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Research Article

Existence of Solutions of a Riccati Differential System from a General Cumulant Control Problem

Stanley R. Liberty[1] and Libin Mou[2]

[1] *Department of Electrical & Computer Engineering, Kettering University, Flint, MI 48504, USA*
[2] *Department of Mathematics, Bradley University, Peoria, IL 61625, USA*

Correspondence should be addressed to Libin Mou, mou@bradley.edu

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We study a system of infinitely many Riccati equations that arise from a cumulant control problem, which is a generalization of regulator problems, risk-sensitive controls, minimal cost variance controls, and k-cumulant controls. We obtain estimates for the existence intervals of solutions of the system. In particular, new existence conditions are derived for solutions on the horizon of the cumulant control problem.

1. Introduction

Consider a linear control system and a quadratic cost function:

$$\begin{gathered} dx = (Ax + Bu)dt + Gdw, \quad t \in [t_0, t_f]; \quad x(t_0) = x_0, \\ J(u) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} \left(x^T Q x + u^T R u \right) dt, \end{gathered} \tag{1.1}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $Q(t) \in \mathbb{S}^n$, and $R(t) \in \mathbb{S}^m$ are continuous matrix functions for $t \in [t_0, t_f]$, $Q_f \in \mathbb{S}^n$ ($\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices.), $x(t) \in \mathbb{R}^n$ is the state with known initial state x_0, $u(t) \in \mathbb{R}^m$ the control, and $w(t) \in \mathbb{R}^p$ a standard Wiener process. Because x is completely determined by the first equation in (1.1) in terms of u, the cost function J is only a function of u.

For $\theta \in \mathbb{R}$, denote by $E(e^{\theta J})$ the (general) expectation of the random variable $e^{\theta J}$. For $i \geq 1$, let

$$\kappa_i = \frac{d^i}{d\theta^i} \ln E\left(e^{\theta J}\right)|_{\theta=0} \tag{1.2}$$

be the ith cumulant of the cost $J(u)$. Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers. Consider the following combination of κ_i:

$$\kappa = \sum_{i=1}^{\infty} \frac{\mu_i \kappa_i}{i!}. \tag{1.3}$$

The cumulant control problem, considered in [1], is to find a control u that minimizes the combined cumulant κ defined in (1.3). This problem leads to the following system of (infinitely many) equations of Riccati type:

$$\begin{gathered} H_1' + (A+BK)^T H_1 + H_1(A+BK) + Q + K^T R K = 0, \qquad H_1(t_f) = Q_f, \\ H_i' + (A+BK)^T H_i + H_i(A+BK) + 2\sum_{j=1}^{i-1} H_j W H_{i-j} = 0, \qquad H_i(t_f) = 0, \quad i \geq 2, \end{gathered} \tag{1.4}$$

where $' = d/dt$ denotes the derivative to t, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $Q(t) \in \mathbb{S}^n$, $R(t) \in \mathbb{S}^m$ are as in (1.1), and $W(t) = G(t)G^T(t) \in \mathbb{S}^n$. $K(t) \in \mathbb{R}^{m\times n}$ and $H_i(t) \in \mathbb{S}^n$ are the unknown matrix functions, which are required to be continuously differentiable for $t \in [t_0, t_f]$. For convenience, the time variable t is often suppressed. System (1.4) will be combined with the following equation

$$K = -R^{-1} B^T \sum_{i=1}^{\infty} \mu_i H_i. \tag{1.5}$$

If $k \geq 1$ is a given integer and $\mu_i = 0$ for $i > k$, then $\kappa = \sum_{i=1}^{k} \mu_i \kappa_i / i!$ is the k-cost cumulant investigated in [2, 3]. In particular, if $k = 1$, then $\kappa = \mu_1 E\{J(u)\}$ and the cumulant problem is the classical regulator problem that minimizes $E(J)$. If $k = 2$, then the cumulant problem is the minimal cost variance control considered in [4, 5]. Interested readers are referred to [1–6] for the investigations, generalizations, and applications of cumulant controls.

Another important cumulant control occurs when $\mu_i = \theta^{i-1}$ for $i \geq 1$. In this case, $\kappa = (1/\theta) \ln E(e^{\theta J})$ is precisely the cumulant generating function, and the cumulant problem is the risk-sensitive control; see, for example, [7]. In this case, (1.4) and (1.5) lead an equation for the matrix function $P(\theta; t) = \sum_{i=1}^{\infty} \theta^{i-1} H_i(t)$:

$$P' + A^T P + PA + P\left(2\theta W - BR^{-1}B'\right)P + Q = 0; \qquad P(\theta;\ t_f) = Q_f. \tag{1.6}$$

As shown in [1], the solution $H_i(t)$ of (1.4) is related to $P(\theta; t)$ by the equation:

$$H_i(t) = \frac{1}{i!}\frac{\partial^i}{\partial\theta^i}[\theta P(\theta; t)]\,|_{\theta=0} \tag{1.7}$$

and the equations in (1.4) for $H_i(t)$ can be obtained by differentiating (1.6) to θ at $\theta = 0$.

