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Research Article

On Mixed Problems for Quasilinear Second-Order Systems

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The paper is devoted to the study of initial-boundary value problems for quasilinear second-order systems. Existence and uniqueness of the solution in the space $H^s(\overline{\Omega} \times [0,T])$, with $s > d/2 + 3$, is proved in the case where Ω is a half-space of $\mathfrak{R}^d$. The proof of the main theorem relies on two preliminary results: existence of the solution to mixed problems for linear second-order systems with smooth coefficients, and existence of the solution to initial-boundary value problems for linear second-order operators whose coefficients depend on the variables x and t through a function $v \in H^s(\mathfrak{R}^{d+1})$. By means of the results proved for linear operators, the well posedness of the mixed problem for the quasi-linear system is established by studying the convergence of a suitable iteration scheme.

1. Introduction

The purpose of this paper is the study of initial-boundary value problems for second-order quasi-linear systems. The name initial-boundary value problem or mixed problem refers to the initial data (i.e., at time $t = 0$) and to the boundary data (i.e., on the boundary of the spatial domain). The interest in mixed problems originates from physical issues: in fluid mechanics, for instance, the spatial domain where a phenomenon takes place usually has an entrance, an exit, and walls.

I will be concerned with the study of mixed problems for second-order differential operators of the form

$$Q = \partial_t^2 + S(\cdot)\partial_t + \sum_{\alpha=1}^{d} T^\alpha(\cdot)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}(\cdot,\nabla_x\cdot)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} F^\alpha(\cdot,\nabla_x\cdot)\partial_\alpha + G(\cdot,\nabla_x\cdot), \quad (1.1)$$

where t is a real variable and the variable x denotes a vector of $\Re^d$, with $d > 1$. For all $\alpha, \beta = 1, \ldots, d$, the functions $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$, take values in the space of $d \times d$ real matrices with smooth coefficients.

The subject of my research will be initial-boundary value problems in a half-space $\Omega = \Re^{d-1} \times (0, \infty)$. The second section of the paper deals with the study of the existence and uniqueness of the solution to the following mixed problem in the half-space $\Omega = \Re^{d-1} \times (0, \infty)$

$$\begin{gathered} Qu = J(u), \quad x \in \Omega,\ t > 0, \\ 2\sum_{\beta=1}^{d-1} E^{d,\beta}(u, \nabla_x u)\partial_\beta u + T^d(u)\partial_t u + F^d(u, \nabla_x u)u = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = h(x), \quad x \in \Omega, \\ \partial_t u(x,0) = k(x), \quad x \in \Omega, \end{gathered} \tag{1.2}$$

where h and k are functions defined in Ω with values in $\Re^d$, whereas the function J is defined in $\Re^d$; the unknown vector-valued function is denoted by u.

In problem (1.2), the initial data h and k will be assumed to lie in Sobolev spaces. The main result of the paper states that if the coefficients of the operator Q are sufficiently regular and the operator satisfies suitable estimates, then there exists $T > 0$ and a function $u \in H^s(\overline{\Omega} \times [0, T])$, with $s > d/2 + 3$, that provides a unique solution to the mixed problem (1.2).

In order to achieve this result, we apply techniques already employed to prove existence and uniqueness in Sobolev spaces of nontrivial solutions to second-order quasi-linear systems, with homogeneous boundary conditions, in the half-space $\Omega = \Re^{d-1} \times (0, \infty)$ (see [1, 2]). The proof of Theorem 2.3 as well as the proof of the main theorems in [1, 2] relies on the assumption that the operator Q and its adjoint satisfy estimates which are analogous to the a priori estimates obtained in [3] in the case of linear Cauchy problems for first-order symmetrizable systems. In fact, in the case where the system under consideration is a first-order linear symmetrizable system with smooth coefficients, according to the results proved in [3], energy estimates imply the well posedness of the Cauchy problem. The existence of a unique solution $u \in C([0,T]; H^s(\Re^d))$ for initial data $u(0) \in H^s$ can be obtained by means of a duality argument, applying the a priori estimates satisfied by the adjoint operator associated with the system. This existence result can be extended to the case of a system with H^s-coefficients, provided that s is large enough: in [4], the well posedness of the Cauchy problem for a first-order system whose coefficients depend on the variables x and t through a known function, has been obtained; this result turns out to be a significant tool in non-linear analysis, since it can be employed to state the well posedness of a class of first-order quasi-linear systems (see [3, 5, 6]). As a matter of fact, for general quasi-linear systems, the study of the Cauchy problem turns out to be a difficult task. However, if the system is Friedrichs symmetrizable (see [3]), then H^s well posedness has been proved to hold true, provided that s is large enough: the proof relies on the definition of an iteration scheme and on the well posedness obtained in the linear case.

