Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 304570, 15 pages
doi:10.1155/2011/304570

Research Article

Existence and Uniqueness Theorem of Fractional Mixed Volterra-Fredholm Integrodifferential Equation with Integral Boundary Conditions

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Received 7 May 2011; Accepted 24 May 2011

Academic Editor: Shaher Momani

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We study the existence and uniqueness of the solutions of mixed Volterra-Fredholm type integral equations with integral boundary condition in Banach space. Our analysis is based on an application of the Krasnosel'skii fixed-point theorem.

1. Introduction

In the last century, notable contributions have been made to both the theory and applications of the fractional differential equations. For the theory part, Momani and Hadid have investigated the local and global existence theorem of both fractional differential equation and fractional integrodifferential equations; see [1–6]. Fractional-order differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering.

Integrodifferential equations with integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics and cellular systems. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [7]. In [8], Tidke studied the problem of existence of global solutions to nonlinear mixed Volttera-Fredholm integrodifferential equations with nonlocal condition.

Ahmad and Nieto [9] studied some existence results for boundary value problem involving a nonlinear integrodifferential equation of fractional order with integral equation.

Very recently N'Guérékata [10] discussed the existence of solutions of fractional abstract differential equations with nonlocal initial condition. Anguraj et al. [11] studied the existence and uniqueness theorem for the nonlinear fractional mixed Volterra-Fredholm integrodifferential equation with nonlocal initial condition.

Motivated by these works, we study in this paper the existence of solution of boundary value problem for fractional integrodifferential equations (in the case $1 < \alpha \leq 2$) in Banach spaces by using Banach and Krasnosel'skii fixed-point theorems.

2. Preliminaries

First of all, we recall some basic definitions; see [12–15].

Definition 2.1. For a function f given on the interval $[a,b]$, the Caputo fractional order derivative of f is defined by

$$ {}_a^tD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \tag{2.1}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.2. *Let $\alpha > 0$, then*

$$ {}_a^tD^{-\alpha}\, {}_a^tD^{\alpha} y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.2}$$

for some $c_i \in R$, $i = 0, 1, \ldots, n-1$, $n = [\alpha] + 1$.

Definition 2.3. Let f be a function which is defined almost everywhere (a.e) on $[a,b]$, for $\alpha > 0$, we define

$$ {}_a^bD^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t)dt, \tag{2.3}$$

provided that the integral (Lebesgue) exists.

Theorem 2.4 (Krasnosel'skii fixed point theorem). *Let M be a closed-convex bounded nonempty subset of a Banach space X. Let A and B be two operators such that*

(i) *$Ax + By = M$, whenever $x, y \in M$,*

(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping,*

then there exists $z \in M$ such that $z = Az + Bz$.

Let X be a Banach space with the norm $\| \cdot \|$. Let $C = ([0,T], X)$ be Banach space ofall *continuous functions* $\psi : [0,T] \to X$, *with supermum norm* $\|\psi\| = \sup\{\|\psi(s)\| : s \in [0,T]\}$. *Consider the fractional mixed Volttera-Fredholm integrodifferential equation with boundary conditions, which has the form*

$$D^{\alpha}y(t) = f\left(t, y(t), \int_0^t k(t,s,y(s))ds, \int_0^T h_1(t,s,y(s))ds\right), \tag{2.4}$$

$$y(0) - y'(0) = \int_0^T g(y(s))ds, \qquad y(T) - y'(T) = \int_0^T h(y(s))ds, \tag{2.5}$$

where $1 < \alpha \le 2$, D^{α} *is the Caputo fractional derivative and the nonlinear functions* $f : [0,T] \times X \times X \times X \to X$, $k, h_1 : [0,T] \times [0,T] \times X \to X$ *and* $g, h : X \to X$ *satisfy the following hypotheses:*

(H1) *there exists constants* G_1, G_2 *such that* $\|h(y)\| \le G_1, \|g(y)\| \le G_2$ *for* $y \in X$,

