Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 575383, 26 pages
doi:10.1155/2011/575383

Research Article

Malliavin Calculus of Bismut Type for Fractional Powers of Laplacians in Semi-Group Theory

Rémi Léandre

Institut de Mathématiques, Université de Bourgogne, 21000 Dijon, France

Correspondence should be addressed to Rémi Léandre, remi.leandre@u-bourgogne.fr

Received 2 March 2011; Revised 16 May 2011; Accepted 2 June 2011

Academic Editor: Nikolai Leonenko

We translate into the language of semi-group theory Bismut's Calculus on boundary processes (Bismut (1983), Lèandre (1989)) which gives regularity result on the heat kernel associated with fractional powers of degenerated Laplacian. We translate into the language of semi-group theory the marriage of Bismut (1983) between the Malliavin Calculus of Bismut type on the underlying diffusion process and the Malliavin Calculus of Bismut type on the subordinator which is a jump process.

1. Introduction

Let $X_0^1, X_1^1, \ldots, X_1^m, X_0^2, X_1^2, \ldots, X_2^m$ be $2m+2$ vector fields on $\mathbb{R}^d$ with bounded derivatives at each order. Let

$$\mathbb{L}^1 = \frac{\partial}{\partial s} + X_0^1 + \frac{1}{2}\sum_{i>0}\left(X_i^1\right)^2 \tag{1.1}$$

be an Hoermander's type operator on $\mathbb{R}^{1+d}$. Let

$$\mathbb{L}^2 = \frac{\partial}{\partial s} + X_0^2 + \frac{1}{2}\sum_{i>0}\left(X_i^2\right)^2 \tag{1.2}$$

be a second Hoermander's operator on $\mathbb{R}^{1+d}$. Bismut [1] considers the generator

$$\mathbb{A} = -\frac{1}{2}\sqrt{-2\mathbb{L}^1} - \frac{1}{2}\sqrt{-2\mathbb{L}^2} \tag{1.3}$$

and the Markov semi-group $\exp[t\mathbb{A}]$. This semi-group has a probabilistic representation. We consider a Brownian motion $t \to z_t$ independent of the others Brownian motions B_t^i. Bismut introduced the solution of the stochastic differential equation starting at x in Stratonovitch sense:

$$dx_t(x) = \mathbb{I}_{z_t<0}\left(X_0^1(x_t(x))dt + \sum_{i>0} X_i^1(x_t(x))dB_t^i \right) + \mathbb{I}_{z_t>0}\left(X_0^2(x_t(x))dt + \sum_{i>0} X_i^2(x_t(x))dB_t^i \right), \tag{1.4}$$

where $t \to B_t^i$ are m independent Brownian motions.

Let us introduce the local time $t \to L_t$ associated with $t \to z_t$ and its right inverse $t \to A_t$ (see [2, 3]). Then,

$$\exp[t\mathbb{A}]f(0,x) = E\left[f(A_t, x_{A_t}(x))\right]. \tag{1.5}$$

Such operator is classically related to the Dirichlet Problem [3].

Classically [4],

$$\exp\left[t\mathbb{L}^1\right]f(x) = E\left[f\left(x_t^1(x)\right)\right], \tag{1.6}$$

where $x_t^1(x)$ is the solution of the Stratonovitch differential equation starting at x:

$$dx_t^1(x) = X_0^1\left(x_t^1(x)\right)dt + \sum X_i^1\left(x_t^1(x)\right)dB_t^i, \tag{1.7}$$

The question is as following: is there an heat-kernel associated with the semi-group $\exp[t\mathbb{L}^1]$? This means that

$$\exp\left[t\mathbb{L}^1\right]f(x) = \int_{\mathbb{R}^d} f(y)p_t(x,y)dy. \tag{1.8}$$

There are several approaches in analysis to solve this problem, either by using tools of microlocal analysis or tools of harmonic analysis. Malliavin [5] uses the probabilistic representation of the semi-group. Malliavin uses a heavy apparatus of functional analysis (number operator on Fock space or equivalently Ornstein-Uhlenbeck operator on the Wiener space, Sobolev spaces on the Wiener space) in order to solve this problem.

Bismut [6] avoids using this machinery to solve this hypoellipticity problem. In particular, Bismut's approach can be adapted immediately to the case of the Poisson process [7]. The main difficulty to treat in the case of a Poisson process is the following: in general the solution of a stochastic differential equation with jumps is not a diffeomorphism when the starting point is moving (see [8–10]).

The main remark of Bismut in [1] is that if we consider the jump process $t \to x_{A_t}^1(x)$, then it is a diffeomorphism almost surely in x. So, Bismut mixed the tools of the Malliavin

Calculus for diffusion (on the process $t \to x_t^1(x)$) and the tools of the Malliavin Calculus for Poisson process (on the jump process $t \to A_t$) in order to show that this is the problem if

$$E\left[f(A_t, x_{A_t}(x))\right] = \int_{\mathbb{R}^{1+d}} q_t(s,y) f(s,y)\, ds\, dy. \tag{1.9}$$

Developments on Bismut's idea was performed by Léandre in [9, 11]. Let us remark that this problem is related to study the regularity of the Dirichlet problem (see [1, page 598]) (see [12–14] for related works).

Recently, we have translated into the language of semi-group theory the Malliavin Calculus of Bismut type for diffusion [15]. We have translated in semi-group theory a lot of tools on Poisson processes [16–22]. Especially, we have translated the Malliavin Calculus of Bismut type for Poisson process in semi-group theory in [17]. It should be tempting to translate in semi-group theory Bismut's Calculus on boundary process. It is the object of this work.

On the general problematic on this work, we refer to the review papers of Léandre [23–25]. It enters in the general program to introduce stochastic analysis tools in the theory of partial differential equation (see [26–28]).

2. Statements of the Theorems

Let us recall some basis on the study of fractional powers of operators [29]. Let $\mathbb{L}$ be a generator of a Markovian semi-group P_s. Then,

$$-\sqrt{-\mathbb{L}} = C\int_0^\infty s^{-3/2}(P_s - \mathbb{I})ds. \tag{2.1}$$

The results of this paper could be extended to generators of the type

$$\mathbb{A} = \int_0^\infty g(s)(P_s - \mathbb{I})ds, \tag{2.2}$$

where $\int_0^\infty g(t) \wedge 1dt < \infty$ and $g \geq 0$, but we have chosen the operator of the type (1.3) to be more closely related to the original intuition on Bismut's Calculus on boundary process. Let be $\mathbb{E}_d = \mathbb{R}^{1+d} \times \mathbb{G}_d \times \mathbb{M}_d$ where $\mathbb{G}_d$ is the space of invertible matrices on $\mathbb{R}^d$ and $\mathbb{M}_d$ the space of symmetric matrices on $\mathbb{R}^d$. (s, x, U, V) is the generic element of $\mathbb{E}_d$. V is called the Malliavin matrix.

On $\mathbb{E}_d$, we consider the vector fields:

$$\begin{aligned}\widehat{X}_i^1 &= \left(0, X_i, DX_i^1(x)U, 0\right),\\ \widehat{Y}^1 &= \left(0,0,0,\sum_{i=1}^m \left\langle U^{-1}X_i, \cdot\right\rangle^2\right).\end{aligned} \tag{2.3}$$

We consider the Malliavin generator $\widehat{\mathbb{L}}^1$ on $\mathbb{E}_d$:

$$\widehat{L}^1 = \frac{\partial}{\partial s} + \widehat{X}_0^1 + \frac{1}{2}\sum_{i>0}\left(\widehat{X}_i^1\right)^2 + \widehat{Y}^1. \tag{2.4}$$

We consider the Malliavin semi-group $\widehat{P}_t^1$ associated and $\sqrt{-\widehat{L}^1}$.

We perform the same algebraic considerations on $\mathbb{L}^2$. We get $\widehat{L}^2$, $\widehat{P}_t^2$, and $\sqrt{-\widehat{L}^2}$. Let us consider the total generator

$$\widehat{\mathbb{A}} = -\sqrt{-\widehat{L}^1} - \sqrt{-\widehat{L}^2} \tag{2.5}$$

and the Malliavin semi-group $\exp[t\widehat{\mathbb{A}}]$.

We get a theorem which enters in the framework of the Malliavin Calculus for heat-kernel.

Theorem 2.1. *Let one suppose that the Malliavin condition in x is checked:*

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\det V^{-p}\right](0,x,I,0) < \infty \tag{2.6}$$

holds for all p, then

$$\exp[t\mathbb{A}]f(0,x) = \int_{\mathbb{R}^{1+d}} f(s,y)q_t(s,y)ds\,dy, \tag{2.7}$$

where $q_t(s,y)$ is the density of a probability measure on $\mathbb{R}^{1+d}$.

Theorem 2.2. *If the quadratic form*

$$\sum_{i>0}\left\langle X_i^1(x),\cdot\right\rangle^2 + \sum_{i>0}\left\langle X_i^2(x),\cdot\right\rangle^2 \tag{2.8}$$

is invertible in x, then the Malliavin condition holds in x.

Remark 2.3. We give simple statements to simplify the exposition. It should be possible by the method of this paper to translate the results of [9, part III], got by using stochastic analysis as a tool.

