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Research Article

Mean Square Stability of Impulsive Stochastic Differential Systems

Shujie Yang, Bao Shi, and Mo Li

Institute of System Science and Mathematics, Naval Aeronautical and Astronautical University, Yantai, Shandong 264001, China

Correspondence should be addressed to Shujie Yang, yangshujie@163.com

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Based on Lyapunov-Krasovskii functional method and stochastic analysis theory, we obtain some new delay-dependent criteria ensuring mean square stability of a class of impulsive stochastic equations. Numerical examples are given to illustrate the effectiveness of the theoretical results.

1. Introduction

It is recognized that the theory of impulsive systems provides a natural framework for the mathematical modeling of many real world phenomena, and impulsive dynamical systems have attracted considerable interest in science and engineering during the past decades. Two classical monographs are Lakshmikantham et al. [1] and Bainov and Simeonov [2]. In general, an impulsive dynamical system can be viewed as a hybrid one comprised of three components: a continuous-time differential equation, which governs the motion of the dynamical systems between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs and a criterion for determining when the states of the systems are to be reset, see Chen and Zheng [3]. Stability properties of impulsive systems have been extensively studied in the literatures. We refer to Li et al. [4, 5], Li et al. [6], Yang [7], Autonio, and Alfonso [8] and the references therein.

Besides impulsive effects, a practical system is usually affected by external stochastic perturbations. Stochastic perturbation is also a factor that makes systems unstable. Recently, stochastic modeling has come to play an important role in many branches of science and industry. An area of particular interest has been stability analysis of impulsive systems with stochastic perturbation. In Yang et al. [9] and Chen et al. [10], the stability properties of nonlinear impulsive stochastic systems are studied using Lyapunov function methods. In Mao et al. [11], a linear matrix inequality approach is proposed for stability analysis of linear uncertain impulsive stochastic systems. However, to the best of our knowledge, there are only few results about this problem.

This paper is inspired by Yang et al. [9], in which the authors considered the problems of stability or robust stabilization for impulsive time delay systems. Unfortunately, they need all the impulsive time sequences to satisfy some strict conditions, that is, the length of the intervals between two jumping time instants must have upper bound or lower bound. But in practical systems, it is always impossible or difficult to obtain it. In this article, by using Lyapunov function methods, together with stochastic analysis, we focus on the mean square stability of trival solution of a class of nonlinear impulsive stochastic time-delay differential systems. We obtain some new conditions ensuring mean square stability of trival solution of the impulsive stochastic differential systems with time-delay. This paper improved some related results.

2. Preliminaries

Throughout this paper, unless explicitly given, for symmetric matrices A and B, the notion $A \geq B$ ($A > B$, $A \leq B$, $A < B$) means $A - B$ is positive semidefinite (positive definite, negative semidefinite, negative definite) matrix. $\lambda_{\max}(\cdot)$ ($\lambda_{\min}(\cdot)$) represents the maximum (minimum) eigenvalue of the corresponding matrix, respectively. $\|\cdot\|$ denotes Euclidean norm for vectors or the spectral norm of matrices. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, the filtration contains all P-null sets and is right continuous. Let $\mathrm{PC}([-\tau, 0], \mathbb{R}^n)$ denote the set of piecewise right continuous function $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the norm defined by $\|\phi\|_\tau = \sup_{-\tau\leq\theta\leq 0}\|\phi(\theta)\|$, where τ is a known positive constant, $\mathrm{PC}(\delta) = \{\varphi \mid \varphi \in \mathrm{PC}([-\tau, 0], \mathbb{R}^n), \|\varphi\|_\tau \leq \delta\}$, $\mathrm{PC}^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ denote the family of all $\mathcal{F}_0$-measurable $\mathrm{PC}([-\tau, 0], \mathbb{R}^n)$-valued stochastic process $\varphi = \{\varphi(s) : -\tau \leq s \leq 0\}$ with $\sup_{-\tau\leq s\leq 0}\mathbb{E}\{\|\varphi(s)\|^2\} < \infty$, where $\mathbb{E}\{\cdot\}$ represents the mathematical expectation operator with respect to the probability measure P, $\mathrm{PC}^b_{\mathcal{F}_0}(\delta) = \{\varphi \mid \varphi \in \mathrm{PC}^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n), \sup_{-\tau\leq s\leq 0}\mathbb{E}\{\|\varphi(s)\|^2\} \leq \delta$, $\mathcal{L}$ denote the well-known $\mathcal{L}$-operator given by the Itô's formula.

In this paper, we consider a class of Itô impulsive stochastic differential systems with time delay

$$\begin{gathered} \mathrm{d}x(t) = f(t, x(t), x_t)\mathrm{d}t + g(t, x(t), x_t)\mathrm{d}\omega(t), \quad t \geq t_0,\ t \neq t_k, \\ x(t_k) = H_k\big(x(t_k^-)\big), \quad k = 1, 2, \ldots, \\ x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{gathered} \tag{2.1}$$

where the initial value $\varphi \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$, the fixed impulsive time moments t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ ($t_k \to \infty$ as $k \to \infty$). $x(t) \in \mathbb{R}^n$ is the system state, $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{n\times m})$. $\omega(t) \in \mathbb{R}^m$ is an standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Besides, we assume that $H_k(0) = 0, (k = 1, 2, \ldots)$, $f(t, 0, 0) = 0$, $g(t, 0, 0) = 0$ and

$$\|H_k(x(t_k^-))\| \leq \gamma_k \|x(t_k^-)\|, \quad \gamma_k \geq 0, \ k = 1, 2, \ldots. \tag{2.2}$$

In the following, we will divide three cases to consider the mean square stability of system (2.1). We denote by $\mathcal{N}_{\inf}(\beta)$ and $\mathcal{N}_{\sup}(\beta)$ the class of impulsive time sequences that satisfy $\inf_k\{t_k - t_{k-1}\} \geq \beta$ and $\sup_k\{t_k - t_{k-1}\} \leq \beta$, respectively.

We need the following lemma and definitions.

Lemma 2.1 (Chaplygin Comparison Theorem, see Shi et al. [12]). *Assume that $f, F \in C(G)$, $g \subset \mathbb{R}^2$ and*

$$f(t, x) < F(t, x), \quad (t, x) \in G. \tag{2.3}$$

If $\phi(t)$ $(t \in U_1)$ and $\Phi(t)$ $(t \in U_2)$ are the solutions of Cauchy problems

$$\begin{aligned} \dot{x} &= f(t, x), \quad (t, x) \in G, \\ x(\tau) &= \xi, \\ \dot{x} &= F(t, x), \quad (t, x) \in G, \\ x(\tau) &= \xi, \end{aligned} \tag{2.4}$$

respectively, then for $t \in (\tau, \infty) \cap U_1 \cap U_2$,

$$\phi(t) < \Phi(t) \tag{2.5}$$

and for $t \in (-\infty, \tau) \cap U_1 \cap U_2$,

$$\phi(t) > \Phi(t). \tag{2.6}$$

Definition 2.2. For a given class $\mathcal{N}$ of admissible impulsive time sequence, the solution of (2.1) is called mean squarely stable if for any $\varepsilon > 0$, there exists a scalar $\delta > 0$, such that the initial function $\varphi \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$ implies $\mathbb{E}\{\|x(t)\|^2\} < \varepsilon$, $t \geq t_0$ for all admissible time sequence in $\mathcal{N}$.

Definition 2.3 (see Yang et al. [9]). The function $V : [t_0 - \tau, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ belongs to class $\mathcal{V}^{(1,2)}$ if

(1) the function $V(t, x)$ is continuously differentiable in t and twice continuously differentiable in x on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n, (k = 1, 2, \ldots)$ and for all $t \geq t_0$, $V(t, 0) \equiv 0$,

(2) $V(t, x)$ is locally Lipschitaian in x,

(3) for each $k = 1, 2, \ldots$, there exist finite limits

$$\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x), \qquad \lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+,x),$$
$$V(t_k^+,x) = V(t_k,x). \tag{2.7}$$

3. Main Results

Theorem 3.1. *Assume that there exist scalars $\lambda_2 > \lambda_1 > 0$, $\lambda_\tau > 0$, $\beta > 0$, $\lambda \le 0$, $\rho > 0$ matrix $P > 0$ and Lyapunov-Krasovskii functional $V(t, x(t)) \in \mho^{(1,2)}$, such that*

(C1) $\lambda_1\|x(t)\|^2 \le V(t,x(t)) \le \lambda_2\|x_t\|_\tau^2$,

(C2) $\mathbb{E}\mathcal{L}V(t,x(t)) \le \lambda\mathbb{E}V(t,x(t)) + \lambda_\tau\mathbb{E}V(t,x_t)$, $t \in [t_{k-1},t_k)$, $k = 1,2,\ldots$, *whenever* $\mathbb{E}V(t,x_t) \le (\mu+\rho)\mathbb{E}V(t,x(t))$,

(C3) $\mu = \sup_{k\in\mathbb{N}}\{\lambda_k' = (\lambda_2/\lambda_1)\gamma_k^2\} > 1$ *and* $\lambda + (\mu+\rho)\lambda_\tau \le -((\ln(\mu+\rho))/\beta)$,

then the trivial solution of system (2.1) *is mean squarely stable over* $\mathcal{N}_{\inf}(\tau+\beta)$.