For a feedback control $u = Kx$ with a given matrix function K, it was shown in [8] and [1, Theorem 2] that the ith cumulant κ_i of $J(u)$ has the following representation:

$$\kappa_i = i!\left(x_0^T H_i(t_0)x_0 + \int_{t_0}^{t_f} \operatorname{tr}(H_i(s)W(s))\,ds \right), \tag{1.8}$$

where $\{H_i\}_{i=1}^{\infty}$ is the solution of (1.4). Consequently (1.3) can be written as

$$\kappa = \sum_{i=1}^{\infty}\mu_i\left(x_0^T H_i(t_0)x_0 + \int_{t_0}^{t_f} \operatorname{tr}(H_i(s)W(s))\,ds \right). \tag{1.9}$$

In [1] the cumulant control problem was restated as minimizing κ in (1.9) with K as a control, $\{H_i\}_{i=1}^{\infty}$ as a state, and (1.4) a state equation. Furthermore, the following result is proved in [1, Theorem 3].

Theorem 1.1. *If the control $u = Kx$ is the optimal feedback control of* (1.9)*, then the solution $\{H_i\}_{i=1}^{\infty}$ of* (1.4) *and K must satisfy* (1.5)*.*

By Theorem 1.1, it is necessary to solve (1.4) and (1.5) in order to find a solution of a cumulant control problem. Because of the nonlinearity of (1.4) and (1.5) in $\{H_i\}_{i=1}^{\infty}$, a global solution may not exist on the whole horizon $[t_0, t_f]$ of the cumulant problem. This can be illustrated by a scalar case of (1.6). Suppose $A = 0, B = G = \theta = R = Q = Q_f = 1$, then (1.6) becomes $P' + P^2 + 1 = 0$ and $P(t_f) = 1$. The solution is $P(t) = \tan(\pi/4 + t_f - t)$, which is defined on $(s, t_f]$ with $s = t_f - \pi/4$. So (1.6) has no solution unless $t_0 > t_f - \pi/4$.

By the local existence theory of differential equations, the solutions K and $\{H_i\}_{i=1}^{\infty}$ of (1.4) and (1.5) exist on a maximal subinterval $(s, t_f] \subset [t_0, t_f]$. Our interest is to give an estimate for this interval. In particular, we will obtain conditions that guarantee $s = t_0$. The idea of our approach is to show that the trace $\operatorname{tr}(H)$ of

$$H = \sum_{i=1}^{\infty}\mu_i H_i \tag{1.10}$$

satisfies a scalar differential inequality:

$$z' + az^2 + bz + c \geq 0; \qquad z(t_f) = \operatorname{tr}(Q_f) \tag{1.11}$$

with some functions a, b, c on $[t_0, t_f]$. A key of the proof is Proposition 2.1 below. It follows that $\operatorname{tr}(H)$ is bounded by the solution of the Riccati equation:

$$z' + az^2 + bz + c = 0; \qquad z(t_f) = z_f, \tag{1.12}$$

where $z_f = \mathrm{tr}(Q_f)$. Consequently, the existence interval of (1.12) gives an estimate for that of system (1.4) and (1.5); see Theorems 2.4 and 3.5 below. By a similar argument, we prove that the cumulant problem is well posed under appropriate conditions; see Theorems 2.3 and 3.4 below.

In [9] the norm of a solution of a coupled matrix Riccati equation was shown to satisfy a differential inequality similar to (1.11). Consequently, specific sufficient conditions were derived for the existence of solutions of the Riccati equation in [9]. Estimates for maximal existence interval of a classical Riccati equation had been obtained in [10] in terms of upper and lower solutions. For the coupled Riccati equation associated with the minimal cost variance control, some implicit sufficient conditions had been given in [11] for the existence of a solution. In this paper, we use the trace $\mathrm{tr}(H)$ to bound the solution of system (1.4) and (1.5), which generally leads to a better estimate for the existence interval.

2. Comparison Results for Traces

We start with an assumption and some preparations. In this paper we assume that

$$\begin{gathered} Q_f,\ Q(t) \geq 0 \ \text{(positive semidefinite)} \quad \text{for } t \in [t_0, t_f]; \\ R(t) > 0 \ \text{(positive definite)} \quad \text{for } t \in [t_0, t_f]. \end{gathered} \tag{2.1}$$

For the sequence $\mu = \{\mu_i\}_{i=1}^{\infty}$ in (1.3), we will assume that

$$\mu_1 = 1; \qquad \mu_i \geq 0 \quad \text{for } i \geq 1; \quad \rho(\mu) < \infty, \tag{2.2}$$

where

$$\rho(\mu) = \sup\left\{ \frac{\mu_i}{\mu_j \mu_{i-j}},\ i \geq 2, j = 1, \ldots, i-1 \right\}. \tag{2.3}$$

Note that the assumption $\mu_1 = 1$ is not essential. The assumption that $\mu_i \geq 0$ for $i \geq 1$ and Proposition 2.1 below imply that the matrix H in (1.4) is a positive semidefinite series. The requirement that $\rho(\mu) < \infty$ imposes some growth condition for the sequence μ; see the proofs of Theorems 2.3 and 2.4.