3. Integration by Parts on the Underlying Diffusion

We consider the vector fields on $\mathbb{R}^{1+d+1}$,

$$X_{i,s,t}^{j,1} = \left(0, X_i^j(x), Z_{i,s,t}^j\right), \tag{3.1}$$

where $Z^j_{i,s,t} = \langle \phi(x), h^j_{s,t} \rangle_i$ ($\phi(x)$ is a convenient matrix on $\mathbb{R}^m$ which depends smoothly on x and whose derivatives at each order are bounded. $(s,t) \to h^j_{s,t}$ does not depend on x, and $h^j_{s,t}$ belong to $\mathbb{R}^m$). Let $\tilde{f}$ be a smooth function on $\mathbb{R}^{1+d+1}$, $\tilde{D}\tilde{f}$ denotes its gradient, and $\tilde{D}^2\tilde{f}$ denotes its Hessian.

We consider the generator $\mathbb{L}^{j,1}_{s,t}$ acting on smooth functions on $\mathbb{R}^{1+d+1}$,

$$\begin{aligned}\mathbb{L}^{j,1}_{s,t}\tilde{f} = \frac{\partial}{\partial s}\tilde{f} + \left\langle X^j_0(x), \tilde{D}\tilde{f} \right\rangle + \frac{1}{2}\sum_{i>0}\left\langle DX^j_i(x)X^j_i(x), \tilde{D}\tilde{f} \right\rangle \\ + \frac{1}{2}\sum_{i>0}\left\langle X^{j,1}_{i,s,t}, \tilde{D}^2\tilde{f}, X^{j,1}_{i,s,t} \right\rangle.\end{aligned} \tag{3.2}$$

In (3.2), the generator is written under Itô's form. It generates a time inhomogeneous in the parameter s semi-group $P^{j,1}_{s,t}$. We can consider

$$-\sqrt{-\mathbb{L}^{j,1}_{\cdot,t}} = C\int_0^\infty s^{-3/2}\left(P^{j,1}_{s,t} - \mathbb{I}\right)ds. \tag{3.3}$$

We put

$$\mathbb{A}^1_t = -\sqrt{-\mathbb{L}^{j,1}_{\cdot,t}} - \sqrt{-\mathbb{L}^{j,2}_{\cdot,t}}. \tag{3.4}$$

It generates a semi-group P^1_t.

Let us consider the Hoermander's type generator associated with the smooth Lipschitz vector fields on $\mathbb{R}^{1+d+d}$((s,x,U) on $\mathbb{R}^{1+d+d}$):

$$\begin{aligned}X^{j,2}_i &= \left(0, X^j_i, DX^j_i U\right), \\ Y^{j,2}_{0,s,t} &= \left(0, 0, \sum X^j_i(x)Z^j_{i,s,t}\right) = \left(0,0,Y^j_{i,s,t}\right), \\ \mathbb{L}^{j,2}_{s,t} &= X^{j,2}_0 + \frac{1}{2}\sum_{i>1}\left(X^{j,2}_i\right)^2 + Y^{j,2}_{0,s,t}.\end{aligned} \tag{3.5}$$

We consider the heat semi-group associated with $\mathbb{L}^{j,2}_{s,t}$

$$\frac{\partial}{\partial s}P^{j,2}_{s,t}\tilde{f} = \mathbb{L}^{j,2}_{s,t}P^{j,2}_{s,t}\tilde{f}. \tag{3.6}$$

Let us recall [15, Theorem 2.2] that

$$P^{j,1}_{s,t}[uf](s_0,x_0,0) = P^{j,2}_{s,t}[\langle Df, U\rangle](s_0,x_0,0), \tag{3.7}$$

where f depends only on (s,x). In the left-hand side of (3.7), we apply the enlarged semi-group to the test function $(s,x,u) \to f(s,x)u$ and in the right-hand side we apply the semi-group to the test function $(s,x,U) \to \langle Df,U\rangle$. u belongs to $\mathbb{R}$ and U belongs to $\mathbb{R}^d$. From this, we deduce the following.

Lemma 3.1. *One has the relation*

$$-\mathbb{L}^{j,1}_{\cdot,t}[uf](s_0,x_0,0) = -\mathbb{L}^{j,2}_{\cdot,t}[\langle Df,U\rangle](s_0,x_0,0). \tag{3.8}$$

Let us consider the semi-group P^2_t associated with

$$\mathbb{A}^2_t = -\sqrt{-\mathbb{L}^{1,2}_{\cdot,t}} - \sqrt{-\mathbb{L}^{2,2}_{\cdot,t}}. \tag{3.9}$$

We get, with the same notations for (s,x,u,U) the following.

Theorem 3.2. *For f bounded continuous with compact support in (s,x), one has the following relation:*

$$P^2_t[\langle Df,U\rangle](s_0,x_0,0) = P^1_t[fu](s_0,x_0,0). \tag{3.10}$$

Proof. For the integrability conditions, we refer to the appendix.

We remark that $\partial/\partial u$ commute with $\mathbb{A}^1_t$, therefore with P^1_t. We deduce that

$$P^1_t[fu](s_0,x_0,u_0) = u_0\exp[t\mathbb{A}][f](s_0,x_0) + P^1_t[fu](s_0,x_0,0). \tag{3.11}$$

By the method of variation of constants,

$$P^1_t[fu](s_0,x_0,0) = \int_0^t \exp[(t-s)\mathbb{A}]\Big[\mathbb{A}^1_s[u\exp[s\mathbb{A}][f](\cdot,\cdot,0)]\Big](s_0,x_0)ds. \tag{3.12}$$

In order to show that, we follow the lines of (2.17) and (2.18) in [15]. We apply $\mathbb{A}^1_t$ to (3.11).

By Lemma 3.1,

$$\mathbb{A}^1_s[u\exp[s\mathbb{A}][f](\cdot,\cdot)](s_1,x_1,0) = \mathbb{A}^2_s[\langle D(\exp[s\mathbb{A}]),U\rangle](s_1,x_1,0). \tag{3.13}$$

Let us consider the vector fields on $\mathbb{R}^{1+d}\times\mathbb{G}_d$,

$$X^{j,3}_i = \left(0, X^j_i, DX^j_iU\right). \tag{3.14}$$

We consider the Hoermander's type operator associated with these vector fields:

$$\mathbb{L}^{j,3} = X^{j,3}_0 + \frac{1}{2}\sum_{i>0}\left(X^{j,3}_i\right)^2. \tag{3.15}$$

We consider the generator

$$\mathbb{A}_t^3 = -\sqrt{-\mathbb{L}_{s,t}^{1,3}} - \sqrt{-\mathbb{L}_{s,t}^{2,3}}. \tag{3.16}$$

It generates a semi-group P_t^3. By lemma 3.2 of [15], we have

$$D\ \exp[s\mathbb{A}][f](s_1, x_1) = P_s^3[DfV](s_1, x_1, I). \tag{3.17}$$

By [15, Equation (3.18)],

$$P_{s,t}^{j,2}\left[P_t^3[DfU](\cdot, I)V\right](s_1, x_1, 0) = \sum_i \int_0^s P_{s-v,t}^j \left[\sum_i \left\langle Y_{i,v,t}^j, P_{v,t}^{j,3}\left[P_t^3[DfU](\cdot, I)\right]\right\rangle\right](s_1, x_1, 0). \tag{3.18}$$

In [15, Equation (3.18)], we consider the semi-group $\overline{P}_t^j$ instead of the semi-group $P_{s,t}^{j,2}$ and the test function Df instead as of the test function $P_t^3[DfU](\cdot, I)$ here. $Y_{i,v,t}^j$ is considered as an element of $\mathbb{R}$ and not as a one-order differential operator:

$$\frac{\partial}{\partial s} P_{s,t}^{j,3} \tilde{f} = \mathbb{L}_{s,t}^{j,3} P_{s,t}^{j,3} \tilde{f}. \tag{3.19}$$

Therefore,

$$\begin{aligned} &\mathbb{A}_t^2\left[P_t^3[DfV](\cdot, I)V\right](s_1, x_1, 0) \\ &\quad = \sum_{i,j} C \int_0^\infty s^{-3/2} \int_0^s P_{s-v,t}^j \left[\left\langle \sum_i Y_{i,v,t}^j, P_{v,t}^{j,3}\left[P_t^3[DfU](\cdot, I)\right]\right\rangle\right](s_1, x_1, 0)\, dv\, ds. \end{aligned} \tag{3.20}$$

We write

$$\mathbb{A}_t^2 = \mathbb{A}_t^3 + \widetilde{A}_t^3, \tag{3.21}$$

where

$$\widetilde{A}_t^3[fU](s_0, x_0, U_0) = \sum_j C \int_0^\infty s^{-3/2}\left(P_{s,t}^{j,2} - P_{s,t}^{j,3}\right)[fU](s_0, x_0, U_0)\, ds. \tag{3.22}$$

The Volterra expansion (see [15, Equation (3.17)]) if it converges gives the following formula:

$$\begin{aligned} P_{s,t}^{j,2}[fU](s_0, x_0, U_0) = \sum \int_{0<s_1<s_2<\cdots<s_n<t} & ds_1 \cdots ds_n P_{s_1}^{j,3} \sum Y_{i,s_1,t}^j \cdots P_{s_n-s_{n-1}}^{j,3} \\ & \times \sum Y_{i,s_n,t}^j \cdots P_{t-s_n}^{j,3}[fU](s_0, x_0, U_0). \end{aligned} \tag{3.23}$$

But $u_0 \to P^{j,3}_{s,t}[fU](s_0,x_0,U_0)$ is linear in u_0. Therefore:

$$P^{j,2}_{s,t}[fU](s_0,x_0,U_0) = P^{j,3}_{s,t}[fU](s_0,x_0,U_0) + \int_0^s P^j_v\left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}[fU]\right\rangle(s_0,x_0,U_0)dv. \tag{3.24}$$

In this last formula, $Y^j_{i,s,t}$ are considered as differential operators.