By following the strategy successfully applied to prove existence and uniqueness of the solution to the initial value problem for first-order systems, I will be concerned in this paper with initial-boundary value problems for second-order quasi-linear systems. As a preliminary result, Proposition 2.1 states the existence in $H^s(\Re^{d+1})$ of the solution to mixed problems for linear second-order operators, whose coefficients depend smoothly on the space variable x and on the time variable t. Subsequently, by means of a similar procedure, I will prove the existence of the solution to the initial-boundary value problem with homogeneous boundary conditions, in the case where the coefficients of the operator depend on the variables x and t through a function $v \in H^s(\Re^{d+1})$, with $s > d/2 + 3$. Thus this result enables me to state the main Theorem 2.3 regarding existence and uniqueness of the solution to the quasi-linear problem in the space $H^s(\overline{\Omega} \times [0,T])$, in the case where the domain is a half-space of $\Re^d$. An iteration scheme, which consists of a sequence of mixed problems for linear systems, will be defined to approximate the solution. Thus, as a result of the estimates satisfied by the operator Q, one can prove the convergence of the scheme and establish existence of a unique solution to problem (1.2) in the space $H^s(\overline{\Omega} \times [0,T])$. The solution turns out to be classical, for this function belongs to Sobolev spaces with sufficiently large exponent.

2. Mixed Problem in a Half-Space

Consider the following initial-boundary value problem in the half-space $\Omega = \Re^{d-1} \times (0,\infty)$:

$$\begin{gathered}
\partial_t^2 u + \overline{S}(x,t)\partial_t u + \sum_{\alpha=1}^{d} \overline{T}^{\alpha}(x,t)\partial_t\partial_\alpha u + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \overline{E}^{\alpha,\beta}(x,t)\partial_\alpha\partial_\beta u + \sum_{\alpha=1}^{d} \overline{F}^{\alpha}(x,t)\partial_\alpha u \\
+\overline{G}(x,t)u = \overline{J}(x,t), \quad x \in \Omega,\ t > 0, \\
2\sum_{\beta=1}^{d-1} \overline{E}^{d,\beta}(x,t)\partial_\beta u + \overline{T}^{d}(x,t)\partial_t u + \overline{F}^{d}(x,t)u = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = h(x), \quad x \in \Omega, \\
\partial_t u(x,0) = k(x), \quad x \in \Omega,
\end{gathered} \tag{2.1}$$

where for all $\alpha,\beta = 1,\ldots,d$, the functions $\overline{S},\overline{T}^{\alpha},\overline{E}^{\alpha,\beta},\overline{F}^{\alpha},\overline{G},\overline{J}$ are defined in $\Omega \times \Re$, $\overline{S},\overline{T}^{\alpha},\overline{E}^{\alpha,\beta},\overline{F}^{\alpha},\overline{G}$ belong to the space of $d \times d$ real matrices, whereas $\overline{J}$ takes values in $\Re^d$. We assume that for every $\alpha,\beta = 1,\ldots,d$, $\overline{E}^{\alpha,\beta} = \overline{E}^{\beta,\alpha}$.

Let us denote by $\overline{Q}$ the linear second-order operator associated to system (2.1):

$$\overline{Q} = \partial_t^2 + \overline{S}(x,t)\partial_t + \sum_{\alpha=1}^{d} \overline{T}^{\alpha}(x,t)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \overline{E}^{\alpha,\beta}(x,t)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} \overline{F}^{\alpha}(x,t)\partial_\alpha + \overline{G}(x,t), \tag{2.2}$$

and by $\overline{Q}^*$ the corresponding adjoint operator:

$$\begin{aligned}
\overline{Q}^* = {}& \partial_t^2 - \partial_t\overline{S}^T - \overline{S}^T\partial_t + \sum_{\alpha=1}^{d}\left(\partial_\alpha\partial_t\overline{T}^{\alpha T} + \partial_t\overline{T}^{\alpha T}\partial_\alpha + \partial_\alpha\overline{T}^{\alpha T}\partial_t + \overline{T}^{\alpha T}\partial_\alpha\partial_t\right) \\
& + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\left(\partial_\beta\partial_\alpha\overline{E}^{\alpha,\beta T} + 2\partial_\alpha\overline{E}^{\alpha,\beta T}\partial_\beta + \overline{E}^{\alpha,\beta T}\partial_\alpha\partial_\beta\right) - \sum_{\alpha=1}^{d}\left(\overline{F}^{\alpha T}\partial_\alpha + \partial_\alpha\left(\overline{F}^{\alpha T}\right)\right) + \overline{G}^T.
\end{aligned} \tag{2.3}$$