(H2) *there exists constants* b_1, b_2 *such that* $\| h(x) - h(y)\| \le b_1\| x - y \|$ *and*

$$\| g(x) - g(y)\| \le b_2\| x - y \|, \quad \forall x,\ y \in X, \tag{2.6}$$

(H3) *there exists continuous functions* $p : [0,T] \to R^+ = [0,\infty)$ *and* $p_1 : [0,T] \to R^+$ *such that* $\| \int_0^t (k(t,s,x) - k(t,\ s,\ y))ds\| \le p(t)\ \|x - y\|$ *and* $\| \int_0^t k(t,\ s,\ y)ds\| \le p_1(t)\ \|y\|$, *for every* $t, s \in [0,T]$ *and* $x, y \in X$,

(H4) *there exists continuous functions* $q : [0,T] \to R^+$ *and* $q_1 : [0,T] \to R^+$ *such that* $\| \int_0^T (h_1(t,s,x) - h_1(t,s,y)ds\| \le q(t)\ \|x - y\|$ *and* $\| \int_0^T h_1(t,\ s,\ y)ds\| \le q_1(t)\|y\| for\ every\ t,s \in [0,T] and x, y \in X$

(H5) *there exists continuous function* $L : [0,T] \to R^+$, *and* N_1 *is positive constant such that* $\|f(t,\ x_1,\ y_1,\ z_1) - f(t,\ x_2,\ y_2,\ z_2)\| \le L(t)K(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|)$ *and* $N_1 = \sup_{t\in[0,T]}\|f(t,0,0,0)\|$, *for every* $t \in [0,T]$ *and* $x_1, y_1, z_1, x_2, y_2, z_2 \in X$, *where* $K : R^+ \to (0,\infty)$ *is continuous nondecreasing function satisfying* $K(\gamma(t)x) \le \gamma(t)K(x)$, *where* γ *is a continuous function* $\gamma : [0,T] \to R^+$.

Lemma 2.5. *Let* $1 < \alpha \le 2$ *and* $f : J \times X \to X$, *where* $J = [0,T]$, *be a continuous function, then the solution of fractional differential equation* (2.4) *with the boundary condition* (2.5) *is*

$$\begin{aligned} y(t) &= \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1 - \frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\ &\quad - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) \\ &\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds. \end{aligned} \tag{2.7}$$

Proof. By Lemma 2.2, we reduce the problem (2.4)-(2.5) to an equivalent integral equation

$$y(t) = {}_0^tI^{\alpha} f + C_1 + C_2 t,$$
$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds + C_1 + C_2 t. \tag{2.8}$$

In view of the relations ${}^cD^{\alpha} I^{\alpha} y(t) = y(t)$ and $I^{\alpha} I^{\beta} y(t) = I^{\alpha+\beta} y(t)$, for $\alpha, \beta > 0$, we obtain

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds + C_2. \tag{2.9}$$

Applying the boundary condition (2.5), we find that

$$\begin{aligned} y(0) &= C_1, \qquad y(T) = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds \\ &\qquad + C_1 + C_2 T, \\ y'(0) &= C_2, \qquad y'(T) = \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds \\ &\qquad + C_2, \end{aligned} \tag{2.10}$$

that is,

$$\begin{aligned} C_2 &= \frac{1}{T}\int_0^T h(y(s))ds - \frac{1}{T}\int_0^T g(y(s))ds \\ &\quad - \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds \\ &\quad + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds, \\ C_1 &= \frac{1}{T}\int_0^T h(y(s))ds + \left(1 - \frac{1}{T}\right) \int_0^T g(y(s))ds \\ &\quad - \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds \\ &\quad + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) ds. \end{aligned} \tag{2.11}$$

Therefore the solution of (2.4)-(2.5) is

$$\begin{aligned}
y(t) &= \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\
&\quad - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds,
\end{aligned} \tag{2.12}$$

which completes the proof. □

3. The Main Result

Theorem 3.1. *If the hypotheses (H1)–(H5) are satisfied, then the fractional integrodifferential equation* (2.4)-(2.5) *has a unique solution on* J.