Also note that if $\theta = \rho(\mu) < \infty$, then $\mu_i \leq \theta \mu_{i-1}$ and $\mu_i \leq \theta^{i-1}$ for all $i \geq 1$. Theorem 3.4 below shows that the cumulant control problem is well posed for any sequence μ with a small $\rho(\mu)$. Some examples of $\rho(\mu)$ are as follows:

(i) $\rho(\{1, \theta, \theta^2, \theta^3, 0, \ldots\}) = \theta$;

(ii) $\rho(\{1, \mu_2, 0, 0, \ldots\}) = \mu_2$;

(iii) $\rho(\{1, \mu_2, \mu_3, 0, \ldots\}) = \max\{\mu_2,\ \mu_3/\mu_2\}$.

We need the following properties of $\{H_i\}_{i=1}^{\infty}$.

Proposition 2.1. *Suppose $\{H_i\}_{i=1}^{\infty}$ is a solution of* (1.4) *on some interval $[s, t_f]$ with a given K, then each $H_i(t) \geq 0$ for $t \in [s, t_f]$.*

Proof. The formula of the ith cumulant κ_i in [8] implies that κ_i is nonnegative for all x_0. It follows from the representation (1.9) that $H_i(t_0)$ must be positive semidefinite. This argument continues to hold with t_0 replaced by any $s \in [t_0, t_f]$. □

Next we verify some properties related to matrix trace that are needed for our analysis. For $A \in \mathbb{R}^{n\times n}$, denote by $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$ the trace of A, and $\lambda_1(A)$ and $\lambda_n(A)$ the smallest and largest eigenvalue of A, respectively.

Proposition 2.2. (a) *For all* $A, B \in \mathbb{R}^{n\times n}$, $\operatorname{tr}(A) = \operatorname{tr}(A^T)$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(b) *For all* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{S}^n$, $B \geq 0$,

$$\lambda_1(A)\operatorname{tr}(B) \leq \operatorname{tr}(AB) \leq \lambda_n(A)\operatorname{tr}(B). \tag{2.4}$$

(c) *If* $A, B, C \in \mathbb{S}^n$, *then*

$$\operatorname{tr}(ABC) = \operatorname{tr}(ACB). \tag{2.5}$$

(d) *If* $A,\ B,\ C \in \mathbb{S}^n$ *are all* ≥ 0, *then*

$$\operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B), \tag{2.6}$$

$$\operatorname{tr}(ABC) \leq \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(C), \tag{2.7}$$

$$\frac{1}{n}\lambda_1(A)\operatorname{tr}(B)^2 \leq \operatorname{tr}\left(AB^2\right). \tag{2.8}$$

Proof. The properties in (a) are obvious by the definitions of trace and matrix multiplication. Some of the inequalities in (b)–(d) might be known, but the authors were not able to find proofs in existing literature. So we include our proofs of (b)–(d) below for readers' convenience.

To prove (b), let T be a unitary matrix such that $B = T^*DT$ where D is diagonal with eigenvalues $\lambda_i(B)$ of B, $i = 1, \ldots, n$. Then

$$\operatorname{tr}(AB) = \operatorname{tr}(AT^*DT) = \sum_{i=1}^{n}(TAT^*)_{ii}\lambda_i(B), \tag{2.9}$$

where $(TAT^*)_{ii}$ is the entry of TAT^* at (i,i). Let v_i be the ith row of T, which is a unit vector. Then $(TAT^*)_{ii} = v_i A v_i^* \in [\lambda_1(A), \lambda_n(A)]$. Since $\lambda_i(B) \geq 0$, we have

$$\lambda_1(A)\lambda_i(B) \leq (TAT^*)_{ii}\lambda_i(B) \leq \lambda_n(A)\lambda_i(B). \tag{2.10}$$

This implies (2.4).

To show (c), use the symmetry of A, B, and C; we get that $\mathrm{tr}(ABC) = \sum_{ijk} a_{ij} b_{jk} c_{ki} = \sum_{ijk} a_{ji} c_{ik} b_{kj} = \mathrm{tr}(ACB)$.

Inequality (2.6) follows from part (b) and the fact that $\lambda_n(A) \leq \mathrm{tr}(A)$ since $A \geq 0$.

