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Research Article

Oscillation for Certain Nonlinear Neutral Partial Differential Equations

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We present some new oscillation criteria for second-order neutral partial functional differential equations of the form $(\partial/\partial t)\{p(t)(\partial/\partial t)[u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)]\} = a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u(x,t-\rho_k(t))-q(x,t)f(u(x,t))-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j))$, $(x,t)\in\Omega\times R^+\equiv G$, where Ω is a bounded domain in the Euclidean N-space R^N with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in R^N. Our results improve some known results and show that the oscillation of some second-order linear ordinary differential equations implies the oscillation of relevant nonlinear neutral partial functional differential equations.

1. Introduction

In this paper, we consider the oscillatory behavior of solutions to the neutral partial functional differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{p(t)\frac{\partial}{\partial t}\left[u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right]\right\} \\
&\quad= a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u\big(x,t-\rho_k(t)\big)-q(x,t)f(u(x,t)) \\
&\qquad-\sum_{j=1}^{m}q_j(x,t)f_j\big(u\big(x,t-\sigma_j\big)\big),\quad (x,t)\in\Omega\times R^+\equiv G,
\end{aligned}\tag{1.1}$$

with the boundary condition

$$\frac{\partial u(x,t)}{\partial \nu}+g(x,t)u(x,t)=0,\quad (x,t)\in\partial\Omega\times R^+\equiv G \tag{1.2}$$

or

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R^+ \equiv G, \tag{1.3}$$

where Δ is the Laplacian in Euclidean N-space R^N, $R^+ := (0,+\infty)$, Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, ν denotes the unit exterior normal vector to $\partial\Omega$, and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times R^+$.

Throughout this paper we assume that the following conditions hold:

(C_1) $p \in C^1(R^+, R^+)$, $\int_{t_0}^{\infty}(1/p(s))ds = \infty$, $t_0 > 0$;

(C_2) $\lambda_i \in C^2(R^+, R^+)$, $0 \le \sum_{i=1}^{l} \lambda_i \le 1$, and the numbers τ_i are nonnegative real constants for $i \in I_l = \{1,2,\dots,l\}$;

(C_3) $q, q_j \in C(\overline{G}, R^+)$, $q(t) = \min_{x\in\overline{\Omega}} q(x,t)$, and $q_j(t) = \min_{x\in\overline{\Omega}} q_j(x,t)$, $j \in I_m = \{1,2,\dots,m\}$;

(C_4) $a, a_k \in C(R^+, R^+)$, $\rho_k \in C(R^+, R^+)$, $\lim_{t\to\infty}(t - \rho_k(t)) = \infty$, $k \in I_s = \{1,2,\dots,s\}$, and $\sigma_j (j \in I_m)$ are nonnegative constants;

(C_5) $f, f_j \in C(R,R)$ are convex in R^+ with $f(u)/u \ge \alpha > 0$, $f_j(u)/u \ge \alpha_j > 0$ for $u \ne 0$, where α and α_j are positive constants for $j \in I_m$.

We refer to these five conditions collectively as condition (C).

A function $u \in C^2(G) \cup C^1(\overline{G})$ is called a solution of the problem (1.1), (1.2) (or (1.1), (1.3)), if it satisfies (1.1) in the domain G and the corresponding boundary condition. A solution u of the problem (1.1), (1.2) (or (1.1), (1.3)) is called oscillatory in the domain G if for each positive number b there exists a point $(x_0, t_0) \in \Omega \times [b, \infty)$ such that $u(x_0, t_0) = 0$.

The theory of partial differential equations with deviating arguments has received much attention (see [1]). We mention here [1–7] concerning oscillatory properties of solutions to some parabolic equations and some hyperbolic equations with deviating arguments.

By considering the function $H(t,s)$, in 1999 Li and Cui [4] obtained some oscillation criteria for solutions of the problems (1.1), (1.2) and (1.1), (1.3). One of the theorems in [4] is as follows.