Proof. Define $F : C \to C$ by

$$\begin{aligned}
Fy(t) &= \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\
&\quad - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds.
\end{aligned} \tag{3.1}$$

We show that F has a fixed point on Br. This fixed point is then a solution of (2.4)-(2.5). Firstly, we show that $FBr \subset Br$, where $Br = \{y \in C : \|y\| \le r\}$. For $y \in Br$, we have

$$\begin{aligned}
\|Fy(t)\| &\le \frac{(1+t)}{T}\int_0^T \|h(y(s))\|\, ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(s))\|ds \\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\| ds
\end{aligned}$$

$$+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds$$

$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds,$$

$$\begin{aligned}\|Fy(t)\| \leq{}& \frac{(1+t)}{T}\int_0^T\|h(y(s))\|ds+\left(1-\frac{(1+t)}{T}\right)\int_0^T\|g(y(s))\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds,\end{aligned}$$

$$\begin{aligned}\|Fy(t)\| \leq{}& \frac{(1+t)}{T}G_1T+\left(1-\frac{(1+t)}{T}\right)G_2T\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,0,0,0)\|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,0,0,0)\|ds\\
&+\left(\frac{1+t}{T}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\left(\frac{1+t}{T}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,0,0,0)\|ds\\
\leq{}& \frac{(1+t)}{T}\,G_1T+\left(1-\frac{(1+t)}{T}\right)G_2\,T+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)\end{aligned}$$

$$
\begin{aligned}
& K\left(\|y(s)\| + \left\|\int_0^S k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
& + \frac{(1+t)N_1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s) \\
& K\left(\|y(s)\| + \left\|\int_0^s k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
& + \frac{(1+t)N_1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s) \\
& K\left(\|y(s)\| + \left\|\int_0^s k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
& + N_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& \le (1+t)G_1 + \left(1 - \frac{(1+t)}{T}\right)G_2 T + \frac{(1+t)N_1}{T} \\
& \left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right) + N_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds, \\
\|Fy(t)\| & \le (1+t)G_1 + \left(1 - \frac{(1+t)}{T}\right)G_2 T \\
& + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)(1 + p_1(s) + q_1(s))K(\|y\|)ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1 + p_1(s) + q_1(s))K(\|y\|)ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1 + p_1(s) + q_1(s))K(\|y\|)ds.
\end{aligned}
\tag{3.2}
$$

Since we have $M_1 = \sup\{L(t)(\,1+p_1(t)+q_1(t));\ t\in[0,T]\}$, and $(1-((1+t)/T)) < (1-(1/T))$, we get

$$
\begin{aligned}
&\le (1+t)G_1 + \left(1-\frac{1}{T}\right) G_2\, T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}K(\|y\|)ds + \frac{(1+t)M_1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}K(\|y\|)ds \\
&\quad + M_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}K(\|y\|)ds \\
&\le (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2\, T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1K(r)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{(1+t)M_1K(r)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + M_1K(r)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\le (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1K(r)}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{M_1K(r)T^{\alpha}}{\Gamma(\alpha+1)} \\
&\le (1+t)G_1 + (T-1)G_2 + \frac{(1+t)}{T}(N_1+M_1K(r))\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&\quad + (N_1+M_1K(r))\frac{T^{\alpha}}{\Gamma(\alpha+1)}, \\
\|Fy(t)\| &\le (1+T)G_1 + (T-1)G_2 + \frac{(1+T)}{T}(N_1+M_1K(r))\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&\quad + (N_1+M_1K(r))\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\le G_1(1+T) + G_2(T-1) + \frac{C_0(N_1+M_1K(r))}{\Gamma(\alpha+1)T^{2-\alpha}},
\end{aligned}
\tag{3.3}
$$

where $C_0 = 2T^2 + T + \alpha(T+1)$.

Now, take $x,\ y\in C$ and for each $t\in[0,T]$, we obtain

$$
\begin{aligned}
\|Fx(t)-Fy(t)\| \le{}& \frac{(1+t)}{T}\int_0^T \|h(x)-h(y)\|ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|\, g(x)-g(y)\|ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}
$$

$$\left\| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau \right) \right.$$
$$\left. -f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) \right\| ds$$
$$+ \frac{(1+t)}{T} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}$$
$$\left\| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau \right) \right.$$
$$\left. -f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) \right\| ds$$
$$+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$$
$$\left\| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau \right) \right.$$
$$\left. -f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau \right) \right\| ds, \tag{3.4}$$

by using (H1)–(H5), we get

$$\|Fx(t) - Fy(t)\| \le \frac{b_1(1+t)}{T} \int_0^T \|x-y\| ds + b_2\left(1 - \frac{(1+t)}{T}\right) \int_0^T \|x-y\| \, ds$$
$$+ \frac{(1+t)}{T} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} L(s)$$
$$K\left(\|x(s)-y(s)\| + \left\| \int_0^s (k(s,\tau,x(\tau)) - k(s,\tau,y(\tau)))d\tau \right\| \right.$$
$$\left. + \left\| \int_0^T (h_1(s,\tau,x(\tau)) - h_1(s,\tau,y(\tau)))d\tau \right\| \right) ds$$
$$+ \frac{(1+t)}{T} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)$$
$$K\left(\|x(s)-y(s)\| + \left\| \int_0^s (k(s,\tau,x(\tau)) - k(s,\tau,y(\tau)))d\tau \right\| \right.$$
$$\left. + \left\| \int_0^T (h_1(s,\tau,x(\tau)) - h_1(s,\tau,y(\tau)))d\tau \right\| \right) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)$$

$$
\begin{aligned}
&K\left(\|x(s)-y(s)\|+\left\|\int_0^s(k(s,\tau,x(\tau))-k(s,\tau,y(\tau)))d\tau\right\|\right.\\
&\left.+\left\|\int_0^T(h_1(s,\tau,x(\tau))-h_1(s,\tau,y(\tau)))d\tau\right\|\right)ds\\
\leq\ & \frac{b_1(1+t)}{T}\int_0^T\|x-y\|ds+b_2\left(1-\frac{(1+t)}{T}\right)\int_0^T\|x-y\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)K(\|x-y\|+p(s)\|x-y\|+q(s)\|x-y\|)ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|x-y\|+p(s)\|x-y\|+q(s)\|x-y\|)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|x-y\|+p(s)\|x-y\|+q(s)\|x-y\|)ds\\
\leq\ & \frac{b_1(1+t)}{T}\int_0^T\|x-y\|ds+b_2\left(1-\frac{1}{T}\right)\int_0^T\|x-y\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)(1+p(s)+q(s))K(\|x-y\|)ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1+p(s)+q(s))K(\|x-y\|)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ L(s)(1+p(s)+q(s))K(\|x-y\|)ds\ .
\end{aligned}
\tag{3.5}
$$

Since we have $M(t)=L(t)(1+p(t)+q(t)$, $M^*=\sup\{M(t):t\in[0,T]\}$, and, Let $K(\|x-y\|)\leq w\|x-y\|$, $(w>0)$, then

$$
\begin{aligned}
\|Fx(t)-Fy(t)\|\leq\ & \frac{b_1(1+t)}{T}\int_0^T\|x-y\|ds+b_2\left(1-\frac{1}{T}\right)\int_0^T\|x-y\|ds\\
&+\frac{wM^*(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|x-y\|ds\\
&+\frac{w\ M^*(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|x-y\|ds\\
&+wM^*\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ \|x-y\|ds\\
\leq\ & \left[b_1(1+T)+b_2(T-1)+\frac{wM^*C_1(1+T)}{\Gamma(\alpha+1)\ T^{2-\alpha}}\right]\|\ x-y\ \|,
\end{aligned}
\tag{3.6}
$$

where $C_1 = 2T^2 + T + \alpha(1+T)$.

As $b_1(1+T)+b_2(T-1)+(wM^*C_1(1+T))/(\Gamma(\alpha+1)T^{2-\alpha}) < 1$, therefore f is a contraction. Thus, the conclusion of the theorem is followed by the contraction mapping principle. □

Theorem 3.2. *Assume that (H1)–(H5) hold with*

$$\left\| f\left(t, y(t), \int_0^t k(t,\tau,y(\tau))d\tau, \int_0^T h_1(t,\tau,y(\tau))d\tau\right)\right\| \leq \varphi(t), \quad \text{where } \varphi(t) \in L_1(J). \tag{3.7}$$

Then the boundary value problem (2.4)-(2.5) *has at least one element on* $[0,T]$.