To show (2.7), first note that $\mathrm{tr}(ABC) \leq \lambda_n(AB)\mathrm{tr}(C)$ by (2.4); then it remains to show that $\lambda_n(AB) \leq \lambda_n(A)\lambda_n(B)$. Choose a $v \in \mathbb{R}^n$ with $v^T v = 1$ such that $\lambda_n(AB) = v^T ABv$. By Schwartz inequality,

$$v^T ABv \leq \sqrt{v^T A^2 v}\sqrt{v^T B^2 v} \leq \sqrt{\lambda_n(A^2)}\sqrt{\lambda_n(B^2)}. \tag{2.11}$$

Since $A, B \geq 0$, we have $\lambda_n(A^2) = \lambda_n(A)^2$ and $\lambda_n(B^2) = \lambda_n(B)^2$. Therefore, $\lambda_n(AB) \leq \lambda_n(A)\lambda_n(B)$.

For (2.8), using notations in the proof of (b), we first get

$$\mathrm{tr}\left(AB^2\right) = \mathrm{tr}\left(AT^* D^2 T\right) = \sum_{i=1}^{n} (TAT^*)_{ii} \lambda_i^2(B) \geq \lambda_1(A) \sum_{i=1}^{n} \lambda_i^2(B). \tag{2.12}$$

Then (2.8) follows from the inequalities

$$\sum_{i=1}^{n} \lambda_i^2(B) \geq \frac{1}{n}\left(\sum_{i=1}^{n} \lambda_i(B)\right)^2 \geq \frac{1}{n}\,\mathrm{tr}\,(B)^2. \tag{2.13}$$

□

Now we estimate the existence intervals of solutions of (1.4) and (1.5). First, let K be given and $\{H_i\}_{i=1}^{\infty}$ be the solution of system (1.4). We have the following result, which will be used in the proof of Theorem 3.5

Theorem 2.3. *Suppose $\rho(\mu) < \infty$ and K in* (1.4) *is given. Let a_1, b_1, and c_1 be functions on $[t_0, t_f]$ satisfying*

$$a_1 \geq 2\rho(\mu)\,\mathrm{tr}(W), \qquad b_1 \geq 2\lambda_n(A + BK), \qquad c_1 \geq \mathrm{tr}\left(Q + K^T RK\right). \tag{2.14}$$

(a) *If $\{H_i\}_{i=1}^{\infty}$ is a solution of system* (1.4)*, then $z = \mathrm{tr}(H)$ satisfies the differential inequality*

$$-z' \leq a_1 z^2 + b_1 z + c_1, \qquad z(t_f) = \mathrm{tr}(Q_f). \tag{2.15}$$

(b) *If the equation*

$$-z' = a_1 z^2 + b_1 z + c_1, \qquad z(t_f) = \mathrm{tr}(Q_f) \tag{2.16}$$

has a solution on $[t_0, t_f]$, then system (1.4) *has a solution $\{H_i\}_{i=1}^{\infty}$ on $[t_0, t_f]$ such that $H = \sum_{i=1}^{\infty} \mu_i H_i$ is convergent.*

Proof. (a) Denote $F = A + BK$. Multiplying the equation in (1.4) for H_i by μ_i and sum over i, we obtain

$$H' + F^T H + HF + Q + K^T RK + 2\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} H_j W H_{i-j} = 0. \tag{2.17}$$

Taking traces of both sides of (2.17) gives

$$-z' = \operatorname{tr}\left(F^T H + HF\right) + \operatorname{tr}\left(Q + K^T RK\right) + 2\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} \operatorname{tr}\left(H_j W H_{i-j}\right). \tag{2.18}$$

Note that $W = GG^T \geq 0$ and by Proposition 2.1, H_i, $H \geq 0$ for $t \in [t_f, t_f]$. Proposition 2.2 (a), (b), and (c) imply that

$$\begin{gathered}\operatorname{tr}\left(F^T H + HF\right) = 2\operatorname{tr}(FH) \leq 2\lambda_n(F)\operatorname{tr}(H) = b_1 z, \\ \operatorname{tr}\left(H_j W H_{i-j}\right) \leq \operatorname{tr}(W)\operatorname{tr}\left(H_j\right)\operatorname{tr}\left(H_{i-j}\right).\end{gathered} \tag{2.19}$$

By (1.4) and definition (2.3) of $\rho(\mu)$, we have

$$\begin{aligned} 2\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} \operatorname{tr}\left(H_j W H_{i-j}\right) &\leq 2\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} \operatorname{tr}\left(H_j\right)\operatorname{tr}(W)\operatorname{tr}\left(H_{i-j}\right) \\ &\leq 2\rho(\mu)\operatorname{tr}(W)\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\mu_j\mu_{i-j}\operatorname{tr}\left(H_j\right)\operatorname{tr}\left(H_{i-j}\right) \\ &= 2\rho(\mu)\operatorname{tr}(W)\left(\sum_{i=1}^{\infty}\mu_i\operatorname{tr}(H_i)\right)^2 = 2\rho(\mu)\operatorname{tr}(W)\operatorname{tr}(H)^2 \leq a_1 z^2. \end{aligned} \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18) and using the definition of c_1 in (2.14) we get

