tag{3.13}$$

are not continuous at $t = (6n - 4)\pi$, $(6n - 3)\pi$, $n = 1, 2, \ldots$, and

$$\begin{gathered} \sum_{n=1}^{[t/6\pi]} \int_{(6n-4)\pi}^{(6n-7/2)\pi} \frac{3 + t^2}{4} \left[\frac{2}{t - (6n - 4)\pi} + \frac{1 + t^2}{t} \right]^2 dt = \infty, \\ \int_{(6n-7/3)\pi}^{(6n-3)\pi} \frac{3 + t^2}{4} \left[\frac{2}{(6n - 2)\pi - t} + \frac{1 + t^2}{t} \right]^2 dt = \infty. \end{gathered} \tag{3.14}$$

Though (3.11) and (3.12) hold, but it is the general requirement that for fixed t, $\int_{t_0}^{t} q(s)ds$ is bounded; hence, this example is not appropriate.

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