Proof. For any given $\varepsilon > 0$, choose $0 < \delta \le \sqrt{\lambda_1/(\mu+\rho)\lambda_2}\,\varepsilon$. We assume that the initial function $\varphi \in PC_{\mathcal{F}_0}^b(\delta)$ and denote the solution $x(t,t_0,\varphi)$ of system (2.1) through (t_0,φ) by $x(t)$. In the following, we will prove that $x(t)$ is mean square stable over $\mathcal{N}_{\inf}(\tau+\beta)$. For $V(t,x(t)) \in \mho^{(1,2)}$, by Itô formula, for $t \ne t_k$, $k = 1,2,\ldots$, we have

$$dV(t,x(t)) = \mathcal{L}V(t,x(t))dt + V_x(t,x(t))g(t,x(t))d\omega(t), \tag{3.1}$$

where $\mathcal{L}V(t,x(t)) = V_t(t,x(t)) + V_x(t,x(t))f + (1/2)\,\mathrm{tr}(g^TV_{xx}g)$.

For $t \in [t_{k-1},t_k)$, $k = 1,2,\ldots$, integrate (3.1) from t_{k-1} to t, we have

$$V(t,x(t)) = V(t_{k-1},x(t_{k-1})) + \int_{t_{k-1}}^t \mathcal{L}V(s,x(s))ds + \int_{t_{k-1}}^t V_x(s,x(s))g(s,x(s))d\omega(s). \tag{3.2}$$

Taking the mathematical expectation of both sides of the above equation, we obtain

$$\mathbb{E}V(t,x(t)) = \mathbb{E}V(t_{k-1},x(t_{k-1})) + \int_{t_{k-1}}^t \mathbb{E}\mathcal{L}V(s,x(s))ds. \tag{3.3}$$

So for $s \in [t,t+\Delta t]$ with $t+\Delta t \in [t_{k-1},t_k)$ and $\Delta t > 0$, if $\mathbb{E}V(s,x_s) \le (\mu+\rho)\mathbb{E}V(s,x(s))$, then we have by (C2)

$$\begin{aligned}\mathbb{E}V(t+\Delta t,x(t+\Delta t)) - \mathbb{E}V(t,x(t)) &= \int_t^{t+\Delta t}\mathbb{E}\mathcal{L}V(s,x(s))ds \\ &\le \int_t^{t+\Delta t}(\lambda\mathbb{E}V(s,x(s)) + \lambda_\tau\mathbb{E}V(s,x_s))ds.\end{aligned} \tag{3.4}$$

In what follows, we first prove that for $t \in [t_0 - \tau, t_1)$,

$$\mathbb{E}V(t, x(t)) \leq \lambda_1 \varepsilon^2. \tag{3.5}$$

Obviously, for $t \in [t_0 - \tau, t_0]$, by (C1) and $x_{t_0} \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$, we obtain

$$\mathbb{E}V(t, x(t)) \leq \lambda_2 \mathbb{E}\left\{ \|x_{t_0}\|_\tau^2 \right\} \leq \lambda_2 \delta^2 \leq \frac{\lambda_1}{\mu + \rho} \varepsilon^2 < \lambda_1 \varepsilon^2. \tag{3.6}$$

Now it needs only to prove that for $t \in (t_0, t_1)$, (3.5) holds. Otherwise, there exists $s \in (t_0, t_1)$, such that

$$\mathbb{E}V(s, x(s)) > \lambda_1 \varepsilon^2. \tag{3.7}$$

Set

$$s_1 = \inf\left\{ t \in (t_0, t_1) : \mathbb{E}V(t, x(t)) > \lambda_1 \varepsilon^2 \right\}, \tag{3.8}$$

then by (3.6), (3.7), and the continuity of $\mathbb{E}V(t, x(t))$ on $[t_0, t_1)$, we know that $s_1 \in (t_0, t_1)$,

$$\mathbb{E}V(s_1, x(s_1)) = \lambda_1 \varepsilon^2, \tag{3.9}$$

and for $t \in [t_0 - \tau, s_1]$, (3.5) holds. Set

$$s_2 = \sup\left\{ t \in [t_0, s_1) : \mathbb{E}V(t, x(t)) \leq \frac{\lambda_1}{\mu + \rho} \varepsilon^2 \right\}, \tag{3.10}$$

then by (3.6) and the continuity of $\mathbb{E}V(t, x(t))$, we have $s_2 \in [t_0, s_1)$,

$$\mathbb{E}V(s_2, x(s_2)) = \frac{\lambda_1}{\mu + \rho} \varepsilon^2, \tag{3.11}$$

and for $t \in [s_2, s_1]$,

$$\mathbb{E}V(t, x_t) \leq \lambda_1 \varepsilon^2 \leq (\mu + \rho) \mathbb{E}V(t, x(t)), \tag{3.12}$$

which implies with (3.4) and (C3) that for $t \in [s_2, s_1]$,

$$D^+ \mathbb{E}V(t, x(t)) \leq \lambda \mathbb{E}V(t, x(t)) + \lambda_\tau \mathbb{E}V(t, x_t) \leq (\lambda + \lambda_\tau (\mu + \rho)) \mathbb{E}V(t, x(t)) \leq 0. \tag{3.13}$$

This is a contradiction with (3.9) and (3.11).