Therefore, $\widetilde{A}^3_t[fU](s_0,x_0,U_0)$ does not depend on U_0 and is equal to

$$\sum_{i,j} C\int_0^\infty s^{-3/2}\int_0^s P^j_v\left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}[fU](s_0,x_0,I)\right\rangle ds\,dv, \tag{3.25}$$

where $Y^j_{i,s,t}$ are considered as elements of $\mathbb{R}^d$. We deduce as in [15, Equation (3.17)],

$$P^2_t[fU](s_0,x_0,0) = \int_0^t \exp[(t-s)\mathbb{A}]\widetilde{A}^3_s P^3_s[fU](s_0,x_0,0)ds. \tag{3.26}$$

But $U_0 \to P^3_s[fU](s_0,x_0,U_0)$ is linear. Therefore,

$$\widetilde{A}^3_t P^3_t[fU](s_0,x_0,0) = \sum_{i,j} C\int_0^\infty s^{-3/2}\int_0^s P^j_v\left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}\left[P^3_t[fU]\right](s_0,x_0,I)\right\rangle ds\,dv. \tag{3.27}$$

It remains to replace f by Df in this last equation and to compare (3.26) with (3.13) and (3.20). □

We consider the Malliavin generator $\widehat{\mathbb{A}}$. We can perform the same algebraic construction as in Theorem 3.2. We get two semi-groups $\widehat{P}^2_t$ and $\widehat{P}^1_t$. $\widehat{Y}^j_{i,s,t}$ and $\widehat{Z}^j_{i,s,t}$ are smooth with bounded derivatives in $\widehat{x} = (x, U, U^{-1}, V)$. We get by the same procedure the following.

Theorem 3.3. *If $\widehat{f}$ is bounded with bounded derivatives and with compact support in s, then one gets*

$$\widehat{P}^2_t\left[\left\langle D\widehat{f},\widehat{U}\right\rangle\right](s_0,\widehat{x},0) = \widehat{P}^1_t\left[\widehat{f}\widehat{u}\right](s_0,\widehat{x},0), \tag{3.28}$$

where one take does not derivative in the direction of s in $D\widehat{f}$.

We can perform the same improvements as in [15, page 512]. We define on $\mathbb{R}^d \times \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ some vectors fields:

$$X^{j,\text{tot}}_i = \left(X^{j,1}_i(x_1),\ldots,X^{j,l}_i(x_1,\ldots,x_l), X^{j,k}_i(x_1,\ldots,x_k)\right), \tag{3.29}$$

where

$$X_i^{j,l}\left(x^1,\dots,x^l\right) = X_{1,i}^{j,l}\left(x^1,\dots,x^{l-1}\right)x^l\frac{\partial}{\partial x^l} + X_{2,i}^l\left(x^1,\dots,x^l\right)\frac{\partial}{\partial x^l} + X_{3,i}^l\left(x^1,\dots,x^{l-1}\right) \tag{3.30}$$

where $X_{1,i}^{j,l}$, $X_{2,i}^{j,l}$ have derivatives bounded at each order and $X_{3,i}^{j,l}$ has derivative with polynomial growth.

We can consider the generator $\widehat{\mathbb{A}}^{\text{tot}}$ associated with these vector fields and perform the same algebraic computations as in Theorem 3.2. We get two semi-groups $\widehat{P}_t^{2,\text{tot}}$ and $\widehat{P}_t^{1,\text{tot}} \cdot \widehat{Y}_{i,s,t}^j$ and $\widehat{Z}_{i,s,t}^j$ are smooth with bounded derivatives in $\widehat{x} = (x, U, U^{-1}, V)$. We get by the same procedure the following.

Theorem 3.4. *If $\widehat{f}^{\text{tot}}$ is bounded with bounded derivatives and with compact support in s, then one gets*

$$\widehat{P}_t^{2,\text{tot}}\left[\left\langle D\widehat{f}^{\text{tot}}, \widehat{U}\right\rangle\right](s_0, \widehat{x}^{\text{tot}}, 0) = \widehat{P}_t^{1,\text{tot}}\left[\widehat{f}^{\text{tot}}\widehat{u}\right](s_0, \widehat{x}^{\text{tot}}, 0), \tag{3.31}$$

where $D\widehat{f}^{\text{tot}}$ does not include derivative in the direction of s.

We refer to the appendix for the proof and the subsequent estimates.

Remark 3.5. Let us show from where come these identities, by using (1.4): we consider a time interval $[A_{t-}, A_t]$. On this random time interval, we do the following translation on the leading Brownian motion B_s^i:

(i) if $z_s > 0$ on this time interval, then dB_s^i is transformed in $dB_s^i + \lambda\langle\phi(x_s), h_{s,t}^2\rangle_i ds$ for a small parameter λ,

(ii) if $z_s < 0$ on this time interval, then dB_s^i is transformed in $dB_s^i + \lambda\langle\phi(x_s), h_{s,t}^1\rangle_i ds$ for a small parameter λ.

According to the fact that f has compact support (this means that we consider bounded values of A_t), the transformed Brownian motion has an equivalent law through the Girsanov exponential to the original Brownian motions. The term in u in Theorem 3.2 come that from the fact we take the derivative in $\lambda = 0$ of the Girsanov exponential. When we do this transformation, we get a random process $x_t^\lambda(x)$. Derivation of it in $\lambda = 0$ is done classically according to the stochastic flow theorem, which leads to the study of generators of the type $\mathbb{L}_{s,t}^{j,2}$ and of the type $\mathbb{L}^{j,3}$.

4. Integration by Parts on the Subordinator

Let us consider diffusion type generator of the previous part:

$$\begin{aligned} \mathbb{L} &= Y_0 + \frac{1}{2}\sum Y_i^2, \\ \mathbb{L}^{\sqrt{t}} &= \left(\sqrt{t}\right)^2 Y_0 + \frac{1}{2}\sum_{i>0}\left(\sqrt{t}Y_i\right)^2. \end{aligned} \tag{4.1}$$

Let us consider the semi-group

$$\frac{\partial}{\partial t}P_t = \mathbb{L}P_t \tag{4.2}$$

and the semi-group

$$\frac{\partial}{\partial s}P_s^{\sqrt{t}} = \mathbb{L}^{\sqrt{t}}P_s^{\sqrt{t}}. \tag{4.3}$$

We have classically

$$P_t = P_1^{\sqrt{t}}, \tag{4.4}$$

where the smooth vector fields are Lipschitz.

Therefore, we can write

$$-\sqrt{-\mathbb{L}} = C\int_0^\infty s^{-3/2}\left(P_1^{\sqrt{s}} - \mathbb{I}\right)ds. \tag{4.5}$$

We consider a diffeomorphsim $f_\lambda(s)$ of $[0,\infty[$ with bounded derivative of first order in λ equal to s if $s < \epsilon$ and equals to s if $s > 2$ (we suppose λ small). We can write

$$\sqrt{-\mathbb{A}^\lambda} = C\int_0^\infty \left(\left(f_\lambda(s)\right)^{-3/2}P_1^{\sqrt{f_\lambda(s)}} - s^{-3/2}\mathbb{I}\right)ds. \tag{4.6}$$

We do this operation on the two operators on $\mathbb{R}^{1+d}$ giving $\mathbb{A}$. We get a generator $\mathbb{A}^\lambda$.

According the line of stochastic analysis, we consider a generator $\mathbb{A}^{\lambda,1}$ on $\mathbb{R}^{1+d+1}$. If $\mathbb{L}^1$ is a generator on $\mathbb{R}^{1+d}$ with associated semi-group P_s, then we consider $\mathbb{A}^{\lambda,1}$ the generator on $\mathbb{R}^{1+d+1}$,

$$\mathbb{A}^{\lambda,1}f(s_0,x_0,u_0) = \sum_j C\int_0^\infty \left(f_\lambda(s)^{-3/2}\left[P_1^{j,\sqrt{f_\lambda(s)}}f(s_0,x_0,u_0J_\lambda(s))\right] - s^{-3/2}f(s_0,x_0,u_0)\right)ds, \tag{4.7}$$

where $J_\lambda(s)$ is the Jacobian of the transformation $s \to f_\lambda(s)$. By doing this procedure in (1.3), we deduce a global generator $\mathbb{A}^{\lambda,1}$ and a semi-group $P_t^{\lambda,1}$ associated with it.

It is not clear that $P_t^{\lambda,1}$ is a Markovian semi-group. We decompose

$$\mathbb{A}^{\lambda,1} = \mathbb{A}^{\lambda,1,\epsilon} + \mathbb{A}^{\lambda,1,\epsilon^c}, \tag{4.8}$$

where

$$\mathbb{A}^{\lambda,1,\epsilon}f(s_0,x_0,u_0) = \sum_j C\int_0^\epsilon \left(s^{-3/2}P_1^{j,\sqrt{s}}f(s_0,x_0,u_0) - s^{-3/2}f(s_0,x_0,u_0)\right)ds. \tag{4.9}$$

$\mathbb{A}^{\lambda,1,\epsilon}$ generates a Markovian semi-group $P_t^{\lambda,1,\epsilon}$. But $\mathbb{A}^{\lambda,1,\epsilon^c}$ is a bounded operator on the set of bounded continuous functions on $\mathbb{R}^{1+d+1}$ endowed with the uniform norm. The Volterra expansion converges on this set:

$$P_t^{\lambda,1} f(s_0, x_0, u_0) = P_t^{\lambda,1,\epsilon} f(s_0, x_0, u_0) + \sum_n \int_{0<s_1<\cdots<s_n<t} P_{s_1}^{\lambda,1,\epsilon} \mathbb{A}^{\lambda,1,\epsilon^c} P_{s_2-s_1}^{\lambda,1,\epsilon} \cdots \mathbb{A}^{\lambda,1,\epsilon^c} P_{t-s_n}^{\lambda,1,\epsilon} ds_1 \cdots ds_n. \tag{4.10}$$

Theorem 4.1 (Girsanov). *For f with compact support in (s,x), one has*

$$P_t^{\lambda,1}[uf](s_0, x_0, 1) = \exp[t\mathbb{A}][f](s_0, x_0). \tag{4.11}$$

Proof. By linearity,

$$P_t^{\lambda,1}[uf](s_0, x_0, u_0) = u_0 P_t^{\lambda,1}[uf](s_0, x_0, 1). \tag{4.12}$$

But by an elementary change of variable,

$$\mathbb{A}^{\lambda,1}\left[u P_t^{\lambda,1}[uf](\cdot,\cdot,1)\right] = \mathbb{A}\left[P_t^{\lambda,1}[uf](\cdot,\cdot,1)\right]. \tag{4.13}$$

The result holds by the unicity of the solution of the parabolic equation associated with $\mathbb{A}$. To state the integrability of u, we refer to [16]. □

Remark 4.2. Let us show from where this formula comes. In the previous part, we have done a perturbation of the leading Brownian motion B_t^i. Here, we do a perturbation of ΔA_s into $f_\lambda(\Delta A_s) = \Delta A_s^\lambda$. By standard result on Levy processes, the law of the Levy process A_t^λ is equivalent to the law of A_t. Moreover, A_t^λ and B_t^i are independents. Therefore, the result.

Bismut's idea to state hypoellipticity result is to take the derivative in λ of

$$P_t^{\lambda,1}[uf](s_0, x_0, 1) = \exp[t\mathbb{A}][f](s_0, x_0) \tag{4.14}$$

in order to get an integration by parts.

First of all, let us compute $(\partial/\partial\lambda) P_t^{\sqrt{f_\lambda(s)}} f$ in $\lambda = 0$. It is fulfilled by the next considerations. Let us consider a generator written under Hoermander's form:

$$\mathbb{L}^\lambda = g_\lambda Y_0 + \frac{1}{2} g_\lambda^2 \sum_{i>0} Y_i^2, \tag{4.15}$$

where g_λ are smooth and where the vector fields Y_i are smooth Lipschitz on $\mathbb{R}^{\tilde{d}}$. We consider the semi-group $P_t^{\lambda,\cdot}$ associated with it. Let us introduce the vector fields on $\mathbb{R}^{\tilde{d}+\tilde{d}}$:

$$Y_i^{\lambda,1} = \left(g_\lambda Y_i, g_\lambda DY_i U + \frac{d}{d\lambda} g_\lambda Y_i\right). \tag{4.16}$$

Let us consider the diffusion generator

$$\mathbb{L}^{\lambda,1} = Y_0^{\lambda,1} + \frac{1}{2}\sum_{i>0}\left(Y_i^{\lambda,1}\right)^2. \tag{4.17}$$

Associated with it there is a semi-group $P_t^{\lambda,1,\cdot}$.

Proposition 4.3. *For f smooth with compact support, one has*

$$\frac{\partial}{\partial\lambda}P_t^{\lambda,\cdot}[f](\tilde{x}) = P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0). \tag{4.18}$$

Proof. Let us introduce the vector fields on $\mathbb{R}^{\tilde{d}+\tilde{d}}$:

$$Y_i^{\lambda,2} = (g_\lambda Y_i, g_\lambda DY_i U) \tag{4.19}$$

and the generator

$$\mathbb{L}^{\lambda,2} = Y_0^{\lambda,2} + \frac{1}{2}\sum_{i>0}\left(Y_i^{\lambda,2}\right)^2. \tag{4.20}$$

Associated with it there is a semi-group $P_t^{\lambda,2,\cdot}$.

If the Volterra expansion converges, then

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0) = \sum\int_{0<s_1<\cdots<s_n<t} & ds_1\cdots ds_n P_{s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right) \\ & \times P_{s_2-s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)\cdots\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},0). \end{aligned} \tag{4.21}$$

But $\tilde{U}_0 \to P_t^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ is linear in $\tilde{U}_0$ and therefore the quantity $(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2})P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ does not depend on $\tilde{U}_0$. Therefore the Volterra expansion reads

$$P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0) = \int_0^t P_s^{\lambda}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)P_{t-s}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},0). \tag{4.22}$$

Let us compute $\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}$. It is equal to

$$\sum_{i>0} g_\lambda g'_\lambda \langle DY_i Y_i, D_U\rangle + \sum_{i>0} g_\lambda g'_\lambda \langle Y_i, D_U D_X, Y_i\rangle + g'_\lambda\langle Y_0, D_U\rangle. \tag{4.23}$$

We use the relation (see [15, Lemma 3.2])

$$D_X P_t^{\lambda,\cdot} f(\tilde{x}) = \left\langle P_t^{\lambda,2,\cdot}[\langle Df, U\rangle](\tilde{x}, I), \cdot\right\rangle \tag{4.24}$$

and the relation

$$D_U P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\widetilde{X}, 0\right) = P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\widetilde{X}, \widetilde{\mathbb{I}}\right). \tag{4.25}$$

Therefore,

$$\begin{aligned}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) P_t^{\lambda,2,\cdot}[\langle Df, U\rangle](\tilde{x}_0, U_0) &= g'_\lambda \left\langle Y_0, DP_t^{\lambda,\cdot}\right\rangle + \sum_{i>0} g_\lambda g'_\lambda \left\langle DY_i Y_i, DP_t^{\lambda,\cdot}\right\rangle \\ &\quad + \sum_{i>0} g_\lambda g'_\lambda \left\langle Y_i, D^2 P_t^{\lambda,\cdot}, Y_i\right\rangle.\end{aligned} \tag{4.26}$$

We insert this formula in the right-hand side of (4.23) and we see that $P_t^{\lambda,1,\cdot}(\langle Df, U\rangle)(\tilde{x}, 0)$ satisfies the same parabolic equation as $(\partial/\partial\lambda)P_t^{\lambda,\cdot} f(x)$. □

Remark 4.4. Let us show from where this formula comes. Classically,

$$P_t^{\lambda,\cdot}[f](x) = E\left[f\left(x_t^\lambda(x)\right)\right], \tag{4.27}$$

where x_t^λ is the solution of the Stratonovitch equation starting at x:

$$dx_s^\lambda(x) = g_\lambda Y_0\left(x_s^\lambda(s)\right)ds + \sum_{i>0} g_\lambda Y_i\left(x_s^\lambda(s)\right)dB_s^i. \tag{4.28}$$

Therefore, $U_s = (\partial/\partial s)x_s^\lambda(x)$ is solution starting at 0 of the Stratonovitch differential equation:

$$\begin{aligned}dU_s &= g'_\lambda Y_0\left(x_s^\lambda(s)\right)ds + \sum_{i>0} g'_\lambda Y_i\left(x_s^\lambda(s)\right)dB_s^i \\ &\quad + g_\lambda \left\langle DY_0(x_s\lambda(x)), U_s^\lambda\right\rangle ds + \sum_{i>0} g_\lambda \left\langle DY_i\left(x_s^\lambda(x)\right), U_s^\lambda\right\rangle dB_s^i\end{aligned} \tag{4.29}$$

which can be solved classically by using the method of variation of constant [4].

Let us introduce the generator on $\mathbb{R}^{1+d+1+1+d}\mathbb{A}^{\lambda,2}$:

$$\begin{aligned}&\mathbb{A}^{\lambda,2} f(s_0, x_0, u_0, v_0, U_0) \\ &\quad = \sum_j C \int_0^\infty \left(f_\lambda(s)^{-3/2} P_1^{j,\sqrt{f_\lambda(s)},2,\cdot} f(s_0, u_0, u_0 J_\lambda(s), v_0, U_0) - s^{-3/2} f(s_0, x_0, u_0, v_0, U_0)\right) ds.\end{aligned} \tag{4.30}$$

It generates a semi-group $P_t^{\lambda,2}$. In order to see that, we split the generator by keeping the values od $s\langle\epsilon$ or $s\rangle\epsilon$ and we proceed as for $\mathbb{A}^{\lambda,1}$ (see (4.10)).

We get the following.

Theorem 4.5. *For f smooth with compact support in s and with derivatives of each order bounded, one has the relation if one takes only derivatives in (s_0, x_0) of the considered expressions:*

$$DP_t^{\lambda,1}[f](s_0,x_0,u_0) = P_t^{\lambda,2}[\langle Df, v, U\rangle](s_0,x_0,u_0,1,\mathbb{I}). \tag{4.31}$$

Proof. We have

$$\frac{\partial}{\partial t}DP_t^{\lambda,1} = \sum_j C\int_0^\infty \left(f_\lambda(s)^{-3/2}DP_1^{j,\sqrt{f_\lambda(s)}}P_t^{\lambda,1}[f](s_0,u_0,u_0J_\lambda(s)) - s^{-3/2}DP_t^{\lambda,1}f(s_0,x_0,u_0)\right)ds. \tag{4.32}$$

But by [15, Lemma 3.2.]:

$$DP_1^{j,\sqrt{f_\lambda(s)}}f(s_0,u_0,u_0J_\lambda(s)) = P_1^{j,\sqrt{f_\lambda(s)},2,\cdot}[\langle Df, v, U\rangle](s_0,x_0,u_0J_\lambda(s),1,\mathbb{I}) \tag{4.33}$$

Therefore $DP_t^{\lambda,1}$ satisfies the parabolic equation associated with $P_t^{\lambda,2}[\langle df, v, U\rangle](s_0, x_0, u_0, 1, \mathbb{I})$. Only the integrability of U puts any problem. It is solved by the appendix since f has compact support in s. □

Theorem 4.6. *For f with compact support in $\tilde{x}$ in $\mathbb{R}^{\tilde{d}}$.*

$$P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0,\tilde{U}_0\right) = P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0,\tilde{U}_0\right) + P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0,\tilde{0}\right) \tag{4.34}$$

if $\tilde{U}$, $\tilde{U}_0$ belong to $\mathbb{R}^{\tilde{d}}$.

Proof. If the Volterra expansion converges, then

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x},\tilde{U}_0\right) &= P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x},\tilde{U}_0\right) \\ &\quad + \sum \int_{0<s_1<\cdots<s_n<t} ds_1\cdots ds_n P_{s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) \\ &\quad \times P_{s_2-s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right)\cdots\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right)P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x},\tilde{U}_0\right). \end{aligned} \tag{4.35}$$

But $\tilde{U}_0 \rightarrow P_t^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ is linear in $\tilde{U}_0$ and therefore the quantity $(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2})P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ do not depend of $\tilde{U}_0$. Therefore the Volterra expansion reads

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x},\tilde{U}_0\right) &= P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x},\tilde{U}_0\right) \\ &\quad + \int_0^t P_s^{\lambda}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right)P_{t-s}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},0)ds \end{aligned} \tag{4.36}$$

But the last term in the right-hand side of (4.26) is equal to $P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0)$ by the end of the proof of the Proposition 4.3. □

Remark 4.7. Analogous formula works for $D\exp[t\mathbb{A}]f$.

Let us compute $\alpha_t = (\partial/\partial\lambda)P_t^{0,1}[uf](s_0,x_0,1)$. We remark that

$$P_1^{j,\sqrt{f_\lambda(s)}}[uf](s_0,x_0,u_0J_\lambda(s)) = P_1^{j,\sqrt{f_\lambda(s)}}[f](s_0,x_0)u_0J_\lambda(s). \tag{4.37}$$

Namely, the generator of $P_t^{j,\sqrt{f_\lambda(s)}}$ does not act on the u_0 component such that the two sides of (4.37) satisfy the same parabolic equality.

Therefore,

$$\begin{aligned} d_t\alpha_t &= \mathbb{A}\alpha_t + \sum_j C\int_0^\infty f_0'(s)s^{-5/2}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds + \sum_j C\int_0^\infty s^{-3/2}J_0'(s)P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds \\ &\quad + \sum_j C\int_0^\infty s^{-3/2}\frac{\partial}{\partial\lambda}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds \\ &= \mathbb{A}\alpha_t + a_1(t) + a_2(t) + a_3(t), \end{aligned} \tag{4.38}$$

where $J_0'(s) = (\partial/\partial\lambda)J_0(s)$, Therefore,

$$\alpha_t = \int_0^t \exp[(t-s)\mathbb{A}](a_1(s)+a_2(s)+a_3(s))ds. \tag{4.39}$$

$a_3(t)$ in the previous expression is the only term which contains a derivative of f, because by Proposition 4.3,

$$\frac{\partial}{\partial\lambda}P_1^{j,\sqrt{s}}[\exp\ t\mathbb{A}][f](s_0,x_0) = P_1^{j,\sqrt{s},1,\cdot}[\langle D\exp[t\mathbb{A}][f],u,U\rangle](s_0,x_0,0,0). \tag{4.40}$$

Let $\mathbb{A}^3$ be the generator on $\mathbb{R}^{1+d+1+d}$:

$$\mathbb{A}^3 f(s_0,x_0,u_0,U_0) = C\sum_j\int_0^\infty s^{-3/2}\Big(P_1^{j,\sqrt{s},1,\cdot}[f](s_0,x_0,u_0,U_0) - f(s_0,x_0,u_0,U_0)\Big)ds. \tag{4.41}$$

It generates a semi-group, P_t^3. We get the following.

Theorem 4.8. *For f with compact support in s and with bounded derivatives at each order, we have*

$$P_t^3[\langle df,u,U\rangle](s_0,x_0,0,0) = \int_0^t \exp[(t-v)\mathbb{A}]\left[\sum_j C\int_0^\infty s^{-3/2}P_1^{j,\sqrt{s},1,\cdot}[\langle D\exp[v\mathbb{A}][f],u,U\rangle](s_0,x_0,0,0)ds\right]dv. \tag{4.42}$$

Proof. If the Volterra expansion converges, then

$$P_t^3[\langle df,U\rangle](s_0,x_0,0,0) = \sum_n \int_{0<s_1<\cdots<s_n<t} ds_1\cdots ds_n P_{s_1}^2\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdots P_{s_n-s_{n-1}}^2\left(\mathbb{A}^3-\mathbb{A}^2\right) \times P_{t-s_n}^2[\langle df,u,U\rangle](s_0,x_0,0,0). \tag{4.43}$$

But $P_{t-s_n}^2[\langle df,u,U\rangle](s_0,x_0,u_0,U_0)$ is linear in (u_0,U_0). Let us explain the details of that. We have to compute

$$\left(P_1^{j,\sqrt{s},1,\cdot} - P_1^{j,\sqrt{s},2,\cdot}\right)P_{t-s_n}^2[\langle df,u,U\rangle](s_0,x_0,u_0,U_0). \tag{4.44}$$

By the technique of the beginning of the proof of Proposition 4.3, it does not depend on (u_0,U_0). Therefore, the Volterra expansion reads:

$$P_t^3[\langle df,U\rangle](s_0,x_0,0,0) = \int_{0<v<t} P_v^2\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2[\langle df,u,U\rangle](s_0,x_0,0,0)dv. \tag{4.45}$$

But

$$\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2[\langle df,u,U\rangle](s_0,x_0,u_0,U_0). \tag{4.46}$$

does not depend on (u_0,U_0). Therefore, the right-hand side of formula (4.45) is equal to

$$\int_{0<v<t} \exp[v\mathbb{A}]\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2[\langle df,u,U\rangle](s_0,x_0,0,0)dv. \tag{4.47}$$

But

$$\mathbb{A}^2\cdot P_{t-v}^2[\langle df,u,U\rangle](s_0,x_0,0,0) = \sum_j C\int_0^\infty s^{-3/2}P_s^{j,\sqrt{2},2,0}P_{t-v}^2[\langle df,u,U\rangle](s_0,x_0,0,0) = O \tag{4.48}$$

because $(u_0, U_0) \to P^j_{t-v}[\langle df, u, U\rangle](s_0, x_0, u_0, U_0)$ is linear in (u_0, U_0) and because the vector fields which give the generator of $P_s^{j,\sqrt{s},2,\cdot}$ are linear in u_0, U_0. Therefore,

$$\begin{aligned}
&\int_{0<v<t} \exp[v\mathbb{A}]\left(\mathbb{A}^3 - \mathbb{A}^2\right) \cdot P^2_{t-v}[\langle df, u, U\rangle](s_0, x_0, 0, 0)dv \\
&\quad = \int_{0<v<t} \exp[v\mathbb{A}]dv \sum_j C \int_0^\infty s^{-3/2} P_1^{j,\sqrt{s},1,\cdot}\left[P^2_{t-v}[\langle df, u, U\rangle]\right](s_0, x_0, 0, 0)ds.
\end{aligned} \tag{4.49}$$

But by an analog of Theorem 4.5,

$$P^2_{t-v}[\langle df, u, U\rangle](s_0, x_0, u_0, U_0) = \langle D\exp[t\mathbb{A}], u_0, U_0\rangle. \tag{4.50}$$

□

We can summarize the previous considerations in the next theorem.

Theorem 4.9. *If $f_\lambda(s)$ is a diffeomorphism of $[0,\infty[$ equal to s if $s \in [0,\epsilon[$ and if $s > 1$, then one has the following integration by part formula if f is with compact support in s, bounded with bounded derivatives at each order:*

$$\begin{aligned}
0 &= \sum_j C \int_0^t du \exp[(t-u)\mathbb{A}]\left[\int_0^\infty f_0'(s)s^{-5/2}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]\right](s_0, x_0) \\
&\quad + \sum_j C \int_0^t du \exp[(t-u)\mathbb{A}]\left[\int_0^\infty J_0'(s)s^{-3/2}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]\right](s_0, x_0) \\
&\quad + P_t^3[\langle df, u, U\rangle](s_0, x_0, 0, 0),
\end{aligned} \tag{4.51}$$

where $J_0'(s) = (\partial/\partial\lambda)J_0(s)$.

Theorem 4.10. *Let one suppose that $f_\lambda(s) = s + \lambda s^5$ near 0. Then,* (4.51) *is still true.*

Proof. It is enough to show that wecan approximate $f_\lambda(s)$ by a function $f_\lambda^\epsilon(s)$ equal to s if $s < \epsilon$. Let us give some details on this approximation. We consider a smooth function g from $\mathbb{R}$ into $[0,1]$ equal to zero if $s \le 1/2$ and equal to 1 if $s > 1$. We put

$$\begin{aligned}
f_\lambda(s) &= s + g\left(\frac{s}{\epsilon}\right)\lambda s^5, \\
\frac{\partial}{\partial\lambda} f_0^\epsilon(s) &= g\left(\frac{s}{\epsilon}\right)s^5, \\
\frac{\partial}{\partial\lambda} J_0^\epsilon(s) &= g'\left(\frac{s}{\epsilon}\right)\frac{s^5}{\epsilon} + 5g\left(\frac{s}{\epsilon}\right)s^4.
\end{aligned} \tag{4.52}$$

We remark that

(i) if $s \leq \epsilon/2$, then $g'(s/\epsilon)s^5/\epsilon = 0$,

(ii) if $s > \epsilon$, then $g'(s/\epsilon)s^5/\epsilon = 0$,

(iii) if $s \in [\epsilon/2, \epsilon]$, then $|g'(s/\epsilon)s^5/\epsilon| \leq Cs^4$.

$P_t^{3,\epsilon}$ is the semi-group associated with $\mathbb{A}^{3,\epsilon}$ where we replace in the construction of (4.41) $f_\lambda(s)$ by $f_\lambda^\epsilon(s)$:

$$P_t^{3,\epsilon}\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0) \longrightarrow P_t^3\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0). \tag{4.53}$$

By the appendix,

$$P_s^{3,\epsilon}\left[\left(|u|^p + |U|^p\right)h\right](s, x, u_0, U_0) < \infty \tag{4.54}$$

if h is compact support in s. Let us consider the generator $A^{3,\epsilon}$ associated with f_λ^ϵ. If $g = \langle df, u, U\rangle$, then we have by Duhamel principle

$$P_1^3[g](s_0, x_0, 0, 0) = P_1^{3,\epsilon}[g](s_0, x_0, 0, 0) + \int_0^1 P_s^{3,\epsilon}\left[\left(\mathbb{A} - \mathbb{A}^{3,\epsilon}\right)P_{1-s}^3[g]\right](s_0, x_0, 0, 0). \tag{4.55}$$

By the proof of Theorem 4.8, $P_{1-s}^3[g](s_0, x_0, u_0, U_0)$ is affine in (u_0, U_0). Namely, in the proof of this theorem, we have removed the $P_{1-s}^2[g](s_0, x_0, u_0, U_0)$ which is equal to zero in $u_0 = 0, U_0 = 0$ because this expression is linear in u_0, U_0. Its component in (u_0, U_0) is smooth with bounded derivatives at each order. By Theorem 4.6, $(\mathbb{A}^{3,\epsilon} - \mathbb{A})P_{1-s}^3[g](s_0, x_0, u_0, U_0)$ is still affine in (u_0, U_0) and its components in (u_0, U_0) are smooth with bounded derivatives at each order. Moreover, if g^1 is affine in (u_0, U_0) with components in (u_0, U_0) smooth with bounded derivatives at each order, then we get that, for $s \leq 1$,

$$\sup_{s_0, x_0}\left|\left(P_1^{j,\sqrt{s},1,0} - P_1^{\epsilon,j,\sqrt{s},1,\cdot}\right)\left[g^1\right](s_0, x_0, u_0, U_0)\right| \leq C(\epsilon)s(|u_0| + |U_0|), \tag{4.56}$$

where $C(\epsilon) \to 0$ when $\epsilon \to 0$. This can be seen as an appliation of the Duhamel formula applied to the two semi-groups $t \to P_t^{j,\sqrt{s},1,\cdot}[g^1]$ and $t \to P_t^{\epsilon,j,\sqrt{s},1,\cdot}[g^1]$. Then, the result arises from the Duhamel formula (4.55). □

We can consider vector fields at the manner of (3.30) and $f_\lambda(s) = s + \lambda s^5$ in a neighborhood of 0. We get a generator $\mathbb{A}^{\text{tot}}$ and semi-groups $P_s^{j,\sqrt{s},\text{tot}}$ and $P_t^{3,\text{tot}}$. We have with the extension of Theorem 4.10 the following.

Theorem 4.11 (Bismut). *If $f_\lambda(s) = s + \lambda s^5$ in a neighborhood of 0 and is equal to 1 if $s > 1$, then one has the following integration by parts: let f^{tot} be a function with compact support in s, bounded with bounded derivatives at each order. Then,*

$$0 = \sum_j C \int_0^t du \exp\left[(t-u)\mathbb{A}^{\text{tot}}\right] \left[\int_0^\infty f_0'(s) s^{-5/2} P_1^{j,\sqrt{s},\text{tot}}\left[\exp\left[t\mathbb{A}^{\text{tot}}\right]\left[f^{\text{tot}}\right]\right]\right]\left(s_0, x_0^{\text{tot}}\right)$$
$$+ \sum_j C \int_0^t du \exp\left[(t-u)\mathbb{A}^{\text{tot}}\right] \left[\int_0^\infty J_0'(s) s^{-3/2} P_1^{j,\sqrt{s},\text{tot}}\left[\exp\left[t\mathbb{A}^{\text{tot}}\right]\left[f^{\text{tot}}\right]\right]\right]\left(s_0, x_0^{\text{tot}}\right)$$
$$+ P_t^{3,\text{tot}}\left[\left\langle df^{\text{tot}}, u, U\right\rangle\right]\left(s_0, x_0^{\text{tot}}, 0, 0\right). \tag{4.57}$$

5. The Abstract Theorem

The proof of Theorem 2.1 follows the idea of Malliavin [5]. If there exist C_l such that, for function f with compact support in $[0,1] \times [0,l]^d$,

$$\left|\exp[t\mathbb{A}][Df](0,x)\right| \le C_l \|f\|_\infty \tag{5.1}$$

then the heat kernel $q_t(s,y)$ exists.

There are two partial derivatives to treat:

(i) the partial derivative in the time of the subordinator s,

(ii) the partial derivatives in the space of the underlying diffusion x.

Let us begin by the most original part of Bismut's Calculus on boundary process, that is, the integration by parts in the time s.

We look at (4.42). We remark (see the next part) that

$$P_t^{3,\text{tot}}\left[u^{-p}\right](0,x,0,0) < \infty \tag{5.2}$$

for all p. So, we take $f^{\text{tot}}(s,x,u) = f(s,x)1/u$ and we apply (4.42) for this convenient semigroup. We get

$$\exp[t\mathbb{A}]\left[\frac{\partial}{\partial s} f\right](0,x) = -P_t^{3,\text{tot}}\left[\left\langle D_x f, U\right\rangle \frac{1}{u}\right](0,x,0,0) + R \tag{5.3}$$

R can be estimated by using the appendix by $C_l\|f\|_\infty$ for f with compact support in $[0,l] \times [0,l]^d$ and by (5.2).

Lemma 5.1. *For a conveniently enlarged semi-group in the manner of Theorem 3.4, one has for f with compact support in s*

$$P_t^{2,\mathrm{tot}}\left[\langle Df^{\mathrm{tot}}, U\rangle\right](s_0, x_0^{\mathrm{tot}}, 0) = \exp\left[t\widehat{\mathbb{A}}^{\mathrm{tot}}\right]\left[\langle Df^{\mathrm{tot}}, UV\rangle\right](s_0, x_0^{\mathrm{tot}}, I, 0). \tag{5.4}$$

Proof. If $\widetilde{f}$ is a function with compact support depending only of s, x^{tot} and V, we have

$$P_t^{2,\mathrm{tot}}\left[\widetilde{f}\right](s_0, x_0^{\mathrm{tot}}, 0) = \exp\left[t\widehat{\mathbb{A}}^{\mathrm{tot}}\right]\left[\widetilde{f}(\cdot, UV)\right](s_0, x_0^{\mathrm{tot}}, I, 0). \tag{5.5}$$

We do the change of variable $U \to U$ and $V \to UV$ on the Malliavin generator $\widehat{\mathbb{A}}^{\mathrm{tot}}$. By using Lemma 3.7 of [15], it is transformed in $\mathbb{A}^{2,\mathrm{tot}}\widetilde{f}(s, x^{\mathrm{tot}}, U, V)$ where for $\mathbb{A}^{2,\mathrm{tot}}$ we consider the same type of operator as $\mathbb{A}^2$ but with the modified vector fields:

$$\begin{aligned} X_i^{j,2} &= \left(0, X_i^{j,\mathrm{tot}}, DX_i^j U, DX_i^j V\right), \\ Y_0^{j,2} &= \left(0, 0, 0, \sum \left(X_i^j\right)^t \left(U^{-1} X_i^j\right)\right). \end{aligned} \tag{5.6}$$

It remains to use the appendix to show the Lemma. □

We consider $Z_i^j = {}^t(U^{-1}X_i^j)$. By the previous Lemma and Malliavin hypothesis,

$$P_t^{2,\mathrm{tot}}\left[\det V^{-p} g\right](0, x_0^{\mathrm{tot}}, I, 0) < \infty \tag{5.7}$$

for all p if $g(s)$ has compact support (V is a matrix). After we consider a test function of the type of Bismut, we consider the component u_i of U in (5.3). We consider the Bismut function $fV^{-1}(u_i/u)$. We integrate by parts as in Theorem 3.4. We deduce under Malliavin assumption that

$$\left| P_t^{3,\mathrm{tot}}\left[\langle D_x f, U\rangle \frac{1}{u}\right](0, x, 0, 0)\right| \leq C_l \|f\|_\infty \tag{5.8}$$

if f has compact support in $[0, l] \times [0, l]^d$.

By the same way, we deduce that if f has compact support in $[0, l] \times [0, l]^d$ then

$$\left|\exp[t\mathbb{A}]\left[D_x f\right](0, x, 0, 0)\right| \leq C_l \|f\|_\infty. \tag{5.9}$$

Therefore, the result is obtained .

Remark 5.2. We could do integration by parts to each order in order to show that the semi-group $\exp[t\mathbb{A}]$ has a smooth heat-kernel under Malliavin assumption.

6. Inversion of the Malliavin Matrix

Proof of Theorem 2.2. Let $s_1 < s_2$ and let ξ be of modulus 1. Then,

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](0,x,I,0) \ge \exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](0,x,I,0). \tag{6.1}$$

These two quantities are equal in $t = 0$ when we consider the semi-group $\exp[t\widehat{\mathbb{A}}]$. Let us compute their derivative in time t. The derivative of the left-hand side is bigger than the derivative of the right-hand side because

$$\widehat{\mathbb{A}}\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](s,x,U,V) \ge \widehat{\mathbb{A}}\left[\mathbb{I}_{[0,s_2]}\mathbb{I}_{V(\xi)\le\epsilon}\right](s,x,U,V). \tag{6.2}$$

(These two quantities are negative.)

By the result of the appendix,

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,t]}\left\{\mathbb{I}_{|U^{-1}-I|>C} + \mathbb{I}_{|U-I|>C} + \mathbb{I}_{|\cdot-x|>C} + \mathbb{I}_{V>C}\right\}\right](0,x,I,0) \le C(p)t^p \tag{6.3}$$

for all p. □

Lemma 6.1. *If $|\xi| = 1$, then there exist C and C_0 independent of ξ such that*

$$\exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{|V\xi|<C_0\epsilon}\mathbb{I}_{[0,\epsilon]}\right](0,x,I,0) < 1 - C\epsilon^{1/2}. \tag{6.4}$$

Proof. We consider a convex function decreasing from $[0,\infty[$ into $[0,1]$ equal to 1 in 0 and tending to 0 at infinity. Let us introduce

$$\alpha_s = \exp\left[s\widehat{\mathbb{A}}\right]\left[g\left(\frac{|V\xi|}{\epsilon}\right)\mathbb{I}_{[0,\epsilon]}\right](0,x,I,0). \tag{6.5}$$

In order to consider the derivative in s of α_s, we study the expression

$$\beta_\epsilon = \widehat{\mathbb{A}}\left[g\left(\frac{|V\xi|}{\epsilon}\right)\mathbb{I}_{[0,\epsilon]}\right](s',x',U',V'). \tag{6.6}$$

We have only to consider by (6.3) the case where s' is small enough, $|x'-x|$ is small enough, $|U-I|$ is small enough, and the positive matrix V' is small enough. For that we have to estimate

$$\gamma_u = \sum_j\left(P_u^{j,2}\left[g\left(\frac{|V\xi|}{\epsilon}\right)\right](s',x',U',V') - g\left(\frac{|V'\xi|}{\epsilon}\right)\right) \tag{6.7}$$

for u between 0 and ϵ. The first derivative of γ_u has an equivalent $-C\epsilon^{-1}$ when $\epsilon \to 0$, and its second derivative has a bound $C\epsilon^{-2}$ when $\epsilon \to 0$. Therefore,

$$0 \geq \gamma_u \geq -\frac{Cu}{\epsilon} \tag{6.8}$$

on $[0, \epsilon]$ and

$$\beta_\epsilon \geq -\frac{C}{\epsilon}\int_0^\epsilon s^{-1/2} ds = -C\epsilon^{-1/2}. \tag{6.9}$$

We deduce from that that

$$\alpha_\epsilon \leq 1 - C\epsilon^{1/2}. \tag{6.10}$$

□

Remark 6.2. We could improve (6.4) by showing that

$$\exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{|V\xi|<C_0\epsilon}\mathbb{I}_{[0,\epsilon]}\right](s', x', U', V') < 1 - C\epsilon^{1/2} \tag{6.11}$$

if s' is small enough, $|x' - x|$ is small enough, $|U' - I|$ is small enough, and the positive matrix V' is small enough.

We consider a very small α. We slice the time interval $[0, \epsilon^\alpha]$ in $\epsilon^{\alpha-1}$ intervals of length ϵ. We have

$$\begin{aligned} \exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,l]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) &\leq \exp\left[\epsilon^\alpha\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,l]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) \\ &\leq \exp\left[\epsilon^\alpha\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,\epsilon]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) \\ &\leq \left\{\sup_{|x'-x|\leq C_0,\, |U-I|\leq C_0} \exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,\epsilon]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x', U', 0)\right\}^{\epsilon^{\alpha-1}} + C\epsilon^p \end{aligned} \tag{6.12}$$

for a small C_0. This last quantity is smaller than $C\epsilon^p$ for all p by the previous lemma if α is small enough. The proof of Theorem 2.2 follows from

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[V^p\mathbb{I}_{[0,l]}\right](0, x, I, 0) \leq \infty \tag{6.13}$$

for all p by using the result of the appendix. The result follows by standard methods (see [15, Equations (4.8) and (4.9)].

It remains to show the following.

Theorem 6.3. *For all $p > 0$,*

$$P_t^{3,\mathrm{tot}}\left[u^{-p}\right](0, x, 0, 0) \leq \infty. \tag{6.14}$$

Proof. We remark that if we consider only functions of u, then

$$P_t^{3,\mathrm{tot}}[f](0, x, u, 0) = P_t^4[f](u), \tag{6.15}$$

where P_t^4 is a Lévy semi-group with generator

$$\mathbb{A}^4 f(u) = C\int_0^\infty \frac{ds}{s^{3/2}}\left(f\left(s^5 g(s) + u\right) - f(u)\right), \tag{6.16}$$

where $g(s) = 1$ on a neighborhood of 0, is with compact support and is positive. The result follows from the adaptation in [17, 18] of the proof of [7] in semi-group theory. We remark that

$$P_t^4\left[u^{-p}\right](0) = C\int_0^\infty \beta^{p-1} P_t^4[\exp[-\beta u]](0)d\beta. \tag{6.17}$$

By using the adaptation in semi-group theory of the exponential martingales of Levy process of [17, 18], we have

$$P_t^4[\exp[-\beta u]](0) = \exp\left[t\int_0^\infty \left[\exp\left[-\beta s^5 g(s)\right] - 1\right)\frac{ds}{s^{3/2}}\right]. \tag{6.18}$$

The result holds from the Tauberian theorem of [7, 17, 18]. □

Appendix

Burkholder-Davies-Gundy Inequality

Theorem A.4. *Let $s_0 > 0$ and $p \in \mathbb{N}$. Then,*

$$\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}\left|x^{\mathrm{tot}}\right|^{2p}\right]\left(0, x_0^{\mathrm{tot}}, 0\right) < \infty. \tag{A.1}$$

Proof. Following the idea of [17, Appendix], we consider the auxiliary function

$$F_C\left(x^{\mathrm{tot}}\right) = \frac{\left|x^{\mathrm{tot}}\right|^{2p} + 1}{1 + \left|x^{\mathrm{tot}}\right|^{2k}/C}. \tag{A.2}$$

We get

$$\begin{aligned} &\frac{d}{dt}\widehat{P}_t^{2,\text{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\text{tot}}\right)\right]\left(0,x_0^{\text{tot}},0\right) \\ &\quad = \widehat{P}_t^{2,\text{tot}}\left[\int_0^{s_0-s}\frac{du}{u^{3/2}}\sum_j\left(P_u^{j,2,\text{tot}}\left[F_C\right]\left(x^{\text{tot}}\right)-F_C\left(x^{\text{tot}}\right)\right)\right]\left(0,x_0^{\text{tot}},0\right). \end{aligned} \tag{A.3}$$

Let us consider an improvement of the Gronwall lemma: if $|x_s - x_0| \leq \int_0^s |x_u| du$, then $|x_t - x_0| \leq Kt|x_0|$ if $t \in [0,1]$.

We remark that

$$\left|L^{j,2,\text{tot}}F_C\left(x^{\text{tot}}\right)\right| \leq KF_C\left(x^{\text{tot}}\right) \tag{A.4}$$

for K independent of C. Then, by the modified Gronwall lemma,

$$\begin{gathered} \left|P_u^{j,2,\text{tot}}|F_C|\left(x^{\text{tot}}\right)-F_C\left(x^{\text{tot}}\right)\right| \leq KuF_C\left(x^{\text{tot}}\right), \\ \left|\frac{d}{dt}\widehat{P}_t^{2,\text{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\text{tot}}\right)\right]\left(0,x_0^{\text{tot}},0\right)\right| \leq KF_C\left(x^{\text{tot}}\right)+K\widehat{P}_t^{2,\text{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\text{tot}}\right)\right]\left(0,x_0^{\text{tot}},0\right), \end{gathered} \tag{A.5}$$

where K does not depend on C.

By Gronwall lemma,

$$\widehat{P}_t^{2,\text{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\text{tot}}\right)\right]\left(0,x_0^{\text{tot}},0\right) \leq K < \infty, \tag{A.6}$$

where K does not depend on C. The result arises by doing $C \to \infty$. □

By the same procedure, we get the following.

Theorem A.5. *Let be $s_0 > 0$ and $p \in \mathbb{N}$. Then*

$$\widehat{P}_t^{2,\text{tot}}\left[\mathbb{I}_{[0,s_0]}|U|^{2p}\right]\left(0,x_0^{\text{tot}},U_0\right) < \infty \tag{A.7}$$

and we get the following.

Theorem A.6. *Let $s_0 > 0$ and $p \in \mathbb{N}$:*

$$P_t^{3,\text{tot}}\left[\mathbb{I}_{[0,s_0]}\left(\left|x^{\text{tot}}\right|+|u|+|U|\right)^{2p}\right]\left(0,x_0^{\text{tot}},u_0,U_0\right) < \infty. \tag{A.8}$$

Remark A.7. We can show (6.3) by the same way.

References

[1] J. M. Bismut, "The calculus of boundary processes," *Annales Scientifiques de l'École Normale Supérieure*, vol. 17, no. 4, pp. 507–622, 1984.

[2] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Berlin, Germany, 1991.

[3] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales*, vol. 2, John Wiley & Sons, New York, NY, USA, 1987.

[4] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, The Netherlands, 2nd edition, 1989.

[5] P. Malliavin, "Stochastic calculus of variation and hypoelliptic operators," in *Proceedings of the Stochastic Analysis*, pp. 195–263, Kinokuniya, Kyoto, Japan.

[6] J. M. Bismut, "Martingales, the malliavin calculus and hypoellipticity under general hörmander's conditions," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 56, no. 4, pp. 469–505, 1981.

[7] J. M. Bismut, "Calcul des variations stochastique et processus de sauts," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 63, no. 2, pp. 147–235, 1983.

[8] K. Bichteler, J. Gravereaux, and J. Jacod, *Malliavin Calculus for Processes with Jumps*, Gordon and Breach Science Publishers, New York, NY, USA, 1988.

[9] R. Léandre, *Une généralisation du théoreme de Hoermander a divers processus de sauts*, Ph.D. thesis, Université de Franche-Comte, Besançon, France, 1984.

[10] R. Léandre, "Régularité de processus de sauts dégénérés. II," *Annales de l'Institut Henri Poincaré. Probabilités et Statistique*, vol. 24, no. 2, pp. 209–236, 1988.

[11] R. Léandre, "Calcul des variations sur un brownien subordonné," in *Séminaire de Probabilités, XXII*, J. Azéma, P. A. Meyer, and M. Yor, Eds., vol. 1321 of *Lecture Notes in Math.*, pp. 414–433, Springer, Heidelberg, Germany, 1988.

[12] G. Ben Arous, S. Kusuoka, and D. W. Stroock, "The Poisson kernel for certain degenerate elliptic operators," *Journal of Functional Analysis*, vol. 56, no. 2, pp. 171–209, 1984.

[13] P. Cattiaux, "Hypoellipticité et hypoellipticité partielle pour les diffusions avec une condition frontière," *Annales de l'Institut Henri Poincaré. Probabilités et Statistique*, vol. 22, no. 1, pp. 67–112, 1986.

[14] P. Cattiaux, "Regularité au bord pour les densités et les densités conditionnelles d'une diffusion réfléchie hypoelliptique," *Stochastics*, vol. 20, no. 4, pp. 309–340, 1987.

[15] R. Léandre, "Malliavin calculus of Bismut type without probability," in *Festchrift in Honour of K. Sinha*, vol. 116, pp. 507–518, Indian Academy of Sciences. Proceedings. Mathematical Sciences, 2006.

[16] R. Léandre, "Girsanov transformation for Poisson processes in semi-group theory," in *Proceedings of the International Conference of Numerical Analysis and Applied Mathematics*, T. Simos, Ed., vol. 936, pp. 336–338, American Institute of Physics Proceedings, Corfu, Greece, September 2007.

[17] R. Léandre, "Malliavin calculus of Bismut type for Poisson processes without probability," in *Fractional Order Systems*, J. Sabatier, Ed., vol. 715, pp. 715–733, Journal Europeen des Systemes Automatises, 2008.

[18] R. Léandre, "Regularity of a degenerated convolution semi-group without to use the Poisson process," in *Proceedings of the Non linear Science and Complexity*, A. Luo, Ed., pp. 311–320, Springer, Porto, Portugal, 2010.

[19] R. Léandre, "Wentzel-Freidlin estimates for jump process in semi-group theory: lower bound," in *Proceedings of the International Conference of Differential Geometry and Dynamical Systems*, V. Balan, Ed., vol. 17 of *Balkan Society of Geometers Proceedings*, pp. 107–113, Bucuresti, Romania, 2010.

[20] R. Léandre, "Wentzel-Freidlin estimates for jump process in semi-group theory: upper bound," in *Proceedings of the International Conference on Scientific Computing*, H. Arabnia, Ed., pp. 187–193, CSREA Press, Las Vegas, Nev, USA, 2010, To appear Compueter technology and applications.

[21] R. Léandre, "Varadhan estimates for a degenerated convolution semi-group: upper bound. To appear," in *Proceedings of the Fractional Differentiation and applications*, M. Vinagre, Ed., Badajoz, Spain, October 2010.

[22] R. Léandre, "A criterium for the strict positivity of the density of the law of a Poisson process," *Advances in Difference Equations*, Article ID 803508, 2011.

[23] R. Léandre, "Applications of the Malliavin calculus of Bismut type without probability," in *Proceedings of the Simulation, Modelling and Optimization*, A. M. Madureira, Ed., pp. 559–564, 2006.

[24] R. Léandre, "Applications of the malliavin calculus of bismut type without probability," *WSEAS Transactions on Mathematics*, vol. 5, pp. 1205–1211, 2006.

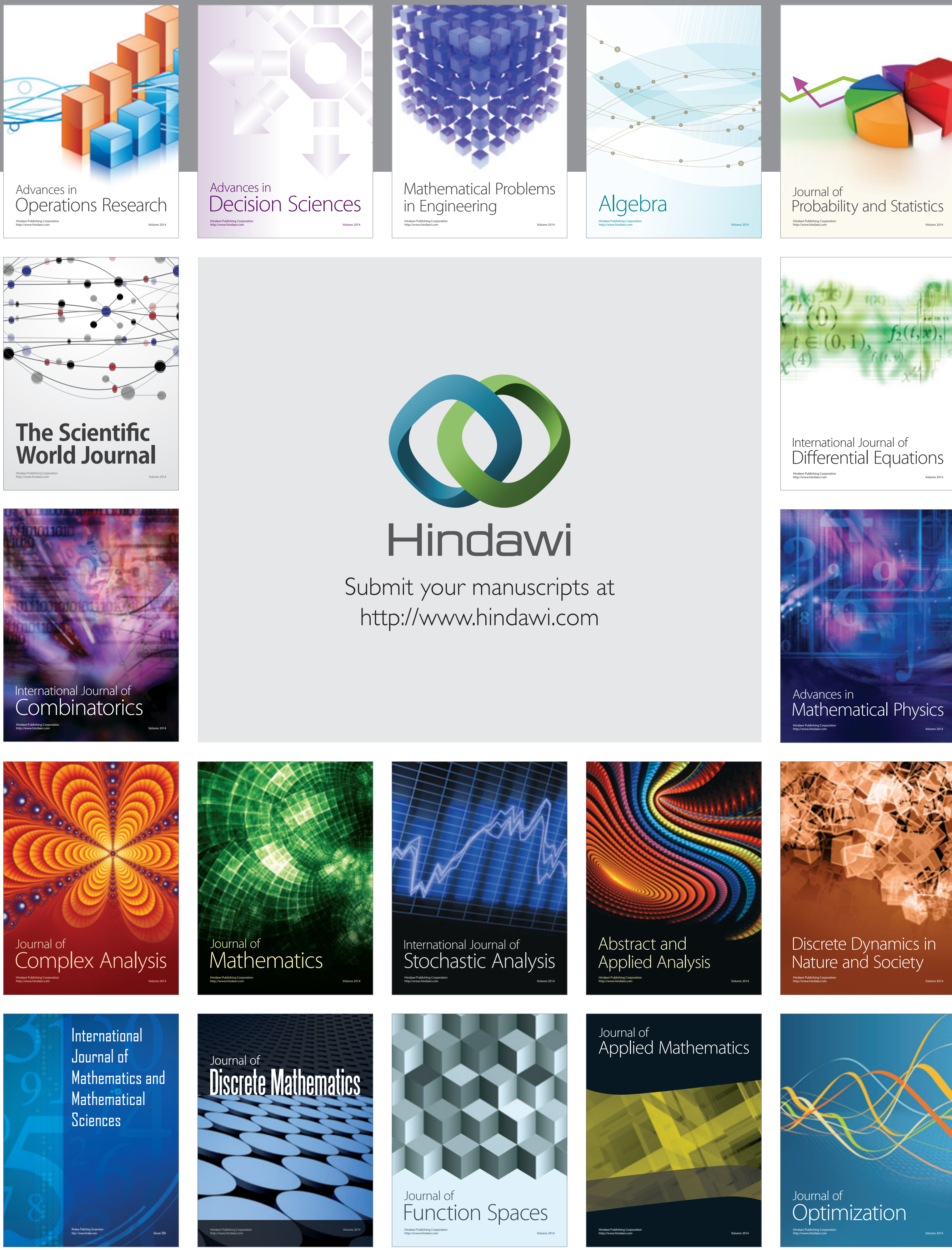
[25] R. Léandre, “Malliavin calculus of Bismut type in semi-group theory,” *Far East Journal of Mathematical Sciences*, vol. 30, no. 1, pp. 1–26, 2008.

[26] R. Léandre, “Stochastic analysis without probability: study of some basic tools,” *Journal of Pseudo-Differential Operators and Applications*, vol. 1, no. 4, pp. 389–400, 2010.

[27] R. Léandre, “A path-integral approach to the Cameron-Martin-Maruyama-Girsanov formula associated to a bilaplacian,” . Preprint.

[28] R. Léandre, “A generalized Fock space associated to a bilaplacian,” in *Proceedings of the International Conference on Applied and Engineering Mathematics*, Shanghai, China, October, 2011.

[29] K. Yosida, *Functional Analysis*, Springer, Heidelberg, Germany, Fourth edition, 1977.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization