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Research Article

Slip Effects on Fractional Viscoelastic Fluids

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Unsteady flow of an incompressible Maxwell fluid with fractional derivative induced by a sudden moved plate has been studied, where the no-slip assumption between the wall and the fluid is no longer valid. The solutions obtained for the velocity field and shear stress, written in terms of Wright generalized hypergeometric functions ${}_p\Psi_q$, by using discrete Laplace transform of the sequential fractional derivatives, satisfy all imposed initial and boundary conditions. The no-slip contributions, that appeared in the general solutions, as expected, tend to zero when slip parameter is $\theta \to 0$. Furthermore, the solutions for ordinary Maxwell and Newtonian fluids, performing the same motion, are obtained as special cases of general solutions. The solutions for fractional and ordinary Maxwell fluid for no-slip condition also obtained as limiting cases, and they are equivalent to the previously known results. Finally, the influence of the material, slip, and the fractional parameters on the fluid motion as well as a comparison among fractional Maxwell, ordinary Maxwell, and Newtonian fluids is also discussed by graphical illustrations.

1. Introduction

There are many fluids in industry and technology whose behavior cannot be explained by the classical linearly viscous Newtonian model. The departure from the Newtonian behavior manifests itself in a variety of ways: non-Newtonian viscosity (shear thinning or shear thickening), stress relaxation, nonlinear creeping, development of normal stress differences, and yield stress [1]. The Navier-Stokes equations are inadequate to predicted the behavior of such type of fluids; therefore, many constitutive relations of non-Newtonian fluids are proposed [2]. These constitutive relations give rise to the differential equations, which, in general, are more complicated and higher order than the Navier-Stokes equations. Therefore, it is difficult to obtain exact analytical solutions for non-Newtonian fluids [3]. Modeling of the equation governing the behaviors of non-Newtonian fluids in different circumstance is important from many points of view. For examples, plastics and polymers are extensively handeled by the chemical industry, whereas biological and biomedical devices like hemodialyser make use of the rheological behavior [4]. In general, the analysis of the behavior of the fluid motion of non-Newtonian fluids tends to be much more complicated and subtle in comparison with that of the Newtonian fluids [5].

The fractional calculus, almost as old as the standard differential and integral one, is increasingly seen as an efficient tool and subtle frame work within which useful generalization is quite long and arguments almost yearly. It includes fractal media, fractional wave diffusion, fractional Hamiltonian dynamics, and biopolymer dynamics as well as many other topics in physics. Fractional calculus is useful in the field of biorheology and bioengineering, in part, because many tissue-like materials (polymers, gels, emulsions, composites, and suspensions) exhibit power-law responses to an applied stress or strain [6, 7]. An example of such power-law behavior in elastic tissue was observed recently for viscoelastic measurements of the aorta, both in vivo and in vitro [8, 9], and the analysis of these data was most conveniently performed using fractional order viscoelastic models. The starting point of the fractional derivative model of non-Newtonian model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville/Caputo fractional calculus operators. This generalization allows one to define precisely noninteger order integrals or derivatives. In general, fractional model of viscoelastic fluids is derived from well-known ordinary model by replacing the ordinary time derivatives, to fractional order time derivatives and this plays an important role to study the valuable tool of viscoelastic properties. We include here some investigation [10–18] in which the fractional calculus approach has been adopted for the flows of non-Newtonian fluids. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of some polymers in the glass transition and the glass state [19]. In other cases, it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [20]. The use of fractional derivatives within the context of viscoelasticity was firstly proposed by Germant [21]. Later, Bagley and Torvik [22] demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives, and Makris et al. [23] achieved a very good fit of the experimental data when the fractional derivative Maxwell model has been used instead of the Maxwell model for the silicon gel fluid. Furthermore, it is worth pointing out that Palade et al. [24] developed a fully objective constitutive equation for an incompressible fluidreducible to the linear fractional derivative Maxwell model under small deformations hypothesis.

A general view of the literature shows that the slip effects on the flows of non-Newtonian fluids has been given not much attention. Especially, polymer melts exhibit a macroscopic wall slip. The fluids exhibiting a boundary slip are important in technological applications, for example, the polishing of artificial heart valves, rarefied fluid problems, and flow on multiple interfaces. In the study of fluid-solid surface interactions, the concept of slip of a fluid at a solid wall serves to describe macroscopic effects of certain molecular phenomena. When the molecular mean free path length of the fluid is comparable to the distance between the plates as in nanochannels or microchannels, the fluid exhibits non-continuum effects such as slipflow, as demonstrated experimentally by Derek et al. [25]. Experimental observations show that [26–28] non-Newtonian fluids, such as polymer melts, often exhibit macroscopic wall slip, which, in general, is described by a nonlinear and nonmonotone relation between the wall slip velocity and the traction. A more realistic class of slip flows are those in which the magnitude of the shear stress reaches some critical value, here called the slip yield stress, before slip occurs. In fact, some experiments show that the onset slip and slip velocity may also depend on the normal stress at the boundary [26, 29]. Much of the research involving slip presumes that the slip velocity depends on the shear stress. The slip condition is an important factor in sharskin, spurt, and hysteresis effects, but the existing theory for non-Newtonian fluids with wall slippage is scant. We mention here some recent attempt regarding exact analytical solutions of non-Newtonian fluids with slip effects [30–36].

The objective of this paper is twofold. Firstly, is to give few more exact analytical solutions for viscoelastic fluids with fractional derivative approach, which is more natural and appropriate tool to describe the complex behavior of such fluids. Secondly, is to study the slip effects on viscoelastic fluid flows, which is important due to their practical applications. More precisely, our aim is to find the velocity field and the shear stress corresponding to the motion of a Maxwell fluid due to a sudden moved plate, where no-slip assumption is no longer valid. However, for completeness, we will determine exact solutions for a larger class of such fluids. Consequently, motivated by the above remarks, we solve our problem for Maxwell fluids with fractional derivatives. The general solutions are obtained using the discrete Laplace transforms. They are presented in series form in terms of the Wright generalized hypergeometric functions ${}_p\Psi_q$ and presented as sum of the slip contribution and the corresponding no-slip contributions. The similar solutions for ordinary Maxwell fluids can easily be obtained as limiting cases of general solutions. The Newtonian solutions are also obtained as special cases of fractional and ordinary Maxwell fluids. Furthermore, the solutions for fractional and ordinary Maxwell fluid for no-slip condition also obtained as a special cases, and they are similar with previously known results in the literature. Finally, the influence of the material, slip and fractional parameters on the motion of fractional and ordinary Maxwell fluids is underlined by graphical illustrations. The difference among fractional Maxwell, ordinary Maxwell, and Newtonian fluid models is also highlighted.

2. The Differential Equations Governing the Flow

The equations governing the flow of an incompressible fluid include the continuity equation and the momentum equation. In the absence of body forces, they are

$$\nabla \cdot \mathbf{V} = 0, \qquad \nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho (\mathbf{V} \cdot \nabla)\mathbf{V}, \tag{2.1}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity field, t is the time, and ∇ represents the gradient operator. The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is given by [10, 11, 14–17]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\left(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right) = \mu \mathbf{A}, \tag{2.2}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extrastress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericsen tensor, μ is the dynamic viscosity of the fluid, λ is relaxation time, the superscript T indicates the transpose operation, and the superposed dot indicates the material time derivative. The model characterized by the constitutive equations (2.2) contains as special case the Newtonian fluid model for $\lambda \to 0$. For the problem under consideration, we assume a velocity field $\mathbf{V}$ and an extrastress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)\mathbf{i}, \qquad \mathbf{S} = \mathbf{S}(y,t), \tag{2.3}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. For these flows, the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(y,0) = \mathbf{0}, \qquad \mathbf{S}(y,0) = \mathbf{0}, \tag{2.4}$$

and (2.1)–(2.3) yield the meaningful equation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = -\frac{1}{\rho}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u(y,t)}{\partial y^2}, \qquad \left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t) = \mu\frac{\partial u(y,t)}{\partial y}, \tag{2.5}$$

where $\tau(y,t) = S_{xy}(y,t)$ is the nonzero shear stress and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The governing equations corresponding to an incompressible Maxwell fluid with fractional derivatives, performing the same motion in the absence of a pressure gradient in the flow direction, are (cf. [4, 15, 17])

$$(1+\lambda^\alpha D_t^\alpha)\frac{\partial u(y,t)}{\partial t} = \nu\frac{\partial^2 u(y,t)}{\partial y^2}, \qquad (1+\lambda^\alpha D_t^\alpha)\tau(y,t) = \mu\frac{\partial u(y,t)}{\partial y}, \tag{2.6}$$

where α is the fractional parameter and the fractional differential operator so-called Caputo fractional operator D_t^α is defined by [37, 38]

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}d\tau, & 0<\alpha<1, \\ \dfrac{df(t)}{dt}, & \alpha = 1, \end{cases} \tag{2.7}$$

and $\Gamma(\bullet)$ is the Gamma function. In the following, the system of fractional partial differential equations (2.6), with appropriate initial and boundary conditions, will be solved by means of Fourier sine and Laplace transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [10–18].

3. Statement of the Problem

Consider an incompressible Maxwell fluid with fractional derivatives occupying the space lying over an infinitely extended plate which is situated in the (x,z) plane and perpendicular to the y-axis. Initially, the fluid is at rest, and at the moment $t = 0^+$, the plate is impulsively brought to the constant velocity U in its plane. Here, we assume the existence of slip boundary between the velocity of the fluid at the wall $u(0,t)$ and the speed of the wall, and the relative velocity between $u(0,t)$ and the wall is assumed to be proportional to the shear rate at the wall. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form $(2.3)_1$ while the governing equations are given by (2.6). The appropriate initial and boundary conditions are [39]

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0; \quad \tau(y,0) = 0, \quad y > 0, \tag{3.1}$$

$$u(0,t) = UH(t) + \theta H(t) \left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0}; \qquad t \geq 0, \tag{3.2}$$

where $H(t)$ is the Heaviside function and θ is the slip strength or slip coefficient. If $\theta = 0$, then the general assumed no-slip boundary condition is obtained. If θ is finite, fluid slip occurs at the wall, but its effect depends upon the length scale of the flow. Furthermore, the natural conditions

$$u(y,t), \qquad \frac{\partial u(y,t)}{\partial y} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \; t > 0 \tag{3.3}$$

have to be also satisfied. They are consequences of the fact that the fluid is at rest at infinity, and there is no shear in the free stream.

4. Solution of the Problem

4.1. Calculation of the Velocity Field

Applying the Laplace transform to $(2.6)_1$, using the Laplace transform formula for sequential fractional derivatives [37, 38], and taking into account the initial conditions $(3.1)_{1,2}$, we find that

$$\frac{\partial \overline{u}(y,q)}{\partial y^2} - \frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\overline{u}(y,q) = 0, \tag{4.1}$$

subject to the boundary conditions

$$\overline{u}(0,q) = \frac{U}{q} + \theta \left.\frac{\partial \overline{u}(y,q)}{\partial y}\right|_{y=0}, \quad \overline{u}(y,q), \frac{\partial \overline{u}(y,q)}{\partial y} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \tag{4.2}$$

where $\overline{u}(y,q)$ is the image function of $u(y,t)$ and q is a transform parameter. Solving (4.1) and (4.2), we get

$$\overline{u}(y,q) = \frac{U}{q\left\{1 + \theta\left[q(1+\lambda^\alpha q^\alpha)/\nu\right]^{1/2}\right\}} \exp\left\{-\left[\frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\right]^{1/2} y\right\}. \tag{4.3}$$

In order to obtain $u(y,t) = \mathcal{L}^{-1}\{\overline{u}(y,q)\}$ and to avoid the lengthy and burdensome calculations of residues and contours integrals, we apply the discrete inverse Laplace transform method [10–18]. However, for a suitable presentation of the velocity field, we firstly rewrite (4.3) in series form

$$\overline{u}(y,q) = \frac{U}{q} + U\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}\sum_{m=0}^{\infty}\frac{\Gamma(m-(k/2))(-\lambda^{-\alpha})^{m}}{m!\Gamma(-k/2)}\frac{1}{q^{-(\alpha+1)(k/2)+\alpha m+1}}$$
$$+U\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}\sum_{n=0}^{\infty}\frac{\Gamma(n-((k+m)/2))(-\lambda^{-\alpha})^{n}}{n!\Gamma(-(k+m)/2)}\frac{1}{q^{-(\alpha+1)(k+m/2)+\alpha n+1}}, \tag{4.4}$$

where we use the fact that

$$(-1)^{k}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}. \tag{4.5}$$

Inverting (4.4) by means of discrete inverse Laplace transform, we find that

$$u(y,t) = UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}t^{-(\alpha+1)(k/2)}\sum_{m=0}^{\infty}\frac{\Gamma(m-(k/2))(-t^{\alpha}/\lambda^{\alpha})^{m}}{m!\Gamma(-k/2)\Gamma(-(k/2)(\alpha+1)+\alpha n+1)}$$
$$+UH(t)\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}t^{-(\alpha+1)((k+m)/2)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(n-((k+m)/2))(-t^{\alpha}/\lambda^{\alpha})^{n}}{n!\Gamma(-(k+m)/2)\Gamma(-((k+m)/2)(\alpha+1)+\alpha n+1)}. \tag{4.6}$$

In term of Wright generalized hypergeometric function [40], we rewrite the above equation as a simpler form

$$u(y,t) = UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}t^{-(\alpha+1)k/2}{}_{1}\Psi_{2}\left[-\frac{t^{\alpha}}{\lambda^{\alpha}}\Bigg|{}^{(-k/2,1)}_{(-k/2,0),(-(k/2)(\alpha+1)+1,\alpha)}\right]$$
$$+UH(t)\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}t^{-(\alpha+1)(k+m/2)}\,{}_{1}\Psi_{2}\left[-\frac{t^{\alpha}}{\lambda^{\alpha}}\Bigg|{}^{(-(k+m)/2,1)}_{(-(k+m)/2,0),(-((k+m)/2)(\alpha+1)+1,\alpha)}\right], \tag{4.7}$$

where the Wright generalized hypergeometric function ${}_p\Psi_q$ is defined as

$$
{}_p\Psi_q\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]=\sum_{n=0}^{\infty}\frac{(z)^n\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\prod_{j=1}^{q}\Gamma(b_j+B_jn)}. \tag{4.8}
$$

In order to justify the initial conditions $(3.1)_{1,2}$, we use the initial value theorem of Laplace transform [41]

$$
\begin{aligned}
u(y,0)&=\lim_{t\to 0}u(y,t)=\lim_{q\to\infty}\left[q\overline{u}(y,q)\right]=0,\\
\partial_t u(y,0)&=\lim_{t\to 0}\partial_t\, u(y,t)=\lim_{q\to\infty}\left[q^2\overline{u}(y,q)-qu(y,0)\right]=0.
\end{aligned} \tag{4.9}
$$

Furthermore, to justify the boundary condition (3.2), we have

$$
\begin{aligned}
\theta\frac{\partial u(y,t)}{\partial y}\bigg|_{y=0}&=UH(t)\sum_{k=0}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+1}t^{-(\alpha+1)((k+1)/2)}\ {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-(k+1)/2,0),(-((k+1)/2)(\alpha+1)+1,\alpha)}^{(-(k+1)/2,1)}\right]\\
&=UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k}t^{-(\alpha+1)(k/2)}\ {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-(k+1)/2,0),(-((k+1)/2)(\alpha+1)+1,\alpha)}^{(-(k+1)/2,1)}\right],\\
u(0,t)&=UH(t)+UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k}t^{-(\alpha+1)(k/2)}{}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-k/2,0),(-(k/2)(\alpha+1)+1,\alpha)}^{(-(k/2),1)}\right].
\end{aligned} \tag{4.10}
$$

It is easy to see that the exact solution (4.7) satisfies the boundary condition (3.2).

4.2. Calculation of the Shear Stress

Applying the Laplace transform to $(2.6)_2$ and using the initial condition $(3.1)_3$, we find that

$$
\overline{\tau}(y,q)=\frac{\mu}{1+\lambda^\alpha q^\alpha}\frac{\partial\overline{u}(y,q)}{\partial y}, \tag{4.11}
$$

where $\overline{\tau}(y,q)$ is the Laplace transform of $\tau(y,t)$. Using (4.3) in (4.11), we find that

$$
\overline{\tau}(y,q)=-\frac{\mu U\left[q(1+\lambda^\alpha q^\alpha)\right]^{-1/2}}{\sqrt{\nu}\left\{1+\theta\left[q(1+\lambda^\alpha q^\alpha)/\nu\right]^{1/2}\right\}}\exp\left\{-\left[\frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\right]^{1/2}y\right\}, \tag{4.12}
$$

in order to obtain $\tau(y,t)$ under the suitable form, we write (4.12) in series form

$$\overline{\tau}(y,q) = -\frac{\mu U}{\nu}\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1} \times \sum_{n=0}^{\infty}\frac{\Gamma(n-(k+m-1)/2)(-\lambda^{-\alpha})^n}{n!\Gamma(-(k+m-1)/2)}\frac{1}{q^{-(\alpha+1)((k+m-1)/2)+\alpha n}}. \tag{4.13}$$

Inverting (4.13) by means of the discrete inverse Laplace transform, we find the shear stress $\tau(y,t)$ under simple form

$$\tau(y,t) = -\rho U H(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1} t^{-(\alpha+1)((m+k-1)/2)-1} \times \sum_{n=0}^{\infty}\frac{\Gamma(n-(k+m-1)/2)(-t^\alpha/\lambda^\alpha)^n}{n!\Gamma(-(k+m-1)/2)\Gamma(-(\alpha+1)(k+m-1)/2+\alpha n)}, \tag{4.14}$$

or equivalently

$$\tau(y,t) = -\rho U H(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1} t^{-(\alpha+1)((m+k-1)/2)-1} \times {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\middle|{}^{(-(k+m-1)/2,1)}_{(-(k+m-1)/2,0),(-((k+m-1)/2)(\alpha+1),\alpha)}\right]. \tag{4.15}$$

5. The Special Cases

5.1. Ordinary Maxwell Fluid with Slip Effects

Making $\alpha \to 1$ into (4.7) and (4.15), we obtain the velocity field

$$\begin{aligned} u(y,t) &= UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda}{\nu}}\right)^k t^{-k}\ {}_1\Psi_2\left[-\frac{t}{\lambda}\middle|{}^{(-k/2,1)}_{(-k/2,0),(-k+1,1)}\right], \\ &+ UH(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=1}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda}{\nu}}\right)^{k+m} t^{-(k+m)}\ {}_1\Psi_2\left[-\frac{t}{\lambda}\middle|{}^{(-(k+m)/2,1)}_{(-(k+m)/2,0),(-(k+m)+1,1)}\right], \end{aligned} \tag{5.1}$$

and the associated shear stress

$$\tau(y,t) = -\rho U H(t) \sum_{k=0}^{\infty} \theta^k \sum_{m=0}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda}{\nu}} \right)^{k+m} t^{-(k+m)} \; {}_1\Psi_2 \left[-\frac{t}{\lambda} \middle|_{(-(k+m-1)/2,0),(-(k+m)+1,1)}^{(-(k+m-1)/2,1)} \right], \quad (5.2)$$

corresponding to an ordinary Maxwell fluid performing the same motion.

5.2. *Fractional Maxwell Fluid without Slip Effects*

Making $\theta \to 0$ into (4.7) and (4.15), we obtain the solutions for velocity field

$$u(y,t) = UH(t) + UH(t) \sum_{m=1}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda^\alpha}{\nu}} \right)^{m} t^{-(\alpha+1)(m/2)} \; {}_1\Psi_2 \left[-\frac{t^\alpha}{\lambda^\alpha} \middle|_{(-m/2,0),(-(m/2)(\alpha+1)+1,\alpha)}^{(-m/2,1)} \right] \quad (5.3)$$

and the associated shear stress

$$\tau(y,t) = -\rho U H(t) \sum_{m=0}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda^\alpha}{\nu}} \right)^{m-1} t^{-(\alpha+1)((m-1)/2)-1} \; {}_1\Psi_2 \left[-\frac{t^\alpha}{\lambda^\alpha} \middle|_{(-(m-1)/2,0),(-((m-1)/2)(\alpha+1),\alpha)}^{(-(m-1)/2,1)} \right], \quad (5.4)$$

they are equivalent to the known solutions obtained in [42, 43] for Sokes' first problem of fractional Maxwell fluid.

5.3. *Ordinary Maxwell Fluid without Slip Effects*

Making $\alpha \to 1$ into (5.3) and (5.4), we recover the solutions for velocity field shear stress for Stokes' first problem of ordinary Maxwell fluid.

5.4. *Newtonian Fluid with Slip Effects*

Finally, making $\lambda \to 0$ into (4.3) and (4.12), the solutions for a Newtonian fluid with slip effects are obtained

$$\begin{aligned} u(y,t) &= UW_{-1/2,1}\left(-\frac{y}{\sqrt{\nu t}}\right) + U\sum_{k=1}^{\infty}\left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k/2)+1}\left(-\frac{y}{\sqrt{\nu t}}\right), \\ \tau(y,t) &= -\frac{\mu U}{\sqrt{\nu t}} W_{-1/2,-1/2}\left(-\frac{y}{\sqrt{\nu t}}\right) - \frac{\mu U}{\sqrt{\nu t}}\sum_{k=1}^{\infty}\left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k-1)/2}\left(-\frac{y}{\sqrt{\nu t}}\right), \end{aligned} \quad (5.5)$$

in which

$$W_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(an+b)}, \qquad z \in C \quad (5.6)$$

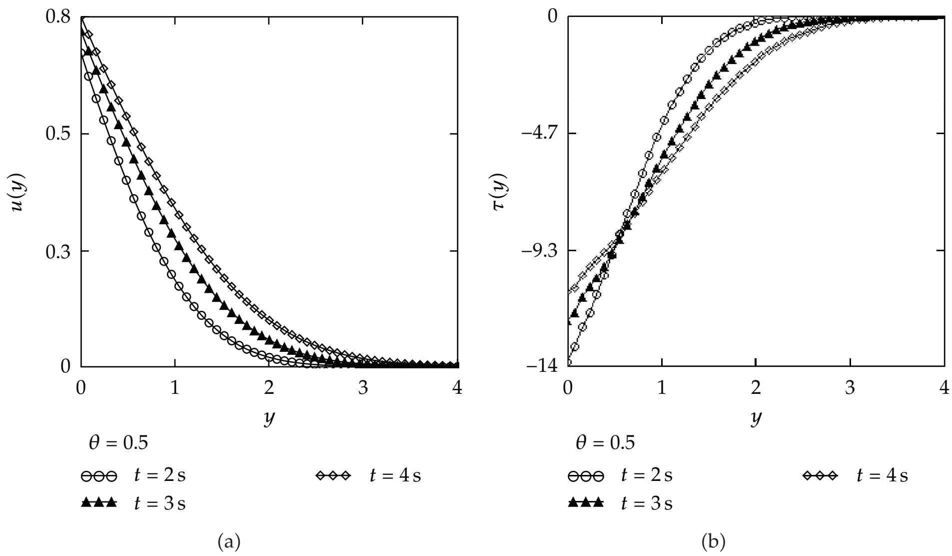


Figure 1: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.5$, and different values of t.

is the Wright function [40]. Using the definition of Wright function and the series expression of error function, we can easily prove that

$$W_{-1/2,1}(-z) = \operatorname{erfc}\left(\frac{z}{2}\right), \qquad W_{-1/2,1/2}(-z) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right). \tag{5.7}$$

Substituting (5.7) into (5.5), we can reduce to

$$\begin{aligned} u(y,t) &= u_N(y,t) + U\sum_{k=1}^{\infty}\left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k/2)+1}\left(-\frac{y}{\sqrt{\nu t}}\right), \\ \tau(y,t) &= \tau_N(y,t) - \frac{\mu U}{\sqrt{\nu t}}\sum_{k=1}^{\infty}\left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k-1)/2}\left(-\frac{y}{\sqrt{\nu t}}\right), \end{aligned} \tag{5.8}$$

where

$$u_N(y,t) = U\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \qquad \tau_N(y,t) = -\frac{\mu U}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right) \tag{5.9}$$

are classical solutions for Stokes' first problem of Newtonian fluid [42, 44].

6. Numerical Results and Conclusions

In this paper, the unsteady flow of fractional Maxwell fluid over an infinite plate, where the no-slip assumption between the wall and the fluid is no longer valid, is studied by means

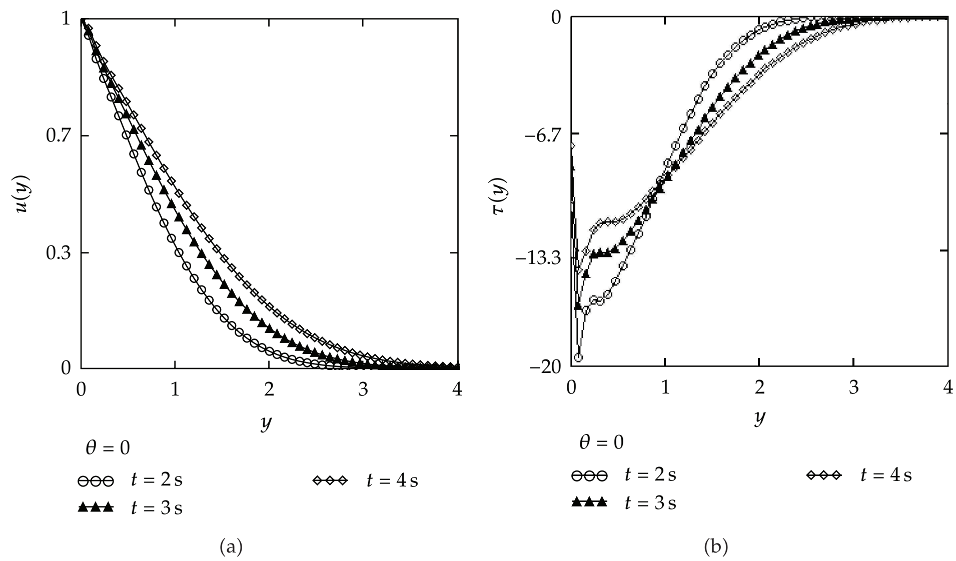


Figure 2: Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.0$, and different values of t.

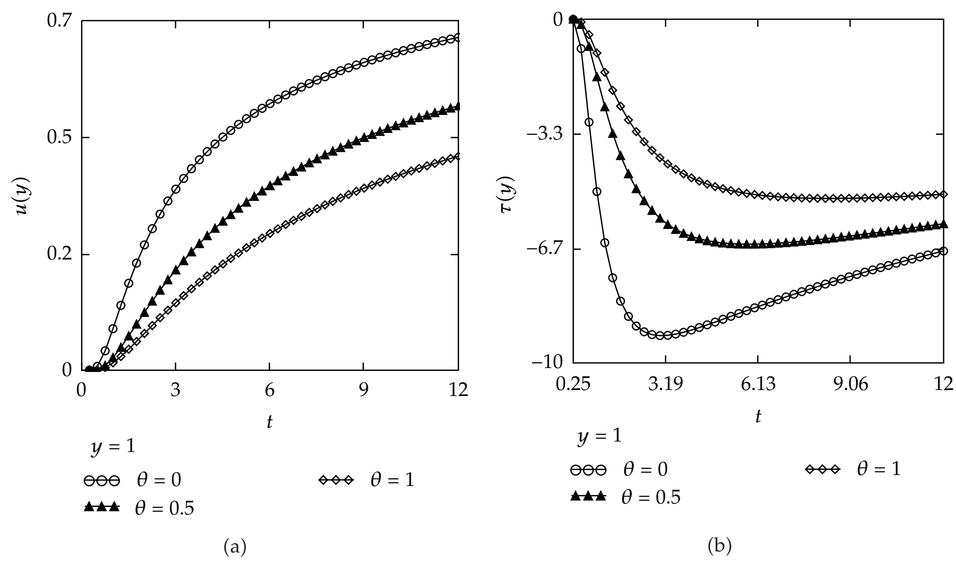


Figure 3: Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, and different values of θ.

of the discrete Laplace transforms. The motion of the fluid is due to the plate that at time $t = 0^+$ is suddenly moved with a constant velocity U in its plane. Closed-form solutions are obtained for the velocity $u(y, t)$ and the shear stress $\tau(y, t)$ in series form in terms of the Wright generalized hypergeometric functions. These solutions, presented as a sum of the slip contribution and the corresponding no-slip contributions, satisfy all imposed initial and boundary conditions. The corresponding solutions for ordinary Maxwell fluids are also

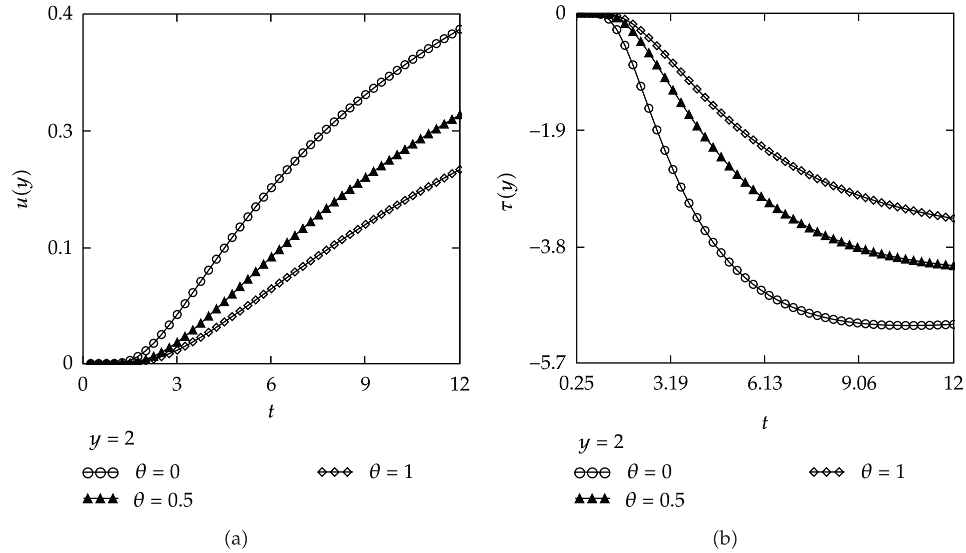


Figure 4: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 2$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, and different values of θ.

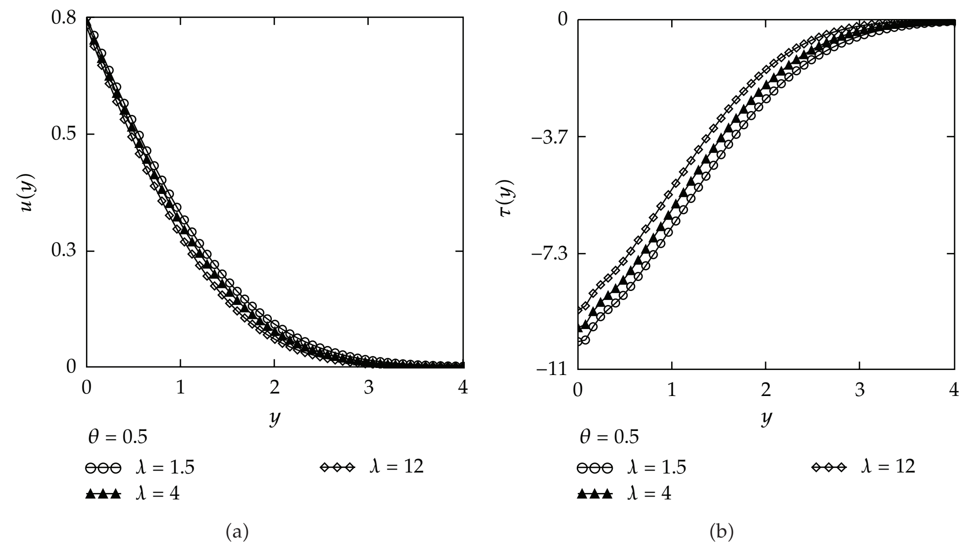


Figure 5: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\alpha = 0.2$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of λ.

obtained from general solutions for $\alpha \rightarrow 1$. In the special case when $\theta \rightarrow 0$, the general solution reduces to previously known results for Stokes' first problem.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ have been drawn against y and t for different values of t, material constants ν, λ, slip parameter θ, and fractional parameter α. From all figures, it is clear that increasing the slip parameter at the wall the velocity decreases at the wall. Figures 1 and 2 are prepared to show the effect of time on velocity and shear

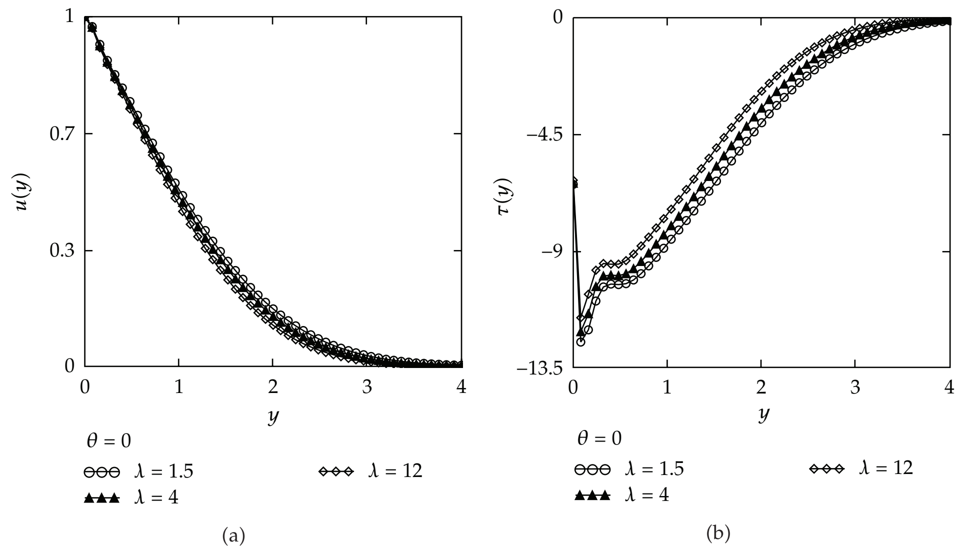


Figure 6: Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\alpha = 0.2$, $\theta = 0.0$, $t = 5\,\mathrm{s}$, and different values of λ.

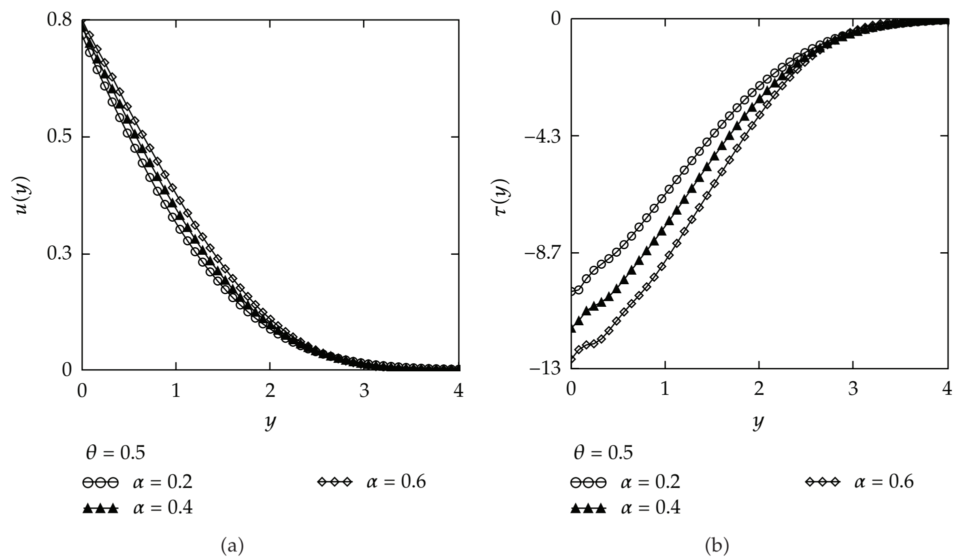


Figure 7: Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of α.

stress profiles with and without slip effects. It is clear that velocity and shear stress (in absolute value) are smaller when slip parameter is nonzero. It is also noted that velocity on the whole flow domain while the shear stress (in absolute value) on large part of flow domain are increasing functions of time t. Figures 3 and 4 are sketched to see the influence of slip effects on fluid motion for two different values of y. It is noted that velocity and shear stress (in absolute value), as expected are decrease when slip parameter θ and y increase.

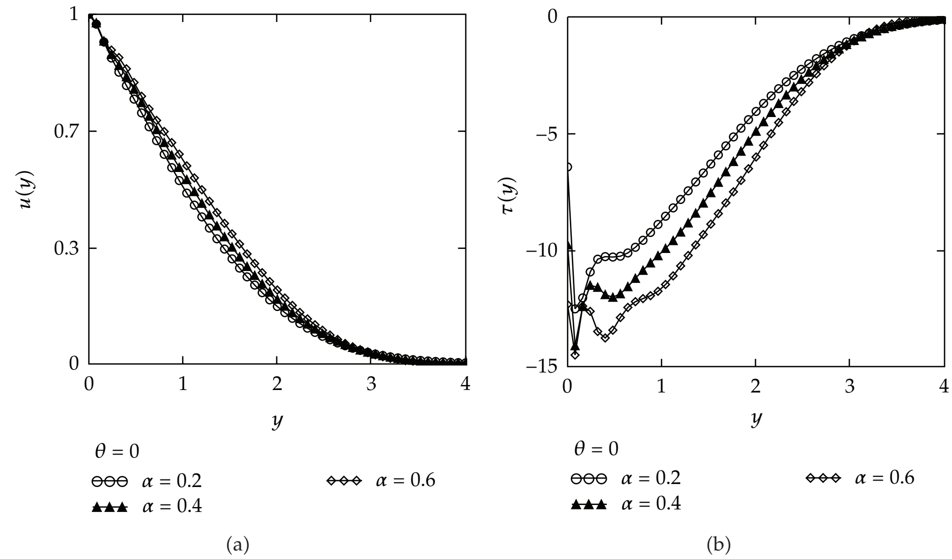


Figure 8: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\theta = 0.0$, $t = 5\,\mathrm{s}$, and different values of α.

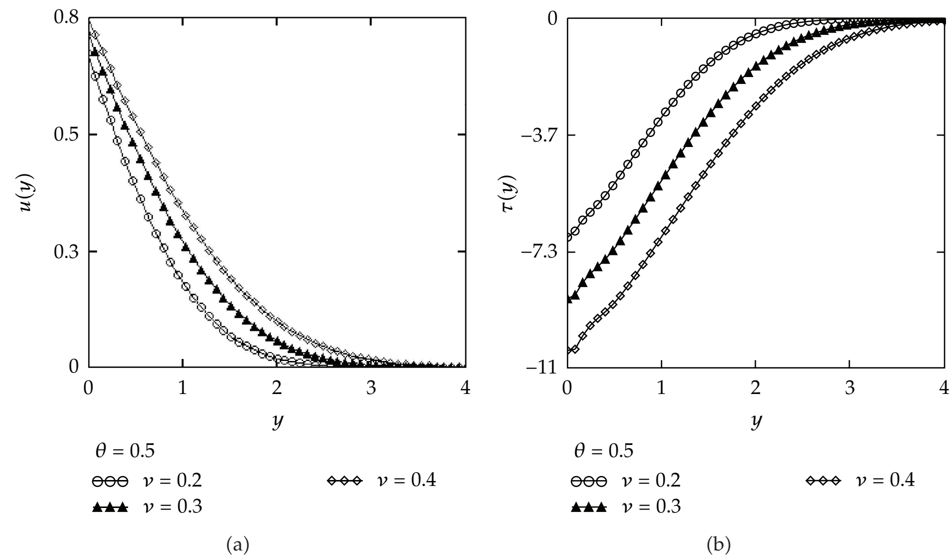


Figure 9: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\rho = 87.071$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of ν.

The influence of relaxation time and kinematic viscosity ν on fluid motion are presented in Figures 5–8. As expected, the two material parameter have opposite effects on fluid motion. For instance, the velocity and shear stress decreases with respect to λ. More important for us is to see the effects of fractional and slip parameter on fluid motion. It is observed that velocity and shear stress either slip effects present or not are increasing functions of fractional parameter, as shown in Figures 9 and 10. The effect of slip parameter is clear from Figure 11.

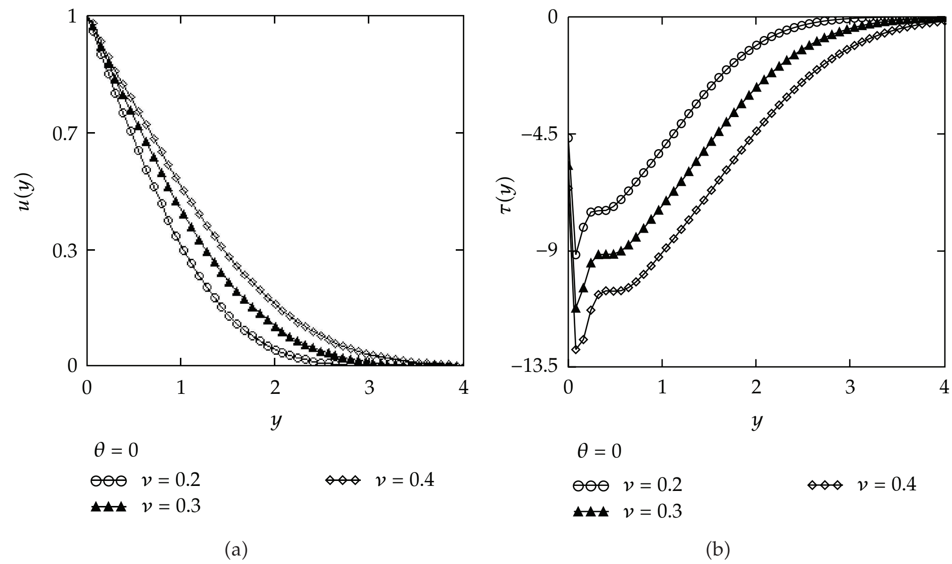


Figure 10: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\rho = 87.071$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.0$, $t = 5\,\text{s}$, and different values of ν.

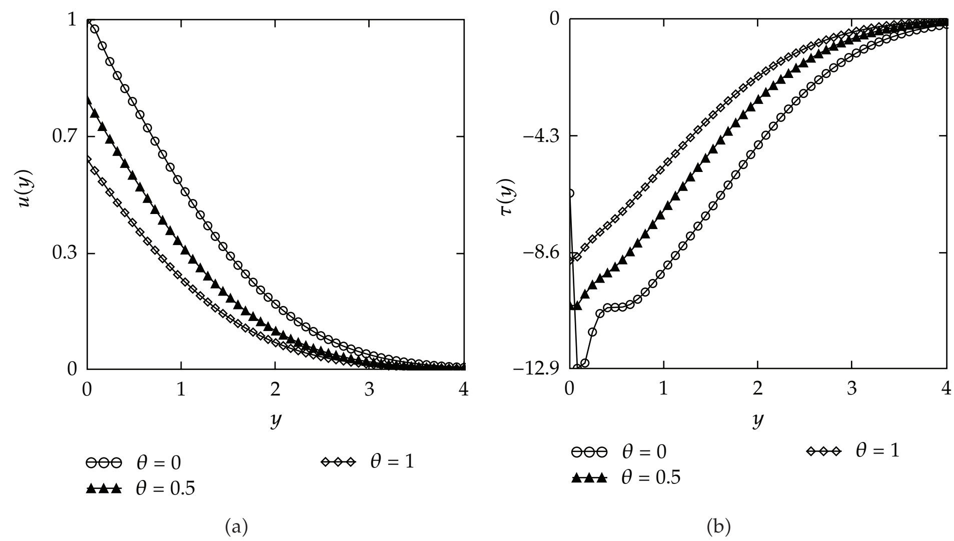
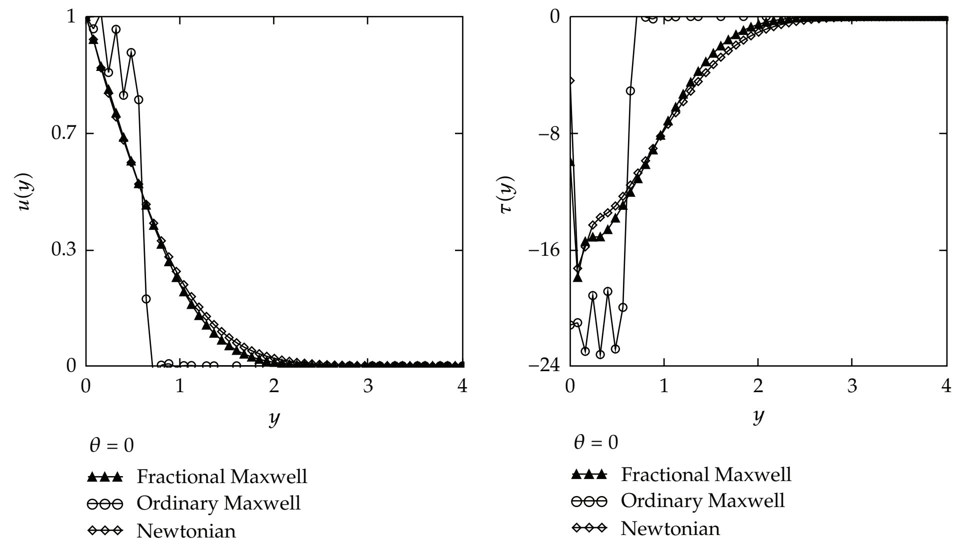