Now, we assume that, for $t \in [t_{m-1}, t_m)$, $m = 1, 2, \ldots, k$, (3.5) holds. For $m = k+1$, we will show that (3.5) holds. To this end, we first prove that for $\theta \in [-\tau, 0]$,

$$\mathbb{E}V\left(t_k^- + \theta, x\left(t_k^- + \theta\right)\right) \leq \frac{\lambda_1}{\mu + \rho}\varepsilon^2. \tag{3.14}$$

Noticing $[t_k - \tau, t_k) \subset [t_{k-1}, t_k)$, we assume that there exists some $s \in [t_k - \tau, t_k]$, such that

$$\mathbb{E}V\left(s^-, x\left(s^-\right)\right) > \frac{\lambda_1}{\mu + \rho}\varepsilon^2, \tag{3.15}$$

then there are two cases to be considered.

(i) For all $t \in [t_{k-1}, s]$, $\mathbb{E}V(t^-, x(t^-)) > (\lambda_1/(\mu + \rho))\varepsilon^2$. Hence, for $t \in [t_{k-1}, s]$, (3.12) and (3.13) hold, which follows by (C3), (3.5), and Lemma (2.1),

$$\begin{aligned}
\mathbb{E}V\left(s^-, x\left(s^-\right)\right) &\leq \exp\left\{\left(\lambda + (\mu + \rho)\lambda_\tau\right)(s - t_{k-1})\right\}\mathbb{E}V(t_{k-1}, x(t_{k-1})) \\
&\leq \lambda_1 \varepsilon^2 \exp\left\{\left(\lambda + (\mu + \rho)\lambda_\tau\right)(t_k - t_{k-1} - \tau)\right\} \\
&\leq \frac{\lambda_1}{\mu + \rho}\varepsilon^2,
\end{aligned} \tag{3.16}$$

this is a contradiction with the assumption.

(ii) There exists some $t \in [t_{k-1}, s)$, such that $\mathbb{E}V(t, x(t)) \leq (\lambda_1/(\mu + \rho))\varepsilon^2$. Set

$$s_1 = \sup\left\{t \in [t_{k-1}, s) : \mathbb{E}V(t, x(t)) \leq \frac{\lambda_1}{\mu + \rho}\varepsilon^2\right\}, \tag{3.17}$$

then $s_1 \in [t_{k-1}, s)$,

$$\mathbb{E}V(s_1, x(s_1)) = \frac{\lambda_1}{\mu + \rho}\varepsilon^2 \tag{3.18}$$

and for $t \in [s_1, s]$, (3.12) and (3.13) hold, which is a contradiction with (3.15) and (3.18), that is, (3.14) holds.

By (2.1), (2.2), and (3.14), we have

$$\begin{aligned}
\mathbb{E}V(t_k, x(t_k)) &= \mathbb{E}V\left(t_k, H_k\left(x\left(t_k^-\right)\right)\right) \leq \lambda_2 \mathbb{E}\left\{\left\|H_k\left(x\left(t_k^-\right)\right)\right\|_\tau^2\right\} \leq \lambda_2 \gamma_k^2 \mathbb{E}\left\{\left\|x\left(t_k^-\right)\right\|_\tau^2\right\} \\
&\leq \frac{\lambda_2 \gamma_k^2}{\lambda_1} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}V\left(t_k + \theta, x\left(t_k^- + \theta\right)\right) \leq \lambda_k' \frac{\lambda_1}{\mu + \rho}\varepsilon^2 < \lambda_1 \varepsilon^2.
\end{aligned} \tag{3.19}$$

Now we will prove that (3.5) holds for $t \in [t_k, t_{k+1})$. Otherwise, there exists some $t \in (t_k, t_{k+1})$, such that (3.7) holds. Let

$$s_1 = \inf\left\{t \in (t_k, t_{k+1}) : \mathbb{E}V(t, x(t)) > \lambda_1 \varepsilon^2\right\}. \tag{3.20}$$

Then by (3.14), (3.19), and the continuity of $\mathbb{E}V(t, x(t))$ on $[t_k, t_{k+1})$, we know that $s_1 \in (t_k, t_{k+1})$ and $\mathbb{E}V(s_1, x(s_1)) = \lambda_1 \varepsilon^2$. If there exists $t \in [t_k, s_1]$, such that $\mathbb{E}V(t, x(t)) \leq (\lambda_1/\mu)\varepsilon^2$, then let

$$s_2 = \sup\left\{t \in [t_k, s_1) : \mathbb{E}V(t, x(t)) \leq \frac{\lambda_1}{\mu + \rho}\varepsilon^2\right\}. \tag{3.21}$$

Otherwise, let $s_2 = t_k$. Then for $t \in [s_2, s_1]$, we obtain (3.12) and (3.13), which follows a contradiction.