$$-z' \leq b_1 z + c_1 + a_1 z^2. \tag{2.21}$$

(b) Suppose that (2.16) has a solution $z^*(t)$ on $[t_0, t_f]$. By local existence theory, system (1.4) has a solution $\{H_i\}_{i=1}^{\infty}$ on a maximal interval $(s, t_f] \subset [t_0, t_f]$. By (a), $\operatorname{tr}(H)$ satisfies inequality (2.15); that is $\operatorname{tr}(H)$ is a lower solution of (2.16). By a comparison theorem of lower-upper solutions, $\operatorname{tr}(H) \leq z^*(t)$ on $(s, t_f]$. Since series (1.10) is positive semidefinite, it follows that H and H_i are all bounded and (1.10) is convergent on $(s, t_f]$. Since $\{H_i\}_{i=1}^{\infty}$ satisfies system (1.4), each H_i is in fact continuously differentiable on $[s, t_f]$. If $s > t_0$, then the local existence theory implies that $\{H_i\}_{i=1}^{\infty}$ can be extended further to the left of s, a contradiction to the maximality of $(s, t_f]$. Therefore $s = t_0$ and (1.4) has a solution on $(t_0, t_f]$, which can be extended to $[t_0, t_f]$. □

Now consider system (1.4) and (1.5). We have

Theorem 2.4. *Denote* $\widehat{R} = BR^{-1}B^T$. *Let* a_2, b_2, *and* c_2 *be functions on* $[t_0, t_f]$ *satisfying*

$$a_2 \geq -\frac{1}{n}\lambda_1\left(\widehat{R}\right) + 2\rho\,\mathrm{tr}(W), \qquad b_2 \geq 2\lambda_n(A), \qquad c_2 \geq \mathrm{tr}(Q). \tag{2.22}$$

(a) *Suppose* K *and* $\{H_i\}_{i=1}^{\infty}$ *are solutions of* (1.4) *and* (1.5) *on some* $(s, t_f]$. *Then on* $(s, t_f]$, $z = \mathrm{tr}(H)$ *satisfies*

$$-z' \leq a_2 z^2 + b_2 z + c_2, \qquad z(t_f) = \mathrm{tr}(Q_f). \tag{2.23}$$

(b) *Suppose the equation*

$$-z' = a_2 z^2 + b_2 z + c_2, \qquad z(t_f) = \mathrm{tr}(Q_f) \tag{2.24}$$

has a solution on $[t_0, t_f]$, *then system* (1.4) *and* (1.5) *have solutions* K *and* $\{H_i\}_{i=1}^{\infty}$ *on* $[t_0, t_f]$.

Proof. (a) Substituting $K = -R^{-1}B^T H$ into system (2.17) we get

$$-H' = A^T H + HA - H\widehat{R}H + Q + 2\sum_{i=2}^{\infty}\mu_i\sum_{j=1}^{i-1}H_j W H_{i-j}, \tag{2.25}$$

where $\widehat{R} = BR^{-1}B^T$. Taking traces of both sides of (2.25) gives

$$-\mathrm{tr}(H') = \mathrm{tr}\left(A^T H + HA\right) - \mathrm{tr}\left(H\widehat{R}H\right) + \mathrm{tr}(Q) + 2\sum_{i=2}^{\infty}\mu_i\sum_{j=1}^{i-1}\mathrm{tr}(H_j W H_{i-j}). \tag{2.26}$$

As in the proof of previous theorem, we have

$$\begin{gathered}\mathrm{tr}\left(A^T H + HA\right) = 2\,\mathrm{tr}(AH) \leq 2\lambda_n(A)\,\mathrm{tr}(H) \leq b_2 z, \\ 2\sum_{i=2}^{\infty}\mu_i\sum_{j=1}^{i-1}\mathrm{tr}(H_j W H_{i-j}) \leq 2\rho(\mu)\,\mathrm{tr}(W) z^2.\end{gathered} \tag{2.27}$$

Using the fact that $H, \widehat{R} \geq 0$ and (2.8), we have

$$\mathrm{tr}\left(H\widehat{R}H\right) = \mathrm{tr}\left(\widehat{R}H^2\right) \geq \frac{\lambda_1\left(\widehat{R}\right)\mathrm{tr}\,(H)^2}{n}. \tag{2.28}$$

By combining (2.28), (2.27), and (2.26) we obtain (2.23).

(b) Suppose that (2.24) has solution $z^*(t)$ on $[t_0, t_f]$. By local existence theory, system (1.5) and (1.4) have solutions K and $\{H_i\}_{i=1}^{\infty}$ on a maximal interval $(s, t_f] \subset [t_0, t_f]$. By part (a), $\mathrm{tr}(H)$ satisfies inequality (2.23) on $(s, t_f]$. It follows that $\mathrm{tr}(H) \leq z^*$ on $(s, t_f]$, which implies that K and $\{H_i\}_{i=1}^{\infty}$ are continuously differentiable on $[s, t_f]$. If $s > t_0$, then local existence theory implies that K and $\{H_i\}_{i=1}^{\infty}$ can be extended further to the left of s, a contradiction to the maximality of $(s, t_f]$. Therefore $s = t_0$, and system (1.4) and (1.5) have solutions on $[t_0, t_f]$. □

3. Well-Posedness and Sufficient Existence Conditions

In this section we will derive specific conditions that ensure that the scalar equations (2.16) and (2.24) have solutions on $[t_0, t_f]$. Consequently we will obtain sufficient conditions for the well-posedness of the cumulant control and the existence of solutions of (1.4) and (1.5).

First we consider an autonomous scalar equation

$$z' + h(z) = 0, \qquad t \leq t_f; \qquad z(t_f) = z_f, \tag{3.1}$$

where z is a polynomial with degree ≥ 2. Assume for some $k \geq 0$ that $h(z)$ has k distinct zeros $z_1 < \cdots < z_k$. Let $z_0 = -\infty$ and $z_{k+1} = \infty$. Since $h(z)$ is locally Lipschitz, the solution $z(t)$ of (3.1) exists and is unique for every z_f for t in a maximal interval, say $(\sigma, t_f]$. If $z_f = z_i$ for some $i = 1, \ldots, k$, then $z(t) = z_i$ for all t. If $z_f \in (z_i, z_{i+1})$ for some $i = 0, 2, \ldots, k$, then $z(t) \in (z_i, z_{i+1})$ for $t \in (\sigma, t_f]$. This implies that for $t \in (\sigma, t_f]$, $-z'(t) = h(z(t))$ has the same sign as $h(z_f)$. In particular, as $t \in (\sigma, t_f]$ decreases, $z(t)$ is strictly increasing if $h(z_f) > 0$ and decreasing if $h(z_f) < 0$. Denote $\widehat{z}_f = \lim_{t \to \sigma+} z(t)$. Then

$$\widehat{z}_f = \begin{cases} z_{i+1} & \text{if } h(z_f) > 0, \\ z_i & \text{if } h(z_f) < 0. \end{cases} \tag{3.2}$$

The following is a well-known fact in stability theory of differential equations.

$$\text{If } h'(z_i) < 0, \quad \text{then } \widehat{z}_f = z_i \quad \text{for every } z_f \in (z_{i-1}, z_{i+1}), \tag{3.3}$$

where $' = d/dz$. Indeed, $h'(z_i) < 0$ implies that $h(z_f) < 0$ for $z_f \in (z_i, z_{i+1})$ and $h(z_f) > 0$ for $z_f \in (z_{i-1}, z_i)$. In either cases, $\widehat{z}_f = z_i$ by (3.2).

Consider (1.12) as an example. We have the following.

Proposition 3.1. *Denote $\Delta = b^2 - 4ac$ and $z_{1,2} = (-b \pm \sqrt{\Delta})/2a$ if $\Delta \geq 0$. Then*

(a) $\widehat{z}_f = -c/b$ *for all* z_f *if* $a = 0$ *and* $b < 0$;

(b) $\widehat{z}_f = z_2$ *if* $\Delta \geq 0$ *and* $a(z_f - z_1) < 0$;

(c) $\widehat{z}_f = \operatorname{sgn}(a)\infty$ *if* $\Delta < 0$ *or* $\Delta \geq 0$ *and* $a(z_f - z_1) > 0$.

Proof. If $a = 0$, then $h(z) = bz + c$, which has root $z_1 = -c/b$. Since $h'(z_1) = b < 0$, $\widehat{z}_f = z_1$ by (3.3). This shows (a).

Next we prove (b) and (c). First assume $a = 1$. If $\Delta < 0$, then $h(z) > 0$ for all z. So $\widehat{z}_f = \infty$ by (3.2). Next consider the case $\Delta \geq 0$, in which $-\infty < z_2 \leq z_1 < \infty$. If $z_f > z_1$, then $h(z_f) > 0$, which implies that $\widehat{z}_f = \infty$. If $z_f < z_1$, then we have either $h(z_f) > 0$ when $z_f < z_2$ or $h(z_f) < 0$ when $z_2 < z_f < z_1$. In either cases, we have $\widehat{z}_f = \infty$ by (3.2). This finishes the proof of (b) and (c) when $a = 1$. If $a \neq 1$, then consider $y = az$, which satisfies $y' + y^2 + by + ac = 0$ and $y(t_f) = az_f$, and the conclusions follow from the special case just proved. □

Write (3.1) as $dz/h(z) + dt = 0$ and integrate it against t from t_f to σ, then we get

$$t_f - \sigma = \int_{z_f}^{\hat{z}_f} \frac{dz}{h(z)}. \tag{3.4}$$

Note that if $\hat{z}_f$ is finite, then $\hat{z}_f$ must be a zero of $h(z)$. It follows that $\int_{z_f}^{\hat{z}_f} [1/h(z)]dz = \infty$ and $(\sigma, t_f] = (-\infty, t_f]$. If $\hat{z}_f$ is infinite, then $\int_{z_f}^{\hat{z}_f} (dz/(h(z)))$ must converge because $h(z)$ has a degree ≥ 2. In summary, we have the following.