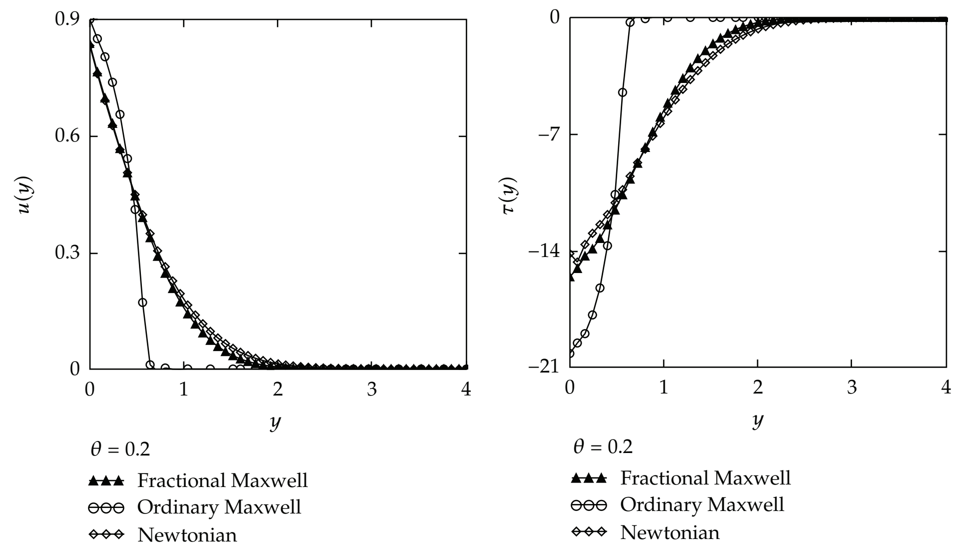
Figure 11: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 5\,\text{s}$, and different values of θ.

Finally, for comparison, the velocity field and the shear stress corresponding to the three models (fractional Maxwell, ordinary Maxwell, and Newtonian) are together depicted in Figures 12–14 for three different values of slip parameter and the same values of t and of the material constants. It is clearly seen from Figures 12 and 13 that the ordinary Maxwell fluid swiftest and the fractional Maxwell fluid is the slowest near the moving plate for slip parameters $\theta = 0.0$ and $\theta = 0.2$. However, the monotonicity is change on large part of the flow domain. The shear stress corresponding to ordinary Maxwell fluid is highest near the



(a) (b)

Figure 12: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\mathrm{s}$, and $\theta = 0.0$.



(a) (b)

Figure 13: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\mathrm{s}$, and $\theta = 0.2$.

moving plate. For higher values of slip parameter the fractional Maxwell fluid is swiftest and the ordinary Maxwell is slowest and shear stress corresponding to fractional Maxwell fluid is largest on the whole flow domain as it is clear from Figure 14. It is important to note the difference between fractional and ordinary Maxwell fluid that, when slip effect is not present, the ordinary Maxwell fluid have oscillating behavior near the moving plate as

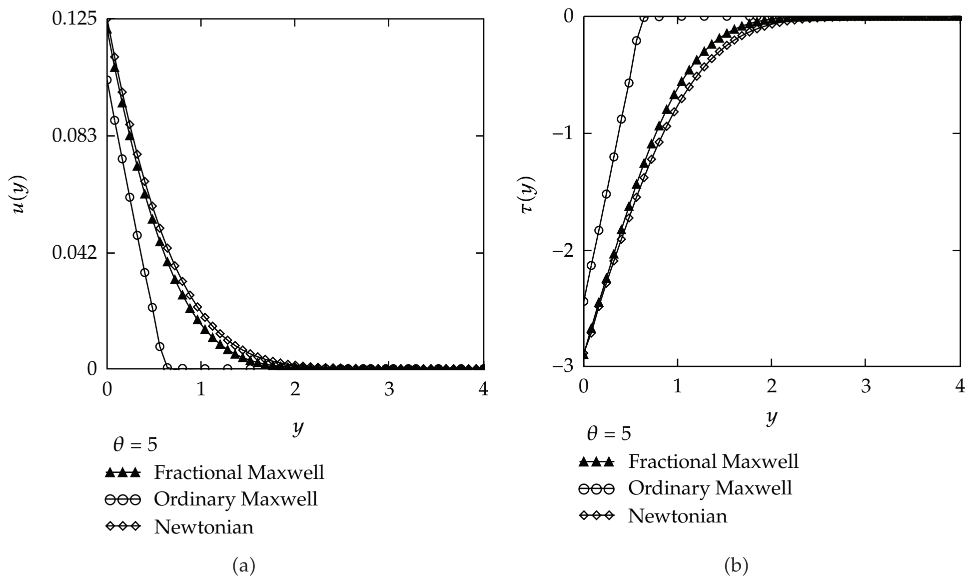


Figure 14: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\mathrm{s}$, and $\theta = 5.0$.

shown in Figure 12, which is the natural one, ordinary Maxwell fluid being the viscoelastic fluid. However the fractional Maxwell fluid have no oscillation. The units of the material constants in all figures are SI units.

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