By mathematical induction, (3.5) holds for any $m = 1, 2, \ldots$, which implies that system (2.1) is mean squarely stable.

If substituting condition

(C′1) $\lambda_1 \|x(t)\|^2 \leq V(t, x(t)) \leq \lambda_2 \|x(t)\|^2$

for (C1) in Theorem (3.1), then we have the following result. □

Theorem 3.2. *Assume that there exist scalars $\lambda_2 > \lambda_1 > 0$, $\lambda_\tau > 0$, $\lambda \leq 0$, $\rho > 0$, matrix $P > 0$ and Lyapunov-Krasovskii functional $V(t, x(t)) \in \mathcal{V}^{(1,2)}$, such that conditions (C′1), (C2), and (C3) hold, then the trivial solution of system* (2.1) *is mean square stable over $\mathcal{N}_{\inf}(\beta)$.*

Proof. The proof is similar to Theorem (3.1), so we omit it. The proof is complete. □

Remark 3.3. Comparing the results in Theorems (3.1) and (3.2), we find the influence of the time delay on the mean square stability of system (2.1).

Remark 3.4. When $\mu > 1$, the impulses which may be destabilizing, so we require the impulses should not happen so frequently.

When $\mu = 1$, we have the following results.

Theorem 3.5. *Assume that there exist scalars $\lambda_2 > \lambda_1 > 0$, $\lambda_\tau > 0$, $\lambda \leq 0$, matrix $P > 0$ and Lyapunov-Krasovskii functional $V(t, x(t)) \in \mathcal{V}^{(1,2)}$, such that condition (C1) and*

(C′2) $\mathbb{E}\mathcal{L}V(t, x(t)) \leq \lambda \mathbb{E}V(t, x(t)) + \lambda_\tau \mathbb{E}V(t, x_t)$, $t \in [t_{k-1}, t_k), k = 1, 2, \ldots$ *whenever* $\mathbb{E}V(t, x_t) \leq \mathbb{E}V(t, x_t)$,

(C′3) $\mu = \sup_{k \in \mathbb{N}}\{\lambda'_k = (\lambda_2/\lambda_1)\gamma_k^2\} = 1$, $\lambda + \lambda_\tau \leq 0$

hold, then the trivial solution of system (2.1) *is mean squarely stable over any impulsive sequences.*

Proof. For any given $\varepsilon > 0$, choose $0 < \delta \leq \sqrt{\lambda_1/\lambda_2}\varepsilon$. We assume that the initial function $\varphi \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$. In what follows, we first prove that for $t \geq t_0$, (3.5) holds.

Obviously, for $t \in [t_0 - \tau, t_0]$, by (C1) and $x_{t_0} \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$, we obtain

$$\mathbb{E}V(t, x(t)) \leq \lambda_2 \mathbb{E}\left\{\|x_{t_0}\|_\tau^2\right\} \leq \lambda_2 \delta^2 \leq \lambda_1 \varepsilon^2. \tag{3.22}$$

Now we should prove that (3.5) holds. Otherwise, there exists $s \in (t_0, t_1)$, such that (3.7) holds. By (3.22) and the continuity of $\mathbb{E}V(t, x(t))$ on $[t_0, t_1)$, we know there exist $\bar{t} \in [t_0, t_1)$ and small scalar $\rho > 0$, such that

$$\mathbb{E}V\left(\bar{t}, x\left(\bar{t}\right)\right) = \lambda_1 \varepsilon^2 \tag{3.23}$$

and for every $\bar{t}_1, \bar{t}_2 \in [\bar{t}, \bar{t} + \rho]$, $\bar{t}_1 < \bar{t}_2$,

$$\mathbb{E}V\left(\bar{t}_1, x\left(\bar{t}_1\right)\right) < \mathbb{E}V\left(\bar{t}_2, x\left(\bar{t}_2\right)\right). \tag{3.24}$$

Let $s = \inf\{t \in [t_0, t_1) : \mathbb{E}V(t, x(t)) = \lambda_1 \varepsilon^2, \mathbb{E}V(u, x(u)) > \lambda_1 \varepsilon^2, u \in (t, t + \rho_1] \subset [t_0, t_1)$, $\mathbb{E}V(\bar{t}_1, x(\bar{t}_1)) < \mathbb{E}V(\bar{t}_2, x(\bar{t}_2))$, for every $\bar{t}_1, \bar{t}_2 \in [t, t + \rho_1], \bar{t}_1 < \bar{t}_2\}$, where $\rho_1 > 0$ is some scalar. Then $[s, s + \rho_1] \subset [t_0, \mathrm{t}_1)$ and for $t \in [s, s + \rho_1]$,

$$\mathbb{E}V(t, x_t) \leq \mathbb{E}V(t, x(t)), \tag{3.25}$$

which implies with (C′2) and (C′3) that for $t \in [s, s + \rho_1]$,

$$D^+ \mathbb{E}V(t, x(t)) \leq \lambda \mathbb{E}V(t, x(t)) + \lambda_\tau \mathbb{E}V(t, x_t) \leq (\lambda + \lambda_\tau) \mathbb{E}V(t, x(t)) \leq 0. \tag{3.26}$$

