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*Research Article*

# Homotopy Analysis Method for Solving Foam Drainage Equation with Space- and Time-Fractional Derivatives

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The analytical solution of the foam drainage equation with time- and space-fractional derivatives was derived by means of the homotopy analysis method (HAM). The fractional derivatives are described in the Caputo sense. Some examples are given and comparisons are made; the comparisons show that the homotopy analysis method is very effective and convenient. By choosing different values of the parameters $\alpha, \beta$ in general formal numerical solutions, as a result, a very rapidly convergent series solution is obtained.

## 1. Introduction

Many phenomena in engineering, physics, chemistry, and other science can be described very successfully by models using the theory of derivatives and integrals of fractional order. Interest in the concept of differentiation and integration to noninteger order has existed since the development of the classical calculus [1–3]. By implication, mathematical modeling of many physical systems are governed by linear and nonlinear fractional differential equations in various applications in fluid mechanics, viscoelasticity, chemistry, physics, biology, and engineering.

Since many fractional differential equations are nonlinear and do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. The Adomian decomposition method (ADM) [4], the homotopy perturbation method (HPM) [5], the variational iteration method (VIM) [6], and other methods have been used to provide analytical approximation to linear and nonlinear problems [7, 8]. However, the convergence region of the corresponding results is rather small.

In 1992, Liao [9–13] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely, Homotopy Analysis Method

(HAM). This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [14], the KdV-type equations [15], higher-dimensional initial boundary value problems of variable coefficients [16], and finance problems [17]. The HAM contains a certain auxiliary parameter $h$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.

The study of foam drainage equation is very significant for that the equation is a simple model of the flow of liquid through channels (Plateau borders [18]) and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [19]. It has been studied by many authors [20–22]. The study for the foam drainage equation with time and space-fractional derivatives of this form

$$D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2 D_x^{\beta} u + \left(D_x^{\beta} u\right)^2, \quad 0 < \alpha,\ \beta \leq 1,\ x > 0, \tag{1.1}$$

has been investigated by the ADM and VIM method in [23, 24]. The fractional derivatives are considered in the Caputo sense. When $\alpha = \beta = 1$, the fractional equation reduces to the foam drainage equation of the form

$$u_t = \frac{1}{2} u u_{xx} - 2u^2 u_x + (u_x)^2. \tag{1.2}$$

In this paper, we extend the application of HAM to obtain analytic solutions to the space- and time-fractional foam drainage equation. Two cases of special interest such as the time-fractional foam drainage equation and the space-fractional foam drainage equation are discussed in details. Further, we give comparative remarks with the results obtained using ADM and VIM method (see [23, 24]).

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 we extend the method to solve the space- and time-fractional foam drainage equation. Discussion and conclusions are presented in Section 5.

## 2. Description on the Fractional Calculus

*Definition 2.1.* A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in (0, \infty)$, and it is said to be in the space $C_n^\mu$ l if and only if $h(n) \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\nu$ if $\nu \leq \mu$.

*Definition 2.2.* The Riemann-Liouville fractional integral operator ($J^\alpha$) of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0,\ x > 0, \qquad J^0 f(x) = f(x). \tag{2.1}$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator $J^\alpha$, which we will need here, are as follows:

for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$,

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) &= J^\beta J^\alpha f(x), \\ J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned} \tag{2.2}$$

*Definition 2.3.* For the concept of fractional derivative, there exist many mathematical definitions [2, 25–28]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [2] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n}\, d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{2.3}$$

Here, we also need two basic properties about them:

$$\begin{aligned} D^\alpha J^\alpha f(x) &= f(x), \\ J^\alpha D^\alpha f(x) &= f(x) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \end{aligned} \tag{2.4}$$

*Definition 2.4.* The MittagLeffler function $E_\alpha(z)$ with $a > 0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0,\ z \in C. \tag{2.5}$$

## 3. Basic Idea of HAM

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N\left[D_t^{\alpha} u(x,t)\right] = 0, \quad t > 0, \tag{3.1}$$

where $N$ is nonlinear operator, $D_t^{\alpha}$ stand for the fractional derivative and is defined as in (2.3), $x, t$ denotes independent variables, and $u(x,t)$ is an unknown function, respectively.

By means of generalizing the traditional homotopy method, Liao [9] constructs the so-called zero-order deformation equation

$$(1-q)L\left[\phi(x,t,q) - u_0(x,t)\right] = qhH(t)N\left[D_t^{\alpha}\phi(x,t,q)\right], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $u_0(x,t)$ is initial guesse of $u(x,t)$, and $\phi(x,t,q)$ is unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds that

$$\phi(x,t,0) = u_0(x,t), \phi(x,t,1) = u(x,t), \tag{3.3}$$

respectively. Thus, as $q$ increases from 0 to 1, the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t,q)$ in Taylor series with respect to $q$, we have

$$\phi(x,t,q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m, \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}\bigg|_{q=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$, and the auxiliary function are so properly chosen, the series (3.4) converges at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t), \tag{3.6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [11]. As $h = -1$ and $H(t) = 1$, (3.2) becomes

$$(1-q)L\left[\phi_1(x,t,q) - u_0(x,t)\right] + qN\left[D_t^{\alpha}\phi(x,t,q)\right] = 0, \tag{3.7}$$

which is used mostly in the homotopy perturbation method [29], whereas the solution obtained directly, without using Taylor series. According to definition (3.5), the governing equation can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{3.8}$$

Differentiating (3.2) $m$ times with respect to the embedding parameter $q$ and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hH(t)R_m(\vec{u}_{m-1}), \tag{3.9}$$

where

$$\begin{gathered} R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N\left[D_t^{\alpha}\phi(x,t,q)\right]}{\partial q^{m-1}}\bigg|_{q=0}, \\ \chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \end{gathered} \tag{3.10}$$

Applying the Riemann-Liouville integral operator $J^{\alpha}$ on both side of (3.9), we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^{+})\frac{t^i}{i!} + hH(t)J^{\alpha}R_m(\vec{u}_{m-1}). \tag{3.11}$$

It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$ is governed by the linear equation (3.9) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work.

Liao [10] proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So, it is important to ensure that the solution series is convergent. Note that the solution series contain the auxiliary parameter $h$, which we can choose properly by plotting the so-called $h$-curves to ensure solution series converge.

*Remark 3.1.* The parameters $\alpha$ and $\beta$ can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameter is bigger than 1, we will need more initial and boundary conditions such as $u_0'(x,0), u_0''(x,0), \ldots$ and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameter to $[0,1]$ to discuss.

## 4. Application

In this section we apply this method for solving foam drainage equation with time- and space-fractional derivatives.

*Example 4.1.* Consider the following form of the time-fractional equation:

$$D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2 u_x + u_x^2, \quad 0 < \alpha \leq 1, \ x > 0, \tag{4.1}$$

with the initial condition

$$u(x,0) = -\sqrt{c}\tanh\left(\sqrt{c}x\right), \tag{4.2}$$

where $c$ is the velocity of wavefront [15].

The exact solution of (4.1) for the special case $\alpha = \beta = 1$ is

$$u(x,t) = \begin{cases} -\sqrt{c}\tanh\left(\sqrt{c}(x-ct)\right), & x \leqslant ct, \\ 0, & x > ct. \end{cases} \tag{4.3}$$

For application of homotopy analysis method, in view of (4.1) and the initial condition given in (4.2), it in convenient to choose

$$u_0(x,t) = -\sqrt{c}\tanh\left(\sqrt{c}x\right), \tag{4.4}$$

as the initial approximate. We choose the linear operator

$$L\left[\phi(x,t;q)\right] = D_t^{\alpha}, \tag{4.5}$$

with the property $L(c) = 0$, where $c$ is constant of integration. Furthermore, we define a nonlinear operator as

$$\begin{aligned} N\left[\phi(x,t,q)\right] = D_t^{\alpha}\phi(x,t,q) - \frac{1}{2}\phi(x,t,q)\phi_{xx}(x,t,q) + 2(\phi(x,t,q))^2\phi_x(x,t,q) \\ - \left(\phi_x(x,t,q)\right)^2. \end{aligned} \tag{4.6}$$

We construct the zeroth-order and the $m$th-order deformation equations where

$$\begin{aligned} R_m\left(\vec{u}_{m-1}\right) = D_t^{\alpha} u_{m-1} - \frac{1}{2}\sum_{k=0}^{m-1} u_k \left(u_{m-1-k}\right)_{xx} + 2\sum_{k=0}^{m-1}\sum_{j=0}^{k} u_j u_{k-j}\left(u_{m-1-k}\right)_x \\ - \sum_{k=0}^{m-1}\left(u_k\right)_x\left(u_{m-1-k}\right)_x. \end{aligned} \tag{4.7}$$

We now successively obtain

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{\Gamma(\alpha+1)}\left[h\left(-1+\tanh\left(\sqrt{c}x\right)^2\right)t^\alpha c^2\right], \\
u_2(x,t) &= \frac{1}{\Gamma(\alpha+1)^2}\Big[\sqrt{c}\tanh\left(\sqrt{c}x\right)\Gamma(\alpha+1)^2 + h\sqrt{c}\tanh\left(\sqrt{c}x\right)\Gamma(\alpha+1)^2 \\
&\quad - ht^\alpha c^2\Gamma(\alpha+1) + ht^\alpha c^2\Gamma(\alpha+1)\tanh\left(\sqrt{c}x\right)^2 - h^2t^\alpha c^2\Gamma(\alpha+1) \\
&\quad + h^2t^\alpha c^2\Gamma(\alpha+1)\tanh\left(\sqrt{c}x\right)^2 + h^2t^{2\alpha}c^{7/2}\tanh\left(\sqrt{c}x\right) \\
&\quad -2h^2t^{2\alpha}c^{7/2}\tanh\left(\sqrt{c}x\right)^3\Big], \\
&\ \vdots
\end{aligned}
\tag{4.8}
$$

By taking $\alpha = 1, h = -1$, we reproduce the solution of problem as follows:

$$
\begin{aligned}
u(x,t) &= \frac{1}{\Gamma(\alpha+1)^3}\Big[-\sqrt{c}\tanh\left(\sqrt{c}x\right)\Gamma(\alpha+1)^3 + t^\alpha c^2\Gamma(\alpha+1)^2 \\
&\quad - t^\alpha c^2\Gamma(\alpha+1)^2\tanh\left(\sqrt{c}x\right)^2 + 2t^{2\alpha}c^{7/2}\tanh\left(\sqrt{c}x\right)\Gamma(\alpha+1) \\
&\quad - 2t^{2\alpha}c^{7/2}\tanh\left(\sqrt{c}x\right)^3\Gamma(\alpha+1) + 13t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^2 \\
&\quad -13t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^4 + 3t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^6 - 3t^{3\alpha}c^5\Big].
\end{aligned}
\tag{4.9}
$$

Figures 1 and 2 show the HAM and exact solutions of time-fractional foam drainage equation with $h = -1, n = 3, \alpha = 1$. It is obvious that, when $\alpha = 1$, the solution is nearly identical with the exact solution. Figures 3 and 4 show the approximate solutions of time-fractional foam drainage equation with $h = -1, n = 3, \alpha = 0.5$ and $\alpha = 0.75$, respectively.

*Remark 4.2.* This example has been solved using ADM and VIM in [23, 24]. The graphs drawn and Tables by $h = -1$ are in excellent agreement with that graphs drawn with ADM and VIM.

*Example 4.3.* Considering the operator form of the space-fractional equation

$$
u_t = \frac{1}{2}uu_{xx} - 2u^2D_x^\beta u + \left(D_x^\beta u\right)^2, \quad 0 < \beta \leq 1,\ x > 0, \tag{4.10}
$$

with the initial condition

$$
u(x,0) = x^2. \tag{4.11}
$$

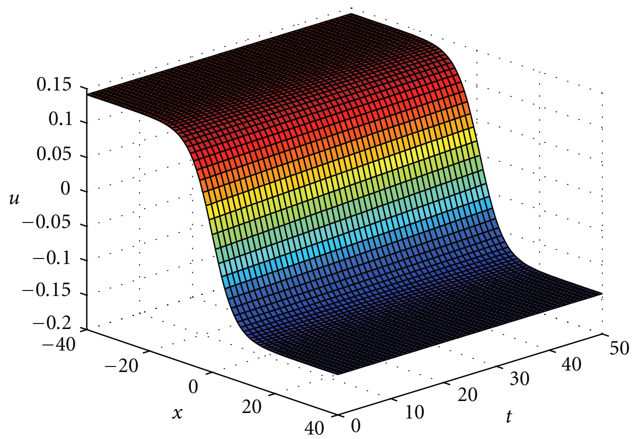


**Figure 1:** HAM solution with $\alpha = 1$.

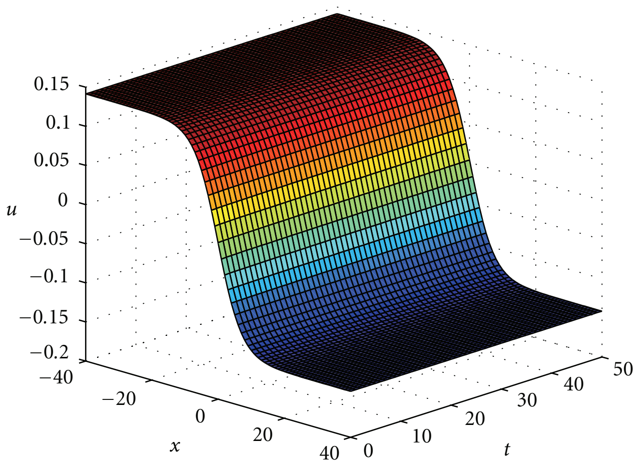


**Figure 2:** Exact solution.

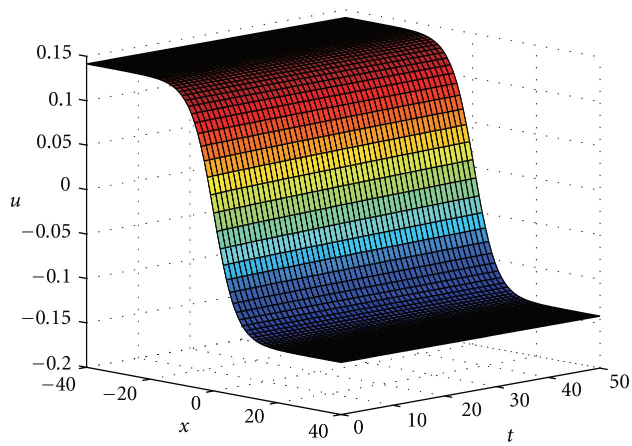


**Figure 3:** HAM solution with $\alpha = \frac{1}{2}$.

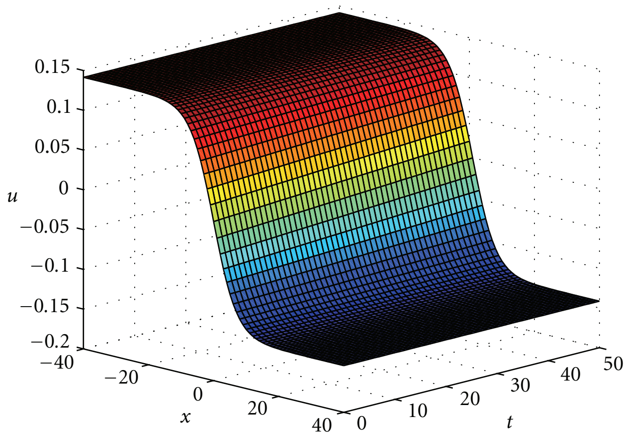


**Figure 4:** HAM solution with $\alpha = \dfrac{3}{4}$.

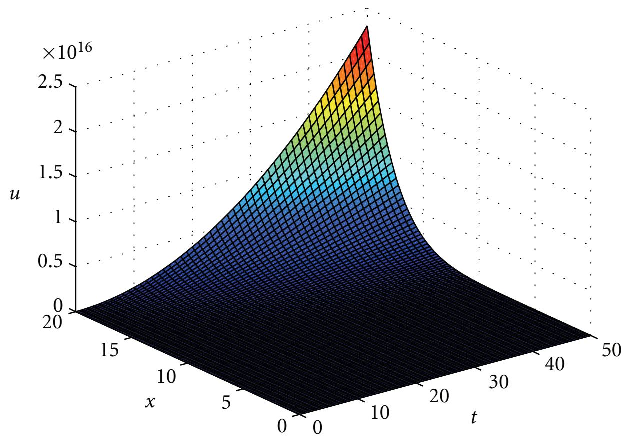


**Figure 5:** HAM solution with $\beta = 1/2$.

For application of homotopy analysis method, in view of (4.10) and the initial condition given in (4.2), it is inconvenient to choose

$$u_0(x,t) = x^2. \tag{4.12}$$

Initial condition has been taken as the above polynomial to avoid heavy calculation of fractional differentiation.
We choose the linear operator

$$L[\phi(x,t,q)] = \frac{\partial \phi(x,t,q)}{\partial t}, \tag{4.13}$$

with the property $L(c) = 0$, where $c$ is constant of integration. Furthermore, we define a nonlinear operator as

$$N[\phi(x,t,q)] = \phi_t(x,t,q) - \frac{1}{2}\phi(x,t,q)\phi_{xx}(x,t,q) + 2(\phi(x,t,q))^2 D_x^\beta \phi(x,t,q) - \left(D_x^\beta \phi(x,t,q)\right)^2. \tag{4.14}$$

We construct the zeroth-order and the $m$th-order deformation equations where

$$R_m(\vec{u}_{m-1}) = (u_t)_{m-1} - \frac{1}{2}\sum_{k=0}^{m-1} u_k (u_{m-1-k})_{xx} + 2\sum_{k=0}^{m-1}\sum_{j=0}^{k} u_j u_{k-j} D_x^\beta u_{m-1-k} - \sum_{k=0}^{m-1} D_x^\beta u_k D_x^\beta u_{m-1-k}. \tag{4.15}$$

We now successively obtain

$$\begin{aligned}
u_1(x,t) &= -hx^2 t + 4\frac{hx^{6-\beta}t}{\Gamma(3-\beta)} - 4\frac{hx^{4-2\beta}t}{\Gamma(3-\beta)^2},\\
u_2(x,t) &= \frac{1}{\Gamma(3-\beta)^2}\Bigg[-14h^2t^2x^{4-2\beta}\beta + 4h^2t^2x^{4-2\beta}\beta^2 + 18hx^{4-2\beta}ht^2\\
&\qquad - 4h^2tx^{4-2\beta} - 4htx^{4-2\beta}\\
&\qquad + \frac{h^2x^{8-3\beta}\pi^{1/2}(-4+\beta)(-5+\beta)(-6+\beta)t^2 4^\beta}{8\Gamma((7/2)-\beta)}\\
&\qquad + \frac{128h^2x^{6-4\beta}4^{-\beta}t^2\Gamma((5/2)-\beta)}{\pi^{1/2}\Gamma(5-3\beta)}\Bigg]\\
&\quad + \frac{1}{\Gamma(3-\beta)}\Bigg[\frac{-h^2x^{10-2\beta}\pi^{1/2}(-4+\beta)(-5+\beta)(-6+\beta)t^2 4^\beta}{16\Gamma((7/2)-\beta)}\\
&\qquad - \frac{64h^2x^{8-3\beta}4^{-\beta}t^2\Gamma((5/2)-\beta)}{\pi^{1/2}\Gamma(5-3\beta)} + 4htx^{6-\beta} + 4h^2tx^{6-\beta}\\
&\qquad -38h^2t62x^{6-\beta} + 11h^2t^2\beta^2x^{6-\beta}\Bigg]\\
&\qquad - \frac{16h^2t^2x^{8-3\beta}}{\Gamma(3-\beta)^3} - hx^2t - h^2x^2t + t^2x^2h^2.\\
&\ \ \vdots
\end{aligned} \tag{4.16}$$

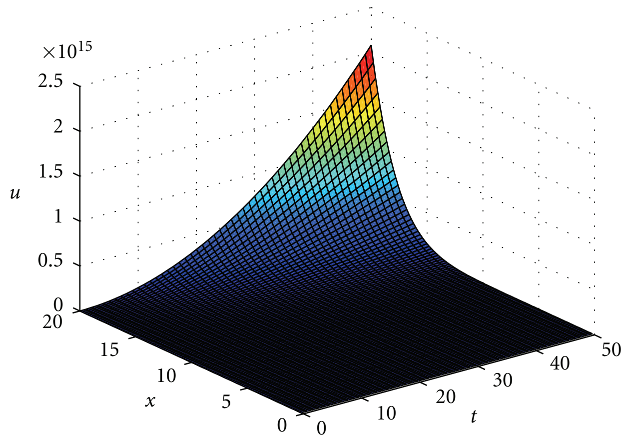


**Figure 6:** HAM solution with $\beta = 1$.

Figures 5 and 6 show the HAM solutions of space-fractional foam drainage equation with $h = -1, n = 3, \beta = 0.5$ and $\beta = 1$, respectively.

*Remark 4.4.* This example has been solved using ADM and VIM in [23, 24]. The graphs drawn and Tables by $h = -1$ are in excellent agreement with that graphs drawn with ADM and VIM.

## 5. Conclusion

In this paper, we have successfully developed HAM for solving space- and time-fractional foam drainage equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting $h$, which is a fundamental qualitative difference in analysis between HAM and other methods. The obtained results demonstrate the reliability of the HAM and its wider applicability to fractional differential equation. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems.

Matlab has been used for computations in this paper.

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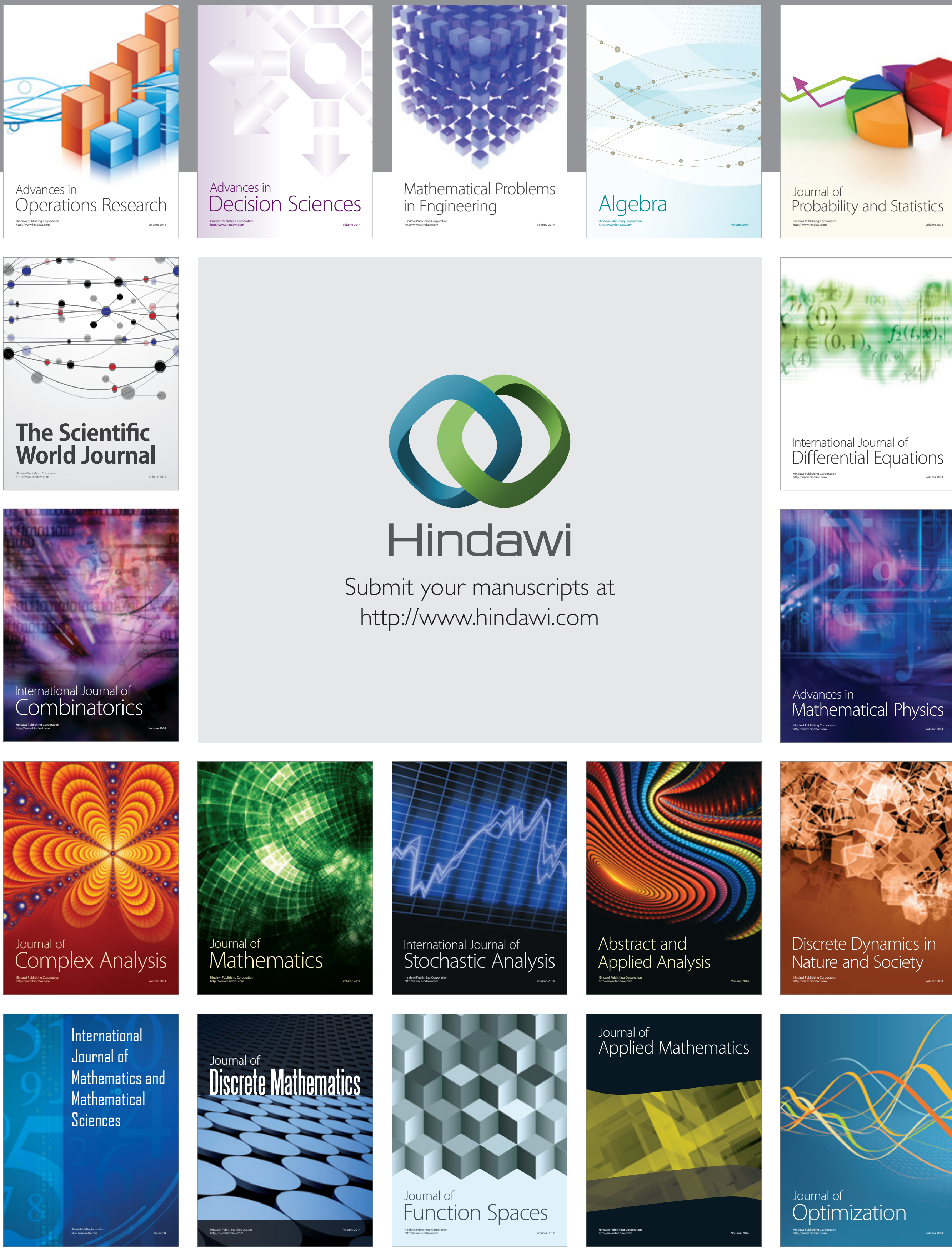
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