Theorem 1.1. *Set $D = \{(t,s) : t \ge s \ge t_0\}$. Let $H \in (D; R)$ satisfy the following conditions:*

(i) *$H(t,t) = 0$ for $t \ge t_0$; $H(t,s) > 0$ for $t \ge s \ge t_0$;*

(ii) *H has a continuous and nonpositive partial derivative on D with respect to the second variable.*

(iii) *$h : D \to R$ is a continuous function with*

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \forall (t,s) \in D. \tag{1.4}$$

If there exists a function $\phi \in C^1[t_0, \infty)$ and there exists some $j_0 \in I_m$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\varphi(s) - \frac{1}{4}\Phi(s)p(s-\sigma_{j_0})h^2(t,s) \right] ds = \infty, \tag{1.5}$$

where $\Phi(s) = e^{-2\int^s \phi(\xi)d\xi}$ *and*

$$\psi(t) = \Phi(t)\left\{\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right] + p(t-\sigma_{j_0})\phi^2(t) - \left[p(t-\sigma_{j_0})\phi(t)\right]'\right\}, \tag{1.6}$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

In this paper, we shall establish some new oscillation results for solutions of the problems (1.1), (1.2) and (1.1), (1.3). Our results are extensive version of Theorem 1.1. Meanwhile, our results show that the oscillation of some second-order linear ordinary differential equations implies the oscillation of relevant nonlinear second-order neutral partial functional differential (1.1), thus we can obtain some new oscillation theorems for (1.1), which do not need the condition of the integrals of the coefficient.

2. Main Results

Theorem 2.1. *Let condition (C) hold, and* $\phi \in C^1[t_0,\infty)(t_0 > 0)$. *Assume that there exists* $j_0 \in I_m$ *such that the inequality*

$$W'(t) + \psi(t) + \frac{W^2(t)}{p(t-\sigma_{j_0})\Phi(t)} \le 0 \tag{2.1}$$

has no eventually positive solution, where $\Phi(s) = \exp\{-2\int^s \phi(\xi)d\xi\}$ *and*

$$\psi(t) = \Phi(t)\left\{\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right] + p(t-\sigma_{j_0})\phi^2(t) - \left[p(t-\sigma_{j_0})\phi(t)\right]'\right\}, \tag{2.2}$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.2) which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0$, $u(x,t-\tau_i) > 0$, $u(x,t-\rho_k(t)) > 0$, and $u(x,t-\sigma_j) > 0$ in $\Omega \times [t_1,\infty)$, $t_1 \ge t_0$, $i \in I_l$, $k \in I_s$, $j \in I_m$.

Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{p(t)\frac{d}{dt}\left[\int_\Omega u(x,t)dx + \sum_{i=1}^{l}\lambda_i(t)\int_\Omega u(x,t-\tau_i)dx\right]\right\} \\
&\quad = a(t)\int_\Omega \Delta u(x,t)dx + \sum_{k=1}^{s}a_k(t)\int_\Omega \Delta u(x,t-\rho_k(t))dx \\
&\qquad - \int_\Omega q(x,t)f(u(x,t))dx - \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))dx, \quad t \ge t_1.
\end{aligned} \tag{2.3}$$

From Green's formula and boundary condition (1.2), it follows that

$$\int_{\Omega} \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial \nu} dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \le 0, \quad t \ge t_1, \tag{2.4}$$

$$\begin{aligned}\int_{\Omega} \Delta u(x,t-\rho_k(t))dx &= \int_{\partial\Omega} \frac{\partial u(x,t-\rho_k(t))}{\partial \nu} dS \\ &= -\int_{\partial\Omega} g(x,t-\rho_k(t))u(x,t-\rho_k(t))dS \le 0, \quad t \ge t_1,\ k \in I_s,\end{aligned} \tag{2.5}$$

where dS is the surface element on $\partial\Omega$. Moreover, from (C_3), (C_5), and Jensen's inequality it follows that

$$\begin{aligned}\int_{\Omega} q(x,t)f(u(x,t))dx &\ge q(t)\int_{\Omega} f(u(x,t))dx \\ &\ge q(t)\int_{\partial\Omega} dx f\left(\int_{\Omega} u(x,t)dx\left(\int_{\Omega} dx\right)^{-1}\right), \quad t \ge t_1,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\int_{\Omega} q_j(x,t)f_j(u(x,t-\sigma_j))dx &\ge q_j(t)\int_{\Omega} f_j(u(x,t-\sigma_j))dx \\ &\ge q_j(t)\int_{\partial\Omega} dx f_j\left(\int_{\Omega} u(x,t-\sigma_j)dx\left(\int_{\Omega} dx\right)^{-1}\right), \quad t \ge t_1.\end{aligned} \tag{2.7}$$