This is a contradiction with the fact $\mathbb{E}V(s + \rho_1, x(s + \rho_1)) > \mathbb{E}V(s, x(s))$, that is, for $t \in [t_0 - \tau, t_1)$, (3.5) holds.

Now, we assume that, for $t \in [t_{m-1}, t_m)$, $m = 1, 2, \ldots, k$, (3.5) holds. For $m = k + 1$, we will show that (3.5) holds. To this end, we first prove that

$$\mathbb{E}V(t_k, x(t_k)) \leq \lambda_1 \varepsilon^2. \tag{3.27}$$

In fact, by (2.1), (2.2), (C1), and (C′3)

$$\begin{aligned} \mathbb{E}V(t_k, x(t_k)) &= \mathbb{E}V\left(t_k, H_k\left(x\left(t_k^-\right)\right)\right) \leq \lambda_2 \mathbb{E}\left\{ \left\| H_k\left(x\left(t_k^-\right)\right) \right\|_\tau^2 \right\} \\ &\leq \lambda_2 \gamma_k^2 \mathbb{E}\left\{ \left\| x\left(t_k^-\right) \right\|_\tau^2 \right\} \leq \frac{\lambda_2 \gamma_k^2}{\lambda_1} \mathbb{E}V\left(t_k, x\left(t_k^-\right)\right) \\ &\leq \lambda_1 \varepsilon^2. \end{aligned} \tag{3.28}$$

Secondly, we assume that there exists $s \in (t_k, t_{k+1})$, such that (3.7) holds. By (3.27) and the continuity of $\mathbb{E}V(t, x(t))$ on $[t_k, t_{k+1})$, we know that there exist $\bar{t} \in [t_k, t_{k+1})$, $\rho_2 > 0$ such that for every $\bar{t}_1, \bar{t}_2 \in [\bar{t}, \bar{t} + \rho_2]$, $\bar{t} < \bar{t}_2$, (3.23) and (3.24) hold.

Let $s = \inf\{t \in [t_k, t_{k+1}) : \mathbb{E}V(t, x(t)) = \lambda_1 \varepsilon^2, \mathbb{E}V(u, x(u)) > \lambda_1 \varepsilon^2, u \in (t, t + \rho_2] \subset [t_k, t_{k+1}), \mathbb{E}V(\bar{t}_1, x(\bar{t}_1)) < \mathbb{E}V(\bar{t}_2, x(\bar{t}_2))$, for every $\bar{t}_1, \bar{t}_2 \in [t, t + \rho_2], \bar{t}_1 < \bar{t}_2\}$, where $\rho_2 > 0$ is some scalar.

Then for $t \in [s, s + \rho_2]$, (3.25) and (3.26) hold. This is a contraction, that is, (3.5) holds for $t \in [t_k, t_{k+1})$. By mathematical induction, (3.5) holds for any $m = 1, 2, \ldots$, which implies that system (2.1) is mean squarely stable. □

Remark 3.6. When $\mu = 1$, both the continuous dynamics and discrete dynamics are stable under the conditions in Theorem (3.5), so the impulse system can be mean squarely stable regardless of how often or how seldom impulses occur.

When $\mu < 1$, we have the following results.

Theorem 3.7. *Assume that there exist scalars $\lambda_2 > \lambda_1 > 0$, $\lambda_\tau > 0$, $\lambda \le 0$, matrix $P > 0$, and Lyapunov-Krasovskii functional $V(t, x(t)) \in \mho^{(1,2)}$, such that (C1), (C2), and*

(C″3) $\mu = \sup_{k\in\mathbb{N}}\{\lambda_k' = (\lambda_2/\lambda_1)\gamma_k^2\} < 1$

hold, then

(i) *if $0 < \mu\lambda + \lambda_\tau \le -\lambda_\tau \ln\mu$, system* (2.1) *is mean squarely stable over impulsive time sequences $\mathcal{N}_{\sup}(-\mu\ln\mu/(\mu\lambda + \lambda_\tau))$;*

(ii) *if $\mu\lambda + \lambda_\tau \le 0$, system* (2.1) *is mean squarely stable over any impulsive time sequences.*

Proof. We prove (i) and omit the proof of (ii).