Proposition 3.2. *Suppose $h(z)$ is a polynomial of degree ≥ 2. Then*

(a) *$\hat{z}_f$ is finite if and only if the solution $z(t)$ of* (3.1) *exists on $(-\infty, t_f]$.*

(b) *$\hat{z}_f = \pm\infty$ if and only the solution $z(t)$ of* (3.1) *exists on a finite maximal interval $(\sigma, t_f]$ with length $t_f - \sigma = \int_{z_f}^{\pm\infty} [1/h(z)]dz$.*

Applying Proposition 3.2 to (1.12) we obtain the following.

Proposition 3.3. (a) *If either $a = 0$ and $b < 0$, or $a(z_f - z_1) < 0$ and $\Delta \geq 0$, then the solution of* (1.12) *exists on $(-\infty, t_f]$.*

(b) *If either $\Delta < 0$ or $\Delta \geq 0$ and $a(z_f - z_1) < 0$, then the solution of* (1.12) *exists on a finite interval $(\sigma, t_f]$ with*

$$t_f - \sigma = \begin{cases} \dfrac{2}{\sqrt{|\Delta|}}\left(\dfrac{\pi}{2} - \arctan\dfrac{2az_f + b}{\sqrt{|\Delta|}}\right) & \text{if } \Delta < 0, \\ \dfrac{2}{2az_f + b} & \text{if } \Delta = 0,\ a(z_f - z_1) > 0, \\ \dfrac{1}{\sqrt{\Delta}} \ln\left|\dfrac{z_f - z_2}{z_f - z_1}\right| & \text{if } \Delta > 0,\ a(z_f - z_1) > 0. \end{cases} \tag{3.5}$$

Proof. Part (a) directly follows from Proposition 3.1(a) and Proposition 3.2 (a) (b).

For part (b), first assume that $\Delta < 0$. Then $\hat{z}_f = \operatorname{sgn}(a)\infty$ and $az^2 + bz + c = a[(z - z_1)^2 + d^2]$, where $d = \sqrt{|\Delta|}/2|a|$, $z_1 = -b/2a$. So

$$\begin{aligned} t_f - \sigma &= \int_{z_f}^{\hat{z}_f} \frac{dz}{az^2 + bz + c} = \frac{1}{ad} \arctan \frac{z - z_1}{d}\bigg|_{z_f}^{\pm\infty} \\ &= \frac{1}{ad}\left(\pm\frac{\pi}{2} - \arctan\frac{z_f - z_1}{d}\right) = \frac{2}{\sqrt{|\Delta|}}\left(\frac{\pi}{2} - \arctan\frac{2az_f + b}{\sqrt{|\Delta|}}\right). \end{aligned} \tag{3.6}$$

Next assume $\Delta = 0$ and $a(z_f - z_1) > 0$. Then $\hat{z}_f = \operatorname{sgn}(a)\infty$ and $az^2 + bz + c = a(z - z_1)^2$, where $z_1 = -b/(2a)$. We have

$$t_f - \sigma = \int_{z_f}^{\pm\infty} \frac{1}{a(z - z_1)^2} dz = -\frac{1}{a(z - z_1)}\bigg|_{z_f}^{\pm\infty} = \frac{1}{a(z_f - z_1)} = \frac{2}{2az_f + b}. \tag{3.7}$$

Finally when $\Delta > 0$ and $\hat{z}_f = \operatorname{sgn}(a)\infty$, we have $az^2 + bz + c = a(z - z_1)(z - z_2)$ and

$$t_f - \sigma = \int_{z_f}^{\pm\infty} \frac{1}{a(z - z_1)(z - z_2)}\, dz = \frac{1}{a(z_1 - z_2)} \ln\left|\frac{z - z_1}{z - z_2}\right|_{z_f}^{\pm\infty} = \frac{1}{\sqrt{\Delta}} \ln\left|\frac{z_f - z_2}{z_f - z_1}\right|. \tag{3.8}$$

□

Now we show that the cumulant control problem is well posed by Theorem 2.3 and Proposition 3.3.

Theorem 3.4. *For any number $L > 0$ there is $\rho_0 > 0$ such that the series $\sum_{i=1}^{\infty} \mu_i \kappa_i / i!$ in* (1.3) *converges for each matrix K and sequence μ with $\|K\|_\infty < L$ and $\rho(\mu) < \rho_0$.*

Proof. Suppose that K is a matrix function with $\|K\|_\infty < L$. Choose a_1, b_1, and c_1 as follows:

$$a_1 = 2\rho(\mu)\|\operatorname{tr}(W)\|_\infty, \qquad b_1 = 2\|\lambda_n(A + BK)\|_\infty + 1, \qquad c_1 = \left\|\operatorname{tr}\left(Q + K^T RK\right)\right\|_\infty. \tag{3.9}$$

Note that $c_1 \le \|\operatorname{tr}(Q)\|_\infty + L^2\|R\|_\infty$, which depends only on L. In addition, $a_1 \to 0$ as $\rho(\mu) \to 0$. It follows that when $\rho(\mu)$ is sufficiently small, $h(z) = a_1 z^2 + b_1 z + c_1$ has two real roots

$$z_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} < 0 \tag{3.10}$$

with $z_2 \to -c_1/b_1$ and $z_1 \to -\infty$ as $\rho(\mu) \to 0$. In particular, since $z_f \ge 0$, we have $h(z_f) > 0$ and so $\hat{z}_f = \infty$. Proposition 3.3 implies that

$$t_f - \sigma = \frac{1}{\sqrt{b_1^2 - 4a_1c_1}} \ln\left|\frac{z_f - z_2}{z_f - z_1}\right|. \tag{3.11}$$

So $t_f - \sigma \to \infty$ as $\rho(\mu) \to 0$. In particular, (2.16) has a solution $z(t)$ on $[t_0, t_f]$ when $\rho(\mu)$ is sufficiently small. By Theorem 2.3 (b), system (1.4) has a solution $\{H_i\}_{i=1}^{\infty}$ such that H converges.

Finally we apply Proposition 3.3 to (2.24) to give a sufficient existence condition for (1.4) and (1.5) and the cumulant control problem. Choose

$$a_2 = \max_{t_0 \le t \le t_f}\left(-\frac{1}{n}\lambda_1\left(\hat{R}\right) + 2\rho \operatorname{tr}(W)\right), \qquad b_2 = 2\max_{t_0 \le t \le t_f}\lambda_n(A), \qquad c_2 = \|\operatorname{tr}(Q)\|_\infty. \tag{3.12}$$

□

Theorem 3.5. *System* (1.4) *and* (1.5) *have solutions K and $\{H_i\}_{i=1}^{\infty}$ on $[t_0, t_f]$ if*

$$t_f - t_0 \le \int_{z_f}^{\hat{z}_f} \frac{1}{a_2 z^2 + b_2 z + c_2}\, dz, \tag{3.13}$$

where $\widehat{z}_f$ *is defined as in* (3.2) *with* $h(z) = a_2 z^2 + b_2 z + c_2$. *In particular, system* (1.4) *and* (1.5) *have solutions* K *and* $\{H_i\}_{i=1}^{\infty}$ *on* $[t_0, t_f]$ *if one of the following holds.*

(a) $a_2 = -\lambda_1(\widehat{R})/n + 2\rho\,\mathrm{tr}(W) < 0$.

(b) $a_2 > 0$, $b_2 < 0$, $\Delta = b_2^2 - 4a_2c_2 \geq 0$ *and* $z_f < z_1$, *where* $z_1 = (-b_2 + \sqrt{b_2^2 - 4a_2c_2})/2a_2$.

Proof. The general conclusion follows directly from Proposition 3.3 and Theorem 2.4. In the case (a), $h(z) = 0$ has two roots $z_1 < 0 < z_2$. Since $z_f = \mathrm{tr}(Q_f) \geq 0$, $a_2(z_f - z_1) < 0$. In the case (b), $h(z) = 0$ has two solutions $z_2 \leq z_1$. So $a_2(z_f - z_1) < 0$ also holds. The conclusion follows from Propositions 3.1 and 3.2 and Theorem 2.4. □

Note that in Theorem 3.5 condition (a) holds if B has full rank (i.e., $\lambda(\widehat{R}) > 0$) and $\rho(\mu)$ is sufficiently small, while condition (b) holds if the system in (1.1) is stable (i.e., $b_2 < 0$) and the product $\rho(\mu)\|\mathrm{tr}(Q)\|_\infty$ is relatively small. The cumulant control problem has an optimal control under each of these conditions.

As an existence theorem, Theorem 3.5 gives one of the very few existence results for a Riccati differential system of infinitely many equations. In terms of the cumulant controls that lead to the system (3) and (4), Theorem 3.5 generalizes the corresponding results in [1, 5] for risk-sensitive controls (where $\mu = \{1, \theta, \theta^2, \ldots\}$) and in [2, 4] for finite cumulant controls (where μ has only finite nonzero components). Numerical examples for risk-sensitive and finite cumulant controls satisfying the conditions in Theorem 3.5 may be found in [3–6].

4. Conclusions

In general it is very difficult to determine the existence interval of a differential Riccati equation (or system). By the approach in this paper, we can at least give an estimate for the existence interval of the Riccati system. Such an estimate leads to sufficient conditions for the existence of solutions to the Riccati system and the cumulant control problem.

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