Since we are looking for a solution to (2.1) in the functional space $H^s(\mathfrak{R}^{d+1})$, with $s > d/2 + 2$, such a function will belong to $C^2(\overline{\Omega})$, due to Sobolev embedding. Thus, it will be continuous up to the boundary of Ω. As a consequence of this property, one has to require compatibility conditions.

Similarly to the proof of Proposition 2.1 in [2], we will establish the following result.

Proposition 2.1. *Let s be an integer number so that $s > d/2+2$. Consider the initial-boundary value problem* (2.1) *and assume the following conditions are fulfilled:*

(i) *for all $\alpha, \beta = 1, \dots, d$, the matrix-valued functions $\overline{S}$, $\overline{T}^{\alpha}$, $\overline{E}^{\alpha,\beta}$, $\overline{F}^{\alpha}$, $\overline{G}$ belong to $C^\infty(\overline{\Omega} \times [0,T])$ and are bounded as well as their derivatives, in addition $\overline{E}^{\alpha,\beta} = \overline{E}^{\beta,\alpha}$ and $\overline{E}^{d,d}$ is the null matrix;*

(ii) *the function $\overline{J}$ belongs to the space $H^s(\mathfrak{R}^{d+1})$; moreover, $h \in H^{s+1}(\overline{\Omega})$; $k \in H^s(\overline{\Omega})$;*

(iii) *there exists a positive real constant λ such that for every vector function $\varphi \in L^2(\mathfrak{R}^{d+1}) \cap C^2(\overline{\Omega} \times [0,T[)$, with $\operatorname{supp}\|\varphi\|$ compact subset of $\overline{\Omega} \times [0,T[$*

$$\|\partial_t\varphi(0)\|^2_{H^{-s}} + \|\varphi(0)\|^2_{H^{-s}} + \|\varphi\|^2_{H^{-s}} \leq \lambda \left\|\overline{Q}^*\varphi\right\|^2_{H^{-s}}; \tag{2.4}$$

(iv) *let us set $r = [s-d/2]$, if $s-d/2 \notin IN$; $r = [s-d/2]-1$, if $s-d/2 \in IN$; and assume the following compatibility conditions to be satisfied: the function $2\sum_{\beta=1}^{d-1}\overline{E}^{d,\beta}\partial_\beta h + \overline{T}^d k + \overline{F}^d h$, and the derivatives of $B[u] = 2\sum_{\beta=1}^{d-1}\overline{E}^{d,\beta}\partial_\beta u + \overline{T}^d\partial_t u + \overline{F}^d u$ with respect to the time variable t of order less than or equal to $r-1$, vanish when evaluated at $t=0$ and $x \in \partial\Omega$.*

Then there exists a function $u \in H^s(\mathfrak{R}^{d+1})$, which provides a solution to the mixed problem (2.1).

Proof. Let us denote by E the space

$$E = \left\{\varphi : \mathfrak{R}^d \times \mathfrak{R} \longrightarrow \mathfrak{R}^d : \varphi \in C^2\left(\overline{\Omega} \times [0,T[\right), \operatorname{supp}\|\varphi\| \subseteq \overline{\Omega} \times [0,T[\text{ and compact}\right\}. \tag{2.5}$$

Let $\varphi \in E$ and define the linear functional L on the space $\overline{Q}^*(E) \subseteq H^{-s}(\mathfrak{R}^{d+1})$ as follows:

$$L\left(\overline{Q}^*\varphi\right) = \left\langle k + \overline{S}(0)h + \sum_{\alpha=1}^{d}\overline{T}^{\alpha}(0)\partial_\alpha h, \varphi(0)\right\rangle_{H^s,H^{-s}} - \langle h, \partial_t\varphi(0)\rangle_{H^s,H^{-s}} + \left\langle \overline{J}, \varphi\right\rangle_{H^s,H^{-s}}. \tag{2.6}$$

Thanks to (iii), the operator $\overline{Q}^*$ is one to one in E. Hence the functional L turns out to be well defined. Furthermore, it turns out to be continuous, due to the assumptions. Thus L can be extended to the space $H^{-s}(\mathfrak{R}^{d+1})$. Let us denote by M this functional. Thanks to

Riesz theorem, there exists a unique function $u \in H^s(\Re^{d+1})$, such that for all $v \in H^{-s}(\Re^{d+1})$, $M(v) = \langle u, v\rangle_{H^s,H^{-s}}$. If $\varphi \in E$, then

$$\begin{aligned} M\left(\overline{Q}^*\varphi\right) &= \left\langle u, \overline{Q}^*\varphi\right\rangle_{H^s,H^{-s}} \\ &= \left\langle k + \overline{S}(0)h + \sum_{\alpha=1}^{d}\overline{T}^{\alpha}(0)\partial_\alpha h, \varphi(0)\right\rangle_{H^s,H^{-s}} - \langle h, \partial_t\varphi(0)\rangle_{H^s,H^{-s}} + \left\langle \overline{J}, \varphi\right\rangle_{H^s,H^{-s}}. \end{aligned} \tag{2.7}$$

Let $\varphi \in C_0^\infty(\Omega \times (0,T))$. Thus $\langle u, \overline{Q}^*\varphi\rangle_{H^s,H^{-s}} = \langle \overline{J}, \varphi\rangle_{H^s,H^{-s}}$; as a consequence, for all $\varphi \in C_0^\infty(\Omega \times (0,T))$, $\langle \overline{Q}u, \varphi\rangle_{L^2,L^2} = \langle \overline{J}, \varphi\rangle_{L^2,L^2}$, and the function u turns out to be solution of system (2.1) in $\Omega \times (0,T)$.

Consider a function $\chi \in C_0^\infty(A\times]0,T[)$, where A is an open subset of $\Re^d$, and define the function $\chi^T \in E$ in the following way:

$$\chi^T(x,t) = \begin{cases} \chi(x,t), & \text{if } x \in \overline{\Omega},\ t \in]0,T[, \\ 0, & \text{if } x \in A \setminus \overline{\Omega},\ t \in]0,T[. \end{cases} \tag{2.8}$$

Thus $\langle u, \overline{Q}^*\chi^T\rangle_{H^s,H^{-s}} = \langle \overline{J}, \chi^T\rangle_{H^s,H^{-s}}$.

Integrating by parts, we deduce

$$\int_0^T \int_{\Re^{d-1}} \left\langle 2\sum_{\beta=1}^{d-1}\overline{E}^{d,\beta}\partial_\beta u + \overline{T}^d\partial_t u + \overline{F}^d u, \chi\right\rangle (y,0,t)\,dy\,dt = 0. \tag{2.9}$$

By means of a standard argument, one proves that the function u satisfies the boundary conditions of problem (2.1).

Let us consider now a function $\psi \in C_0^\infty(\Omega\times]-\infty,T[)$ and define the function $\psi^T \in E$, as

$$\psi^T(x,t) = \begin{cases} \psi(x,t), & \text{if } x \in \Omega,\ 0 \le t < T, \\ 0, & \text{if } x \in \Omega,\ t < 0. \end{cases} \tag{2.10}$$

We have $\langle u, \overline{Q}^*\psi^T\rangle_{H^s,H^{-s}} = M(\overline{Q}^*\psi^T)$.

Integrating by parts, we obtain the following identity:

$$\begin{aligned} &-\langle u(0), \partial_t\psi(0)\rangle_{L^2,L^2} + \left\langle \partial_t u(0) + \sum_{\alpha=1}^{d}\overline{T}^{\alpha}(0)\partial_\alpha u(0) + \overline{S}(0)u(0), \psi(0)\right\rangle_{L^2,L^2} + \left\langle \overline{Q}u, \psi\right\rangle_{L^2,L^2} \\ &\quad= \left\langle k + \overline{S}(0)h + \sum_{\alpha=1}^{d}\overline{T}^{\alpha}(0)\partial_\alpha h, \psi(0)\right\rangle_{L^2,L^2} - \langle h, \partial_t\psi(0)\rangle_{L^2,L^2} + \left\langle \overline{J}, \psi\right\rangle_{L^2,L^2}. \end{aligned} \tag{2.11}$$

Let us consider a vector function ψ, which, in a neighbourhood of the origin $t = 0$, is expressed by the product $\psi(x,t) = t\mu(x)$, where $\mu \in C_0^\infty(\Omega)$. Thus, we prove, by means of a standard argument, that $u(x,0) = h(x)$, in Ω. As a result, we deduce similarly that $\partial_t u(x,0) = k(x)$, in Ω. □

Thanks to the result established in Proposition 2.1, we can prove the existence of the solution to a mixed problem for a second-order linear operator with coefficients that depend on the space variable and on the time-variable through a given function. Let $v \in H^s(\mathfrak{R}^{d+1})$ with $s > d/2 + 3$ and denote by Q_v the operator

$$Q_v = \partial_t^2 + S(v)\partial_t + \sum_{\alpha=1}^{d} T^\alpha(v)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}(v,\nabla_x v)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} F^\alpha(v,\nabla_x v)\partial_\alpha + G(v,\nabla_x v). \tag{2.12}$$

Let us consider the mixed problem

$$\begin{gathered} Q_v u = J(v), \quad x \in \Omega,\ t > 0, \\ 2\sum_{\beta=1}^{d-1} E^{d,\beta}(v,\nabla_x v)\partial_\beta u + T^d(v)\partial_t u + F^d(v,\nabla_x v)u = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = h(x), \quad x \in \Omega, \\ \partial_t u(x,0) = k(x), \quad x \in \Omega, \end{gathered} \tag{2.13}$$

where for all $\alpha,\beta = 1,\dots,d$, the functions $E^{\alpha,\beta}, F^\alpha, G$, are defined in $\mathfrak{R}^d \times \mathfrak{R}^{d^2}$, with values in the space of $d \times d$ real matrices with smooth coefficients, the matrix valued functions S, T^α are defined in $\mathfrak{R}^d$, the function J is also defined in $\mathfrak{R}^d$, with values in the space $\mathfrak{R}^d$.

Proposition 2.2. *Let s be an integer number, $s > d/2 + 3$ and $v \in H^s(\mathfrak{R}^{d+1})$. Assume the following conditions are satisfied:*

(i) *for all $\alpha,\beta = 1,\dots,d$, the matrix valued functions $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$, have C^∞-entries, which are bounded as well as their derivatives; moreover, $E^{\alpha,\beta} = E^{\beta,\alpha}$; $E^{d,d}$ as well as $S(0)$ and $T^\alpha(0)$ are null matrices;*

(ii) *$J \in C^\infty(\mathfrak{R}^d)$, and $J(0)$ is the null matrix;*

(iii) *$h \in H^{s+1}(\overline{\Omega})$, $k \in H^s(\overline{\Omega})$;*

(iv) *there exists a positive real constant λ such that for every $\varphi \in L^2(\mathfrak{R}^{d+1}) \cap C^2(\overline{\Omega} \times [0,T[)$, with $\operatorname{supp}\|\varphi\|$ compact subset of $\overline{\Omega} \times [0,T[$*

$$\|\partial_t\varphi(0)\|_{H^{-s}}^2 + \|\varphi(0)\|_{H^{-s}}^2 + \|\varphi\|_{H^{-s}}^2 \le \lambda \left\|\overline{Q}_v^*\varphi\right\|_{H^{-s}}^2; \tag{2.14}$$

(v) *the initial values h and k satisfy compatibility conditions as in (iv) of the previous proposition.*

Then the initial-boundary value problem (2.13) admits a solution $u \in H^s(\mathfrak{R}^{d+1})$.

Proof. Thanks to Moser estimates [3], the functions $S(v)$ and $T^\alpha(v)$ belong to the space $H^s(\mathfrak{R}^{d+1})$, for $v \in H^s(\mathfrak{R}^{d+1})$, $S, T^\alpha \in C^\infty$, $S(0)$ and $T^\alpha(0)$ are null matrices. As a consequence, $(S(v(\cdot,0))h + \sum_{\alpha=1}^{d} T^\alpha(v(\cdot,0))\partial_\alpha h) \in H^s(\overline{\Omega})$. Therefore, by defining a linear functional L on the space $Q_v{}^*\ (E)$, as in the proof of Proposition 2.1, one can prove the existence of a solution $u \in H^s(\mathfrak{R}^{d+1})$ to the mixed problem (2.13). □

The existence result of Proposition 2.2 allows us to prove the existence of the solution to problem (1.2).

Theorem 2.3. *Let s be an integer number, with $s > d/2 + 3$ and assume that the second-order quasi-linear operator Q (1.1) satisfies the following conditions:*

(i) *for all $\alpha, \beta = 1, \dots, d$, the matrix-valued functions S, T^α, $E^{\alpha,\beta}$, F^α, G, have C^∞-entries, which are bounded as well as their derivatives, moreover, $E^{\alpha,\beta} = E^{\beta,\alpha}$, $E^{d,d}$ as well as $E^{\alpha,\beta}(0,0), F^\alpha(0,0), G(0,0)$, $S(0)$ and $T^\alpha(0)$ are null matrices;*

(ii) *$J \in C^\infty(\mathfrak{R}^d)$, and $J(0)$ is the null matrix, in addition, the function J has bounded derivatives;*

(iii) *$h \in H^{s+1}(\overline{\Omega})$; $k \in H^s(\overline{\Omega})$;*

(iv) *the initial values h and k satisfy compatibility conditions as in (iv) of Proposition 2.1;*

(v) *if $v \in H^s(\mathfrak{R}^{d+1})$, there exists a positive real constant λ such that for every $\varphi \in L^2(\mathfrak{R}^{d+1}) \cap C^2(\overline{\Omega} \times [0,T[)$, with $\operatorname{supp} \|\varphi\|$ compact subset of $\overline{\Omega} \times [0,T[$*

$$\left\|\partial_t\varphi(0)\right\|_{H^{-s}}^2 + \left\|\varphi(0)\right\|_{H^{-s}}^2 + \left\|\varphi\right\|_{H^{-s}}^2 \le \lambda \left\|\overline{Q}_v^*\varphi\right\|_{H^{-s}}^2; \tag{2.15}$$

(vi) *there exists a positive constant Γ such that for any $T > 0$ and $v \in H^s(\mathfrak{R}^{d+1})$ it holds true that for every $\varphi \in H^s(\mathfrak{R}^{d+1})$*

$$\|\varphi\|_{H^s(\overline{\Omega}\times[0,T])}^2 \le \Gamma T \left\|Q_v\varphi\right\|_{H^{s-2}(\overline{\Omega}\times[0,T])}^2 + \left\|\varphi(0)\right\|_{H^{s-1}(\overline{\Omega})}^2 + \left\|\partial_t\varphi(0)\right\|_{H^{s-1}(\overline{\Omega})}^2. \tag{2.16}$$

Then there exist $\widetilde{T} > 0$ and a function $u \in H^s(\overline{\Omega} \times [0,\widetilde{T}])$ which provides a classical solution, to the initial-boundary value problem (1.2), *with initial data $u(x,0) = h(x)$ and $\partial_t u(x,0) = k(x)$. Moreover, the solution turns out to be unique.*

Proof. According to the notation introduced above, we denote by Q_{n-1} the second-order operator $Q_{u^{n-1}}$

$$\begin{aligned} Q_{n-1} &= \partial_t^2 + S\left(u^{n-1}\right)\partial_t + \sum_{\alpha=1}^{d} T^\alpha\left(u^{n-1}\right)\partial_t\partial_\alpha \\ &\quad + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}\left(u^{n-1}, \nabla_x u^{n-1}\right)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} F^\alpha\left(u^{n-1}, \nabla_x u^{n-1}\right)\partial_\alpha + G\left(u^{n-1}, \nabla_x u^{n-1}\right). \end{aligned} \tag{2.17}$$

Let us address the study of the well posedness of the initial-boundary value problem (1.2), by means of an iteration scheme, which consists of the following sequence of mixed problems:

$$\begin{gathered}
Q_{n-1}u^n = J\left(u^{n-1}\right), \quad x \in \Omega,\ t > 0, \\
2\sum_{\beta=1}^{d-1} E^{d,\beta}\left(u^{n-1}, \nabla_x u^{n-1}\right)\partial_\beta u^n + T^d\left(u^{n-1}\right)\partial_t u^n + F^d\left(u^{n-1}, \nabla_x u^{n-1}\right)u^n = 0, \quad x \in \partial\Omega,\ t > 0, \\
u^n(x,0) = h(x), \quad x \in \Omega, \\
\partial_t u^n(x,0) = k(x), \quad x \in \Omega.
\end{gathered} \tag{2.18}$$

Let δ be a positive real number, such that $\|h\|_{H^s(\overline{\Omega})} \leq \delta/4$ and $\|k\|_{H^s(\overline{\Omega})} \leq \delta/4$. Let u^0 be a function in the space $H^s(\mathfrak{R}^{d+1})$ whose norm satisfies $\|u^0\|_{H^s} \leq \delta$. We initialize the scheme by means of the function u^0. Due to the result of Proposition 2.2, since $u^{n-1} \in H^s(\mathfrak{R}^{d+1})$ for all $n \in N$, the mixed problem (2.18) admits a solution $u^n \in H^s(\mathfrak{R}^{d+1})$. This solution turns out to be unique, for the operator Q_{n-1} satisfies condition (vi).

We will prove that the sequence of functions $(u^n)_{n\in IN}$ is convergent in the space $H^s(\overline{\Omega} \times [0,T])$, to a function u, which provides the solution of the mixed problem (1.2). The result can be achieved similarly to the existence result of Theorem 3.1 in [2]. For the sake of completeness, let us outline the proof.

At first, let us prove by induction that for all $n \in \mathbb{N}$, $\|u^n\|_{H^s} \leq \delta$. In accordance with the choice of the function u^0, we have $\|u^0\|_{H^s} \leq \delta$. Assume that for all $n < m$, $\|u^n\|_{H^s} \leq \delta$, and prove that, as long as T is small enough, the estimate is true in the case where $n \geq m$. Since $s > d/2 + 3$ and the functions $u^n \in H^s$, there exists a positive constant μ, which depends on δ, so that for every $\alpha, \beta = 1, \ldots, d$, for all $n < m$

$$\begin{gathered}
\|u^n\|_{L^\infty(\mathfrak{R}^{d+1})} \leq \mu, \qquad \|\nabla_x u^n\|_{L^\infty(\mathfrak{R}^{d+1})} \leq \mu, \qquad \left\|\partial_\alpha\partial_\beta u^n\right\|_{L^\infty(\mathfrak{R}^{d+1})} \leq \mu, \\
\|\partial_t u^n\|_{L^\infty(\mathfrak{R}^{d+1})} \leq \mu, \qquad \|\partial_t\partial_\alpha u^n\|_{L^\infty(\mathfrak{R}^{d+1})} \leq \mu.
\end{gathered} \tag{2.19}$$

Thanks to Moser estimates (see [3]) and to the conditions satisfied by the matrices $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$, for all $\alpha, \beta = 1, \ldots, d$, we obtain

$$\|Q_{m-1}u^m\|_{H^{s-2}(\mathfrak{R}^{(d+1)})} \leq \zeta(s,\mu)\|u^m\|_{H^s(\mathfrak{R}^{(d+1)})} + \rho(s,\mu)\|\partial_t u^m\|_{H^{s-1}(\mathfrak{R}^{d+1})}, \tag{2.20}$$

where $\zeta(s,\mu)$ and $\rho(s,\mu)$ are suitable positive constants that depend on s, μ and on the matrices $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$. Thus,

$$\|Q_{m-1}u^m\|_{H^{s-2}(\overline{\Omega}\times[0,T])} \leq \zeta(s,\mu)\|u^m\|_{H^s(\overline{\Omega}\times[0,T])} + \rho(s,\mu)\|\partial_t u^m\|_{H^{s-1}(\overline{\Omega}\times[0,T])}. \tag{2.21}$$

Because of (vi), we deduce

$$\|u^m\|^2_{H^s(\overline{\Omega}\times[0,T])} \leq \Gamma T\Big(\zeta(s,\mu)\|u^m\|_{H^s(\overline{\Omega}\times[0,T])}+\rho(s,\mu)\|\partial_t u^m\|_{H^{s-1}(\overline{\Omega}\times[0,T])}\Big)^2+\|h\|_{H^s(\overline{\Omega})}+\|k\|_{H^s(\overline{\Omega})}$$
$$\leq 2\Gamma T\max\Big(\zeta^2(s,\mu),\rho^2(s,\mu)\Big)\|u^m\|^2_{H^s(\overline{\Omega}\times[0,T])}+\frac{\delta^2}{4}. \tag{2.22}$$

If $T < 1/(4\Gamma\max(\zeta^2(s,\mu),\rho^2\ (s,\mu)))$, then $\|u^m\|^2_{H^s(\overline{\Omega}\times[0,T])} \leq \delta^2$.

As a consequence of the previous estimate, by induction, we get for all $n \in \mathbb{N}$, $\|u^n\|_{H^s(\overline{\Omega}\times[0,T])} \leq \delta$.

The subsequent step of the proof consists in proving that the sequence $(u^n)_{n\in\mathbb{N}}$ is convergent in the space $H^s(\overline{\Omega}\times[0,T])$.

Let W^n be the difference $u^n - u^{n-1}$, for every $n \in \mathbb{N}$; hence,

$$\begin{aligned} Q_{n-1}W^n &= Q_{n-1}u^n - Q_{n-1}u^{n-1} = J\Big(u^{n-1}\Big) - Q_{n-1}u^{n-1} \pm Q_{n-2}u^{n-1} \\ &= (Q_{n-2} - Q_{n-1})u^{n-1} + J\Big(u^{n-1}\Big) - J\Big(u^{n-2}\Big). \end{aligned} \tag{2.23}$$

Thanks to the mean value theorem,

$$\Big\|\Big(S\Big(u^{n-2}\Big)-S\Big(u^{n-1}\Big)\Big)\partial_t u^{n-1}\Big\|_{H^{s-1}} \leq \text{const }(\delta)\Big\|W^{n-1}\Big\|_{H^s}. \tag{2.24}$$

Similar estimates can be computed for the other terms. Moreover,

$$\Big\|J\Big(u^{n-2}\Big)-J\Big(u^{n-1}\Big)\Big\|_{H^s} \leq \Big\|DJ(\zeta)\cdot W^{n-1}\Big\|_{H^s} \leq \text{const }\Big\|W^{n-1}\Big\|_{H^s} \tag{2.25}$$

Thus, for all $n \in \mathbb{N}$,

$$\|Q_{n-1}W^n\|_{H^{s-2}} \leq \text{const }(\delta)\Big\|W^{n-1}\Big\|_{H^s}. \tag{2.26}$$

Let us notice that if $Q_{n-1}W^n$ was the null function, then the function u^{n-1}, would provide a solution to the mixed problem (1.2) and the proof of the theorem would be completed. In fact, we would obtain $Q_{n-1}u^{n-1} = J(u^{n-1})$ and because of the initial conditions $W^n(x,0) = 0$, $\partial_t W^n(x,0) = 0$, and the estimate (2.16), we would have $\|W^n\|^2_{H^s(\overline{\Omega}\times[0,T])} \leq \Gamma T\|Q_{n-1}W^n\|^2_{H^{s-2}(\overline{\Omega}\times[0,T])}$. Thus $u^n \equiv u^{n-1}$.

If we assume that $Q_{n-1}W^n$ does not vanish identically, thanks to (vi), we obtain

$$\|W^n\|^2_{H^s(\overline{\Omega}\times[0,T])} \leq \Gamma T\Big(\text{const }(\delta)^2\Big\|W^{n-1}\Big\|^2_{H^s(\overline{\Omega}\times[0,T])}\Big). \tag{2.27}$$

The constant in (2.27) is independent of n; hence, provided that T is sufficiently small, the sequence of functions $(u^n)_{n\in\mathbb{N}}$ turns out to be a Cauchy sequence in the space $H^s(\overline{\Omega} \times [0,T])$. Let u be the limit of $(u^n)_{n\in\mathbb{N}}$. Therefore, the limit function u belongs to the space $C^2(\overline{\Omega}\times[0,T])$.

The study of the convergence of the iteration scheme (2.18) is the last step of the proof.

The sequence $(u^n)_{n\in\mathbb{N}}$ is convergent to the function u in $H^s(\overline{\Omega}\times[0,T])$, with $s > d/2 + 3$, thus as $n \to \infty$, the sequence $(Q_{n-1}u^n(x,t))_n$ turns out to be convergent to $Qu(x,t)$, for every $(x,t) \in \Omega \times (0,T)$. Therefore, thanks to Sobolev embedding, the function u provides a classical solution to system (1.2), $Qu = J$, in $\Omega \times (0,T)$.

Furthermore, by continuity of the trace operator, the solution u turns out to satisfy the boundary condition: $2\sum_{\beta=1}^{d-1} E^{d,\beta}(u,\nabla_x u)\partial_\beta u + T^d(u)\partial_t u + F^d(u,\nabla_x u)u = 0$, where $x \in \partial\Omega$, $t \in [0,T]$.

As far as the initial datum is concerned, we get $u(x,0) = h$ and $\partial_t u(x,0) = k(x)$, as $x \in \Omega$.

By means of the same estimates obtained for the sequence of functions $(W^n)_{n\in\mathbb{N}}$, we can prove the uniqueness, in the space $H^s(\overline{\Omega} \times [0,T])$, of the solution u to the problem (1.2), with initial conditions $u(x,0) = h(x)$ and $\partial_t u(x,0) = k(x)$, $x \in \Omega$. □

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