Proof. Consider $Br = \{y \in C : \|y\| \leq r\}$. We define the operators A and B as

$$\begin{aligned}
(Ax)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(t, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds,\\
(Bx)(t) &= \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\quad f\left(s, x(s), \int_0^s k(s,\ \tau,\ x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds.
\end{aligned} \tag{3.8}$$

Let us observe that if $x,\ y \in Br$, then $Ax + By \in Br$,

$$\begin{aligned}
\|Ax + By\| = &\left\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right.\\
&+ \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau\ , \int_0^T h_1(s,\ \tau,\ y(\tau))d\tau\right)ds + \frac{(1+t)}{T}\\
&\left.\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s),\ \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds\right\|\\
\leq &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\| f\left(s, x(s),\ \int_0^s k(s,\tau,x(\tau))d\tau,\ \int_0^T h_1(s,\tau,x(\tau))d\tau\right)\right\| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1+t)}{T}\int_0^T\|h(y(s))\|ds+\left(1-\frac{(1+t)}{T}\right)\int_0^T\|g(y(s))\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\left\|f\left(s,\ y(s),\ \int_0^s k(s,\ \tau,\ y(\tau))d\tau,\ \int_0^T h_1(s,\ \tau,\ y(\tau))d\tau\right)\right\|ds\\
\le{}&\|\psi\|_{L_1}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+(1+t)G_1\\
&+\left(1-\frac{(1+t)}{T}\right)G_2T+\frac{(1+t)\|\psi\|_{L_1}}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\\
&+\frac{(1+t)\|\psi\|_{L_1}}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
\le{}&\frac{\|\psi\|_{L_1}T^\alpha}{\Gamma(\alpha+1)}+(1+t)G_1+\left(1-\frac{1}{T}\right)G_2T+\frac{(1+t)T^{\alpha-1}\|\psi\|_{L_1}}{T\Gamma(\alpha)}+\frac{(1+t)T^{\alpha}\|\psi\|_{L_1}}{T\Gamma(\alpha+1)}\\
\le{}&\frac{\|\psi\|_{L_1}T^\alpha}{\Gamma(\alpha+1)}+\frac{(1+t)T^{\alpha-1}\|\psi\|_{L_1}}{T\Gamma(\alpha)}+\frac{(1+t)T^{\alpha}\|\psi\|_{L_1}}{T\Gamma(\alpha+1)}+(1+T)G_1+(T-1)G_2\\
\le{}&G_1(1+T)+G_2(T-1)+\frac{C_2T^{\alpha-2}}{\Gamma(\alpha+1)}\ \|\psi\|_{L_1},
\end{aligned}
\tag{3.9}
$$

where $C_2=2T^2+T(\alpha+1)+T$.

Now we prove that Bx is contraction mapping,

$$
\begin{aligned}
\|Bx_1-Bx_2\|\le{}&\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau\ ,\ \int_0^T h_1(s,\tau,x_1(\tau))d\tau\right)\right.\\
&\left.-f\left(s,x_2(s)\int_0^s k(s,\tau,x_2(\tau))d\tau\ ,\int_0^T h_1(s,\tau,x_2(\tau))d\tau\right)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau,\int_0^T h_1(s,\tau,x_1(\tau))d\tau\right)\right.\\
&\left.-f\left(s,x_2(s),\int_0^s k(s,\tau,x_2(\tau))d\tau\ ,\int_0^T h_1(s,\tau,x_2(\tau))d\tau\right)\right\|ds
\end{aligned}
$$

$$\leq \frac{(1+t)}{T}\left[\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} L(s)\big(1+p(s)+q(s)\big)K(\|x_1-x_2\|)ds \right.$$
$$\left. + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)\big(\ 1+p(s)+q(s)\big)K(\|x_1-x_2\|)ds\right]. \tag{3.10}$$

Let $K(\|x_1-x_2\|) \leq w\|x_1-x_2\|$, we obtain

$$\begin{aligned}
\|Bx_1-Bx_2\| &\leq \frac{(1+t)wM^*}{T}\|\ x_1-x_2\ \|\left[\ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\ +\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right],\\
\|Bx_1-Bx_2\| &\leq \frac{(1+T)wM^*}{T}\left[\ \frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\|\ x_1-x_2\ \|\\
&\leq \frac{wM^*(1+T)(\alpha+T)}{\Gamma(\alpha+1)T^{2-\alpha}}\|\ x_1-x_2\ \|.
\end{aligned} \tag{3.11}$$

It is clear that B is contraction mapping, since $x(t)$ is continuous, then Ax is continuous

$$\begin{aligned}
\|Ax(t)\| &= \left\|\frac{1}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\|\\
&\leq \|\psi\|_{L_1}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
\|Ax(t)\| &\leq \frac{T^{\alpha}\|\psi\|_{L_1}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{3.12}$$

Hence, A is uniformly bounded on Br. Now, let us prove that $Ax(t)$ is equicontinuous, let $t_1,\ t_2 \in [0,\ T]$ and $x \in Br$. Using the fact that f is bounded on the compact set $J \times Br$, thus $\sup_{(t,s)\in J\times Br}\|f(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))d\tau, \int_0^T h_1(s,\ \tau,\ x(\tau))d\tau)\| = c_0 < \infty$,we get

$$\begin{aligned}
\|Ax(t_1)-Ax(t_2)\| = &\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau,\right.\right.\\
&\left.\int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\ \int_0^{t_2} (t_2-s)^{\alpha-1}\\
&\left.f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\|\\
\leq &\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]\right.
\end{aligned}$$

$$f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\Bigg\|$$
$$+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\ , \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\|$$
$$\leq \frac{c_0}{\Gamma(\alpha+1)}\ \left[2(t_2-t_1)^{\alpha}+({t_1}^{\alpha}-{t_2}^{\alpha})\right]. \tag{3.13}$$

So A is relatively compact. By Arzela-Ascoli theorem, A is compact. Now we conclude the result of the theorem of Krasnosel'skii theorem. □

Example 3.3. Consider the following fractional mixed Volterra-Fredholm integrodifferential equation:

$$y^{(1.5)}(t) = \frac{1}{10}+\frac{1}{10+|y(t)|}+\int_0^t \frac{|y(t)|}{10\ e^{|y(t)|}+t}dt\ +\int_0^1 \frac{|y(t)|e^{-t}}{10\ +|y(t)|^2}dt, \tag{3.14}$$

with integral boundary conditions

$$y(0)-y'(0)=\int_0^1\frac{1}{10+|y(t)|}dt, \qquad y(1)-y'(1)=\int_0^1\frac{1}{10+e^{-|y(t)|}}dt. \tag{3.15}$$

Here,

$$\begin{gathered}
\|g(y(t))\| = \left\|\frac{1}{10+|y(t)|}\right\| \leq \frac{1}{10}, \qquad \|g(x)-g(y)\| \leq \frac{1}{100}\|x-y\|, \\
\|h(y(t))\| = \left\|\frac{1}{10+e^{-|y(t)|}}\right\| \leq \frac{1}{10}, \qquad \|h(x)-h(y)\| \leq \frac{1}{100}\|x-y\|, \\
\left\|\int_0^t (k(t,s,x)-k(t,s,y))ds\right\| \leq \frac{1}{10e^t}\|x-y\ \|, \qquad \left\|\int_0^t k(t,s,y)ds\right\| \leq \frac{1}{10+t}\|y(t)\|, \\
\left\|\int_0^t (h_1(t,s,x)-h_1(t,s,y))ds\right\| \leq \frac{1}{10e^t}\|x-y\ \|, \qquad \left\|\int_0^t h_1(t,s,y)ds\right\| \leq \frac{1}{10+t}\|y(t)\|, \\
\|f(t,x_1y_1,z_1)-f(t,x_2,y_2,z_2)\| \leq \frac{1}{10\ +t}(\|x_1-x_2\ \|+\|y_1-y_2\ \|+\|z_1-z_2\ \|), \\
f(t,0,0,0)=\frac{1}{10}.
\end{gathered} \tag{3.16}$$

Hence, the conditions (H1)–(H5) hold with $G_1 = G_2 = 0.1$, $b_1 = b_2 = 0.01$, $M_1^* = 0.12$, $w = 0.1$, $C_o = 6$, $N_1 = 0.1$, $M^* = 0.12$, and $C_1 = 6$, thus

$$b_1(1+T) + b_2(T-1) + \frac{wM^*C_1(1+T)}{\Gamma(\alpha+1)T^{2-\alpha}} < 1 \Longleftrightarrow 0.01(2) + \frac{(0.1)(0.12)6(2)}{\Gamma(2.5)} < 1. \quad (3.17)$$

We conclude from the above example that the integrodifferential equation has unique solution.

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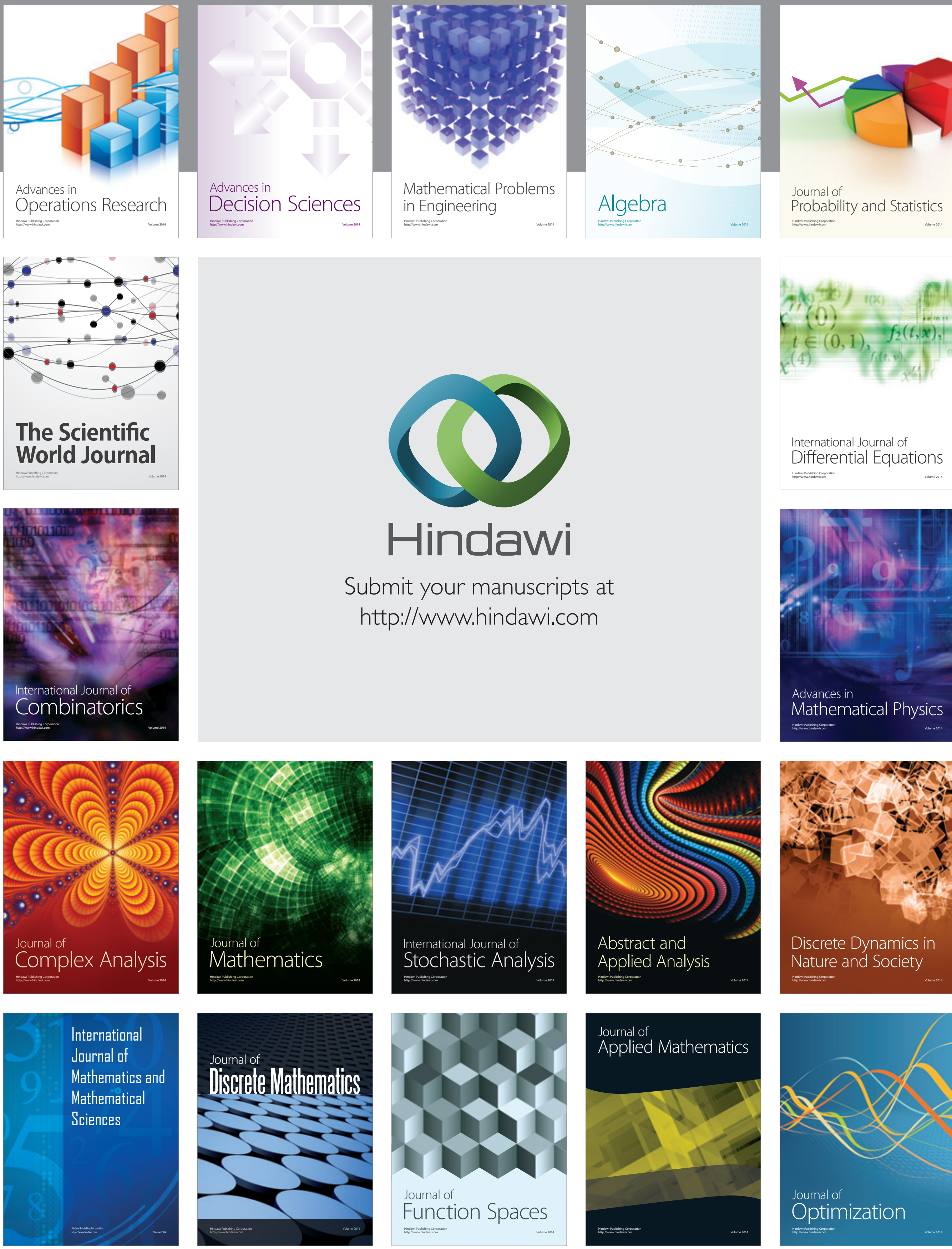
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