Set

$$V_1(t) = \int_{\Omega} u(x,t)dx\left(\int_{\Omega} dx\right)^{-1}, \quad t \ge t_1. \tag{2.8}$$

In view of (2.4)–(2.8), (2.3) yields

$$\frac{d}{dt}\left\{p(t)\frac{d}{dt}\left[V_1(t)+\sum_{i=1}^{l}\lambda_i(t)V_1(t-\tau_i)\right]\right\}+q(t)f(V_1(t)) + \sum_{j=1}^{m} q_j(t)f_j(V_1(t-\sigma_j)) \le 0, \quad t \ge t_1. \tag{2.9}$$

Let $Z(t) = V_1(t) + \sum_{i=1}^{l}\lambda_i(t)V_1(t-\tau_i)$. We have $Z(t) > 0$ and $[p(t)Z'(t)]' < 0$ for $t \ge t_1$. Hence $p(t)Z'(t)$ is a decreasing function in the interval $[t_1,\infty)$. We can claim that $p(t)Z'(t) > 0$ for $t \ge t_1$. In fact, if $p(t)Z'(t) \le 0$ for $t \ge t_1$, then there exists a $T \ge t_1$ such that $p(T)Z'(T) < 0$. This implies that

$$Z'(t) \leq \frac{p(T)Z'(T)}{p(t)} \quad \text{for } t \geq T,$$
$$Z(t) - Z(T) \leq p(T)Z'(T)\int_T^t \frac{1}{p(s)}ds, \quad t \geq T. \tag{2.10}$$

Therefore $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts the fact that $Z(t) > 0$.

From(2.9), for the j_0 in (2.1) we obtain

$$[p(t)Z'(t)]' + q_{j_0}(t)f_{j_0}(V_1(t-\sigma_{j_0})) \leq 0, \quad t \geq t_1. \tag{2.11}$$

Noting condition (C_5), from (2.11) we have

$$[p(t)Z'(t)]' + \alpha_{j_0}q_{j_0}(t)V_1(t-\sigma_{j_0}) \leq 0, \quad t \geq t_1 \tag{2.12}$$

or

$$[p(t)Z'(t)]' + \alpha_{j_0}q_{j_0}(t)\left[1 - \sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]Z(t-\sigma_{j_0}) \leq 0, \quad t \geq t_1. \tag{2.13}$$

Let

$$W(t) = \Phi(t)\left[\frac{p(t)Z'(t)}{Z(t-\sigma_{j_0})} + p(t-\sigma_{j_0})\phi(t)\right]; \tag{2.14}$$

we have

$$W'(t) \leq -2\phi(t)W(t) + \Phi(t)\left\{-\alpha_{j_0}q_{j_0}(t)\left[1 - \sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]\right.$$
$$\left. - \frac{p(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})} + [p(t-\sigma_{j_0})\phi(t)]'\right\}. \tag{2.15}$$

Using the fact that $p(t)Z'(t)$ is decreasing, we get

$$p(t)Z'(t) \leq p(t-\sigma_{j_0})Z'(t-\sigma_{j_0}), \quad \text{for } t \geq t_1. \tag{2.16}$$

Thus

$$\begin{aligned} W'(t) &\leq -2\phi(t)W(t) \\ &\quad + \Phi(t)\left\{ -\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\right] \right. \\ &\qquad \left. - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{p(t)Z'(t)}{Z(t-\sigma_{j_0})}\right)^2 + \left[p(t-\sigma_{j_0})\phi(t)\right]' \right\} \\ &= -2\phi(t)W(t) + \Phi(t) \\ &\quad \times \left\{ -\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\right] \right. \\ &\qquad \left. - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{W(t)}{\Phi(t)} - p(t-\sigma_{j_0})\phi(t)\right)^2 + \left[p(t-\sigma_{j_0})\phi(t)\right]' \right\} \\ &= -\psi(t) - \frac{W^2(t)}{p(t-\sigma_{j_0})\Phi(t)}, \end{aligned} \tag{2.17}$$

that is, $W(t)$ is a positive solution of (2.1), which contradicts the assumption. This completes the proof of Theorem 2.1. □

In order to study oscillation of the problem (1.1) and (1.3), the following fact will be used (see [2]). The smallest eigenvalue η_0 of the Dirichlet problem

$$\begin{gathered} \Delta u(x) + \eta u(x) = 0, \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{2.18}$$

is positive, and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Theorem 2.2. *Let all conditions in Theorem 2.1 hold, then every solution $u(x,t)$ of the problem* (1.1), (1.3) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.3) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we may assume that $u(x,t) > 0$, $u(x, t-\tau_i) > 0$, $u(x, t-\rho_k(t)) > 0$, and $u(x, t-\sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t_1 \geq t_0$, $i \in I_l$, $k \in I_s, j \in I_m$.

Multiplying both sides of (1.1) by $\varphi(x) > 0$ and integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left\{ p(t)\frac{d}{dt}\left[\int_\Omega u(x,t)\varphi(x)dx + \sum_{i=1}^{l}\lambda_i(t)\int_\Omega u(x,t-\tau_i)\varphi(x)dx\right]\right\} \\ &\quad = a(t)\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,t-\rho_k(t))\varphi(x)dx \\ &\qquad - \int_\Omega q(x,t)f(u(x,t))\varphi(x)dx - \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j)\varphi(x))dx, \quad t \geq t_1. \end{aligned} \tag{2.19}$$

From Green's formula and boundary condition (1.3), it follows that

$$\int_{\Omega} \Delta u(x,t)\varphi(x)dx = \int_{\Omega} u(x,t)\Delta\varphi(x)dx = -\beta_0 \int_{\Omega} u(x,t)\varphi(x)dx \le 0, \quad t \ge t_1, \tag{2.20}$$

$$\begin{aligned}\int_{\Omega} \Delta u(x,t-\rho_k(t))\varphi(x)dx &= \int_{\Omega} u(x,t-\rho_k(t))\Delta\varphi(x)dx \\ &= -\beta_0 \int_{\Omega} u(x,t-\rho_k(t))\varphi(x)dx \le 0, \quad t \ge t_1,\ k \in I_s.\end{aligned} \tag{2.21}$$

Moreover, from (C_3) and (C_5) by Jensen's inequality it follows that

$$\begin{aligned}&\int_{\Omega} q(x,t) f(u(x,t))\varphi(x)\mathrm{d}x \\ &\quad \ge q(t)\int_{\Omega} f(u(x,t))\varphi(x)dx \\ &\quad \ge q(t)\int_{\Omega}\varphi(x)dx f\left(\int_{\Omega} u(x,t)\varphi(x)dx\left(\int_{\Omega}\varphi(x)dx\right)^{-1}\right), \quad t \ge t_1,\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\int_{\Omega} q_j(x,t) f_j(u(x,t-\sigma_j))\varphi(x)dx \\ &\quad \ge q_j(t)\int_{\Omega} f_j(u(x,t-\sigma_j))\varphi(x)dx \\ &\quad \ge q_j(t)\int_{\Omega}\varphi(x)dx f_j\left(\int_{\Omega} u(x,t-\sigma_j)\varphi(x)dx\left(\int_{\Omega}\varphi(x)dx\right)^{-1}\right), \quad t \ge t_1.\end{aligned} \tag{2.23}$$

Set

$$V_2(t) = \int_{\Omega} u(x,t)\varphi(x)dx\left(\int_{\Omega}\varphi(x)dx\right)^{-1}, \quad t \ge t_1. \tag{2.24}$$

In view of (2.20)–(2.24), (2.19) yields

$$\frac{d}{dt}\left\{p(t)\frac{d}{dt}\left[V_2(t)+\sum_{i=1}^{l}\lambda_i(t)V_2(t-\tau_i)\right]\right\}+q(t)f(V_2(t))+\sum_{j=1}^{m}q_j(t)f_j(V_2(t-\sigma_j)) \le 0, \quad t \ge t_1. \tag{2.25}$$

Let $Z(t) = V_2(t) + \sum_{i=1}^{l}\lambda_i(t)V_2(t-\tau_i)$; the remainder of the proof is similar to that of Theorem 2.1, so we omit it. □

Theorem 2.3. *Let the condition (C) hold, and $\phi \in C^1[t_0,\infty)$, $F \in C([t_0,\infty),R)$. Suppose that there exists $j_0 \in I_m$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\Phi(s)F^2(s)\right]e^{\int_{t_0}^{s}F(\tau)d\tau}ds=\infty, \tag{2.26}$$

where $\Phi(s)=e^{-2\int^{s}\phi(\tau)d\tau}$ and $\psi(s)$ is defined as in (2.2)*. Then*

(I) *every solution $u(x,t)$ of the problem* (1.1), (1.2) *is oscillatory in G;*

(II) *every solution $u(x,t)$ of the problem* (1.1), (1.3) *is oscillatory in G.*

Proof. (I) From Theorem 2.1, we only need to prove that (2.1) has no eventually positive solution. Suppose to the contrary that there is a solution $w(t)$ of system (2.1) which has no zero in $[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $w(t)>0$ in $[t_1,\infty)$, $t_1\ge t_0$. Hence for all $t\ge t_1$, we have by (2.1)

$$w'(t)\le-\left[\psi(t)-\frac{1}{4}p(t-\sigma_{j_0})\Phi(t)F^2(t)\right]-\left[\frac{w^2(t)}{p(t-\sigma_{j_0})\Phi(t)}+\frac{1}{4}p(t-\sigma_{j_0})\Phi(t)F^2(t)\right], \tag{2.27}$$

that is,

$$\begin{gathered}w'(t)+F(t)w(t)\le-\left[\psi(t)-\frac{1}{4}p(t-\sigma_{j_0})\Phi(t)F^2(t)\right],\\ w(t)e^{\int_{t_0}^{t}F(\tau)d\tau}-w(T)e^{\int_{t_0}^{T}F(\tau)d\tau}\le-\int_{T}^{t}\left[\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\Phi(s)F^2(s)\right]e^{\int_{t_0}^{s}F(\tau)d\tau}ds.\end{gathered} \tag{2.28}$$

Hence

$$\int_{T}^{t}\left[\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\Phi(s)F^2(s)\right]e^{\int_{t_0}^{s}F(\tau)d\tau}ds\le w(T)e^{\int_{t_0}^{T}F(\tau)d\tau}-w(t)e^{\int_{t_0}^{t}F(\tau)d\tau}. \tag{2.29}$$

In view of $w(t)\ge 0$, we get

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\Phi(s)F^2(s)\right]e^{\int_{t_0}^{s}F(\tau)d\tau}ds\le w(T)e^{\int_{t_0}^{T}F(\tau)d\tau}, \tag{2.30}$$

which contradicts assumption (2.26). Hence, (2.1) has no eventually positive solution. By Theorem 2.1, every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G.

(II) According to Theorem 2.2, the remainder of the proof is similar to that of the proof of part (I), so we omit the details. The proof of Theorem 2.3 is complete. □

Set $D=\{(t,s):t\ge s\ge t_0\}$. Let $H\in C(D,R)$ satisfy the following conditions:

(i) $H(t,t)=0$, for $t\ge t_0$, $H(t,s)>0$ for $t>s\ge t_0$;

(ii) H has a continuous and nonpositive partial derivative on D with respect to the second variable;

(iii) $h : D \to R$ is a continuous function with

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)}, \quad \forall (t,s) \in D. \tag{2.31}$$

Taking $F(s) = (\partial H(t,s)/\partial s)/H(t,s)$, we have the following Philo's type theorem in [8].

Theorem 2.4. *Let the condition (C) hold, and $\phi \in C^1[t_0,\infty)$. Suppose that there exists $j_0 \in I_m$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\varphi(s) - \frac{1}{4}\Phi(s)p(s-\sigma_{j_0})h^2(t,s)\right] ds = \infty, \tag{2.32}$$

where $\Phi(s) = e^{-2\int^s \phi(\tau)d\tau}$ and $\Phi(s)$ is defined as in (2.2). *Then*

(I) *every solution $u(x,t)$ of the problem* (1.1), (1.2) *is oscillatory in G;*

(II) *every solution $u(x,t)$ of the problem* (1.1), (1.3) *is oscillatory in G.*

Remark 2.5. We can establish a lot of oscillation criteria from Theorem 2.3 if we choose differential ϕ and F. For example, taking $\phi = 0$, $F = 0$, Theorem 2.3 reduces to a Grammatikopoulos's type criteria in [9].

Next we present another oscillation theorem.

Theorem 2.6. *Let the condition (C) hold. Suppose that there exists $j_0 \in I_m$ such that the following ordinary differential equation*

$$y'' + Q(t)y(t) = 0 \tag{2.33}$$

is oscillatory, where

$$Q(t) = \frac{1}{p(t-\sigma_{j_0})}\left\{\alpha_{j_0}q_{j_0}(t)\left[1 - \sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right] + \frac{[p'(t-\sigma_{j_0})]^2}{4p(t-\sigma_{j_0})} - \frac{p''(t-\sigma_{j_0})}{2}\right\}, \tag{2.34}$$

then

(I) *every solution $u(x,t)$ of the problem* (1.1), (1.2) *is oscillatory in G;*

(II) *every solution $u(x,t)$ of the problem* (1.1), (1.3) *is oscillatory in G.*

Proof. Let $y(t)$ be a nonoscillatory solution of (2.33) Without loss of generality, we assume that $y(t) > 0$, $t \ge T_0 \ge t_0$. Similar to the proof Theorem 2.3, we can get

$$w'(t) \le -Q(t) - w^2(t), \quad \text{for } t \ge T_0, \tag{2.35}$$

where $Q(t)$ is defined as in (2.34). In fact, taking $\phi(t) = (p'(t-\sigma_{j_0}))/(2p(t-\sigma_{j_0}))$ in Theorem 2.3, we obtain (2.35) from (2.1).

Therefore, from (2.35), by using Theorem 7.2 in [10, Chap. XI], we see that (2.33) is nonoscillatory. This contradicts the fact that (2.33) is oscillatory. The proof of Theorem 2.6 is complete. □

Corollary 2.7. *Let the condition (C) hold. Suppose that there exists* $j_0 \in I_m$ *such that*

$$\infty \geq \lim_{r\to\infty} t^2 \frac{1}{p(t-\sigma_{j_0})}\left\{\alpha_{j_0} q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]+\frac{[p'(t-\sigma_{j_0})]^2}{4p(t-\sigma_{j_0})}-\frac{p''(t-\sigma_{j_0})}{2}\right\} > \frac{1}{4}, \tag{2.36}$$

then

(I) *every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G*;*

(II) *every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillatory in* G*.*

Proof. From Theorem 2.5 and Theorem 7.1 in [10, Chap.XI], it is easy to see that the result of Corollary 2.7 is true. □

Corollary 2.8. *Let the condition (C) hold. Suppose that there exists* $j_0 \in I_m$ *such that*

$$\infty \geq \liminf_{r\to\infty} t\int_t^{\infty} \frac{1}{p(t-\sigma_{j_0})} \times\left\{\alpha_{j_0} q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]+\frac{[p'(t-\sigma_{j_0})]^2}{4p(t-\sigma_{j_0})}-\frac{p''(t-\sigma_{j_0})}{2}\right\}dt > \frac{1}{4}, \tag{2.37}$$

then

(I) *every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G*;*

(II) *every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillatory in* G*.*

Corollary 2.9. *Let condition (C) hold. If there exist* $T > t_0$*,* $\alpha > 3-2\sqrt{2}$*, and* $j_0 \in I_m$ *such that for every* $n \in N$*,*

$$\int_{2^nT}^{2^{n+1}T} \frac{1}{p(t-\sigma_{j_0})}\left\{\alpha_{j_0} q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]+\frac{[p'(t-\sigma_{j_0})]^2}{4p(t-\sigma_{j_0})}-\frac{p''(t-\sigma_{j_0})}{2}\right\}dt > \frac{\alpha}{2^nT}, \tag{2.38}$$

then

(I) *every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G*;*

(II) *every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillatory in* G*.*

Corollary 2.10. *Let the condition (C) hold, $\lambda > 1$, and $\alpha_0 = (\sqrt{\lambda} - 1)^2$. If there exist $T > t_0$, $\alpha > \alpha_0$, and $j_0 \in I_m$ such that for every $n \in N$,*

$$\int_{\lambda^n T}^{\lambda^{n+1}T} \frac{1}{p(t-\sigma_{j_0})} \left\{ \alpha_{j_0} q_{j_0}(t) \left[1 - \sum_{i=1}^{l} \lambda_i (t - \sigma_{j_0}) \right] + \frac{[p'(t-\sigma_{j_0})]^2}{4p(t-\sigma_{j_0})} - \frac{p''(t-\sigma_{j_0})}{2} \right\} dt > \frac{\alpha}{(\lambda - 1)\lambda^n T}, \tag{2.39}$$

then

(I) *every solution $u(x,t)$ of the problem* (1.1), (1.2) *is oscillatory in G;*

(II) *every solution $u(x,t)$ of the problem* (1.1), (1.3) *is oscillatory in G.*

Remark 2.11. Corollaries 2.8–2.10 are easy to be proved by Theorem 2.6 of this paper, Theorems A and 2 of Huang [11], or Theorem 2 of Wong [12]. Corollaries 2.9 and 2.10 are different from the most known ones in the sense that they are based on the information only on a sequence of intervals such as $[2^n T, 2^{n+1} T]$, rather than on the whole half-line $[t_0, \infty)$.

Example 2.12. Let constants $c > 0$ and $\mu > 0$. Consider the partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{ \frac{1}{t+\pi+1} \frac{\partial}{\partial t}\left[u(x,t) + \frac{3}{t+2\pi} u(x,t-2\pi) \right] \right\} \\
&\quad = \frac{1}{t+\pi+1} \Delta u(x,t) \\
&\qquad + \left[\frac{1}{(t+\pi+1)^2} + \frac{6}{(t+\pi+1)(t+2\pi)^2} + \frac{3}{(t+\pi+1)^2(t+2\pi)} \right] \Delta u\left(x, t - \frac{3\pi}{2} \right) \\
&\qquad + \left[\frac{6}{(t+\pi+1)(t+2\pi)^3} + \frac{3}{(t+\pi+1)^2(t+2\pi)^2} \right] \Delta u(x, t-\pi) \\
&\qquad - \left[\frac{3}{(t+\pi+1)(t+2\pi)} + \frac{t+\pi}{t+\pi-3} \frac{\mu}{\ln^2(t+1)} \right] u(x,t) \left[1 + \frac{c}{1+u^2(x,t)} \right] \\
&\qquad - \frac{t+\pi}{t+\pi-3} \frac{\mu}{\ln^2(t+1)} u(x,t-\pi), \quad (x,t) \in (0,\pi) \times R^+ \equiv G,
\end{aligned} \tag{2.40}$$

with the boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0. \tag{2.41}$$

A straightforward verification shows that the functions $q_1(t) = ((t+\pi)/(t+\pi-3))(\mu/\ln^2(t+1))$, $\lambda_1(t-\sigma_1) = \lambda_1(t-\pi) = (3/(t+\pi))$, and $p(t-\sigma_1) = p(t-\pi) = (1/(t+1))$. By simple computation, for constant $\mu > 0$ and for each $t \geq 0$, we have

$$Q(t) = (t+1)\left\{ \frac{\mu}{\ln^2(t+1)} + \frac{1}{4(t+1)^3} - \frac{1}{(t+1)^3} \right\} = \frac{\mu(t+1)}{\ln^2(t+1)} - \frac{3}{4(t+1)^2}. \tag{2.42}$$

Then, for constant $\mu > 0$,

$$\lim_{t\to\infty} t^2 Q(t) = \lim_{t\to\infty}\left[\frac{\mu t^2(t+1)}{\ln^2(t+1)} - \frac{3t^2}{4(t+1)^2}\right] = \infty > \frac{1}{4}. \tag{2.43}$$

Hence, by Corollary 2.7, (2.40) is oscillatory if $\mu > 0$. For example, if $c = 0$, $u(x,t) = \sin x \cos t$ is such a solution. However, criteria in [1–6] fail to imply this fact and in [7] fail to apply to (2.40) when $0 < \mu \leq 1$. In addition, those criteria are quite difficult to apply to get oscillation of all solutions of problem (2.40), (2.41) for $c > 0$.

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