Because $\mu < 1$ and $0 < \mu\lambda + \lambda_\tau \le -\lambda_\tau \ln\mu$, then there exist a sufficiently small $\rho_0 > 0$, such that

$$\mu + \rho_0 < 1, \quad \lambda(\mu + \rho_0) + \lambda_\tau > 0,$$
$$\frac{-\ln\mu}{\lambda + \mu^{-1}\lambda_\tau} \le \frac{-\ln(\mu + \rho_0)}{\lambda + (\mu + \rho_0)^{-1}\lambda_\tau}. \tag{3.29}$$

For any given $\varepsilon > 0$, choose $0 < \delta \le \sqrt{((\mu + \rho_0)\lambda_1)/\lambda_2}\varepsilon$. We assume the initial function $\varphi \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$. For $t \in [t_0 - \tau, t_0]$, by (C1), (3.29), and $x_{t_0} \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$, we obtain

$$\mathbb{E}V(t, x(t)) \le \lambda_2\mathbb{E}\left\{\|x_{t_0}\|_\tau^2\right\} \le \lambda_2\delta^2 \le \lambda_1(\mu + \rho_0)\varepsilon^2 < \lambda_1\varepsilon^2. \tag{3.30}$$

Now we will prove that (3.5) holds. Otherwise, there exists $s \in (t_0, t_1)$, such that (3.7) holds. Set

$$t^* = \inf\left\{t \in (t_0, t_1) : \mathbb{E}V(t, x(t)) \ge \lambda_1\varepsilon^2\right\}, \tag{3.31}$$

then by (3.7), (3.30), and the continuity of $\mathbb{E}V(t, x(t))$ on $[t_0, t_1)$, we know that $t^* \in (t_0, t_1)$, $\mathbb{E}V(t^*, x(t^*)) = \lambda_1\varepsilon^2$. Set

$$\bar{t} = \sup\left\{t \in [t_0, t^*) : \mathbb{E}V(t, x(t)) \le \lambda_1(\mu + \rho_0)\varepsilon^2\right\}, \tag{3.32}$$

then by (3.30) and the continuity of $\mathbb{E}V(t, x(t))$, we have $\bar{t} \in [t_0, t^*)$, $\mathbb{E}V(\bar{t}, x(\bar{t})) = \lambda_1(\mu + \rho_0)\varepsilon^2$ and for $t \in [\bar{t}, t^*]$,

$$\mathbb{E}V(t, x_t) \le \lambda_1\varepsilon^2 \le \frac{1}{\rho_0 + \mu}\mathbb{E}V(t, x(t)). \tag{3.33}$$

Conditions (C2) and (C″3) imply that for $t \in [\bar{t}, t^*]$,

$$D^+\mathbb{E}V(t,x(t)) \le \lambda\mathbb{E}V(t,x(t)) + \lambda_\tau\mathbb{E}V(t,x_t) \le \left(\lambda + \frac{\lambda_\tau}{\mu+\rho_0}\right)\mathbb{E}V(t,x(t)). \tag{3.34}$$

By Lemma (2.1), (3.29), (3.26), and $t_1 - t_0 \le (-\ln\mu/(\lambda + (\lambda_\tau/\mu)))$, we have

$$\begin{aligned} \mathbb{E}V(t^*,x(t^*)) &\le \exp\left\{\left(\lambda + \frac{\lambda_\tau}{\mu+\rho_0}\right)\left(t^* - \bar{t}\right)\right\}\mathbb{E}V\left(\bar{t},x\left(\bar{t}\right)\right) \\ &< exp\left\{\left(\lambda + \frac{\lambda_\tau}{\mu+\rho_0}\right)(t_1 - t_0)\right\}(\mu+\rho_0)\lambda_1\varepsilon^2 \\ &\le \lambda_1\varepsilon^2, \end{aligned} \tag{3.35}$$

this is a contradiction with the fact $\mathbb{E}V(t^*,x(t^*)) = \lambda_1\varepsilon^2$.

Now, we assume that, for $t \in [t_{m-1}, t_m)$, $m = 1,2,\dots,k$, (3.5) holds. For $m = k+1$, we will show that (3.5) holds. To this end, we first prove that

$$\mathbb{E}V(t_k,x(t_k)) \le (\mu+\rho_0)\lambda_1\varepsilon^2. \tag{3.36}$$

In fact, by (2.1), (2.2), (C1), and (C″3)

$$\begin{aligned} \mathbb{E}V(t_k,x(t_k)) = \mathbb{E}V\left(t_k,H_k\left(x\left(t_k^-\right)\right)\right) &\le \lambda_2\mathbb{E}\left\{\left\|H_k\left(x\left(t_k^-\right)\right)\right\|_\tau^2\right\} \\ &\le \lambda_2\gamma_k^2\mathbb{E}\left\{\left\|x\left(t_k^-\right)\right\|_\tau^2\right\} \le \frac{\lambda_2\gamma_k^2}{\lambda_1}\mathbb{E}V\left(t_k,x\left(t_k^-\right)\right) \\ &\le (\mu+\rho_0)\lambda_1\varepsilon^2. \end{aligned} \tag{3.37}$$

Secondly, we assume that there exists $s \in (t_k, t_{k+1})$, such that (3.7) holds. Set

$$\begin{aligned} t^* &= \inf\left\{t \in (t_k,t_{k+1}) : \mathbb{E}V(t,x(t)) \ge \lambda_1\varepsilon^2\right\}, \\ \bar{t} &= \sup\left\{t \in [t_k,t^*) : \mathbb{E}V(t,x(t)) \le \lambda_1(\mu+\rho_0)\varepsilon^2\right\}, \end{aligned} \tag{3.38}$$

then by (3.37) and the continuity of $\mathbb{E}V(t,x(t))$ on $[t_k,t_{k+1})$, we have $t^* \in (t_k,t_{k+1})$, $\bar{t} \in [t_k,t^*)$ and $\mathbb{E}V(t^*,x(t^*)) = \lambda_1\varepsilon^2$, $\mathbb{E}V(\bar{t},x(\bar{t})) = (\mu+\rho_0)\lambda_1\varepsilon^2$.

On the other hand, for $t \in [\bar{t}, t^*]$, (3.33) and (3.34) hold, which lead to a contradiction, that is, (3.5) holds for $t \in [t_k, t_{k+1})$. By mathematical induction, (3.5) holds for any $m = 1,2,\dots$, which implies that system (2.1) is mean squarely stable. □

4. Application and Numerical Example

As an application, we consider the stochastic impulsive Hopfield neural network with delays in Yang et al. [9] as follows:

$$\begin{aligned} \mathrm{d}x(t) &= \left[-Cx(t) + Af(x(t)) + Bg(x_t)\right]\mathrm{d}t + \sigma(t, x(t), x_t)\mathrm{d}\omega(t), \quad t \geq t_0,\ t \neq t_k, \\ x(t_k) &= H_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, \\ x(t_0 + \theta) &= \varphi(\theta), \quad s \in [-\tau, 0], \end{aligned} \tag{4.1}$$

where the initial value $\varphi(s) \in \mathrm{PC}^b_{\mathcal{F}_0}(\delta)$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, $c_i > 0$ is the neuron-charging time constant, $A = (a_{ij})_{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix. $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in \mathbb{R}^n$ and $g(x_t) = (g_1(x_{1t}), g_2(x_{2t}), \ldots, g_n(x_{nt}))\mathbb{R}^n$, where $f_i(x_i(t))$ and $g_i(x_{it})$ denote, respectively, the measures of response or activation to its incoming potentials of the unit i at time t and time $t - \tau_i$. We also assume that $H_k(0) = 0$, $(k = 1, 2, \ldots)$, $f(0) = 0$, $g(0) = 0$, and $\sigma(t, 0, 0) = 0$, then system (4.1) admits an equilibrium solution $x(t) \equiv 0$. Moreover, we assume that $H(\cdot)$ satisfies (2.2), and $f(\cdot)$, $g(\cdot)$, $\sigma(\cdot)$ satisfy

$$\left\|f(x(t))\right\| \leq \|Fx(t)\|, \qquad \left\|g(x_t)\right\| \leq \|Gx_t\|, \tag{4.2}$$

$$\mathrm{tr}\left[\sigma^T(t, x(t), x_t)\sigma(t, x(t), x_t)\right] \leq \|Kx(t)\|^2 + \|K_\tau x_t\|^2, \tag{4.3}$$

where F, G, K, and K_τ are known constant matrices with appropriate dimensions.

Corollary 4.1. *Assume that there exist positive scalars ε_1, ε_2, β, symmetric matrix $P > 0$ and $\mu = \sup_{k\in\mathbb{N}}\{\lambda'_k = (\lambda_{\max}(P)/\lambda_{\min}(P))\gamma_k^2\}$. Then the following results hold:*

(i) *if $\mu > 1$, $\lambda + \mu\lambda_\tau < -\ln\mu/\beta$, then system* (4.1) *is mean squarely stable over impulsive time sequence $\mathcal{N}_{\inf}(\tau + \beta)$;*

(ii) *if $\mu = 1$, $\lambda + \lambda_\tau \leq 0$, then system* (4.1) *is mean squarely stable over any impulsive time sequence;*

(iii) *if $\mu < 1$ and $0 < \mu\lambda + \lambda_\tau \leq -\lambda_\tau \ln\mu$, then system* (4.1) *is mean squarely stable over impulsive time sequence $\mathcal{N}_{\sup}(-\mu\ln\mu/(\mu\lambda + \lambda_\tau))$;*

(iv) *if $\mu < 1$, $\mu\lambda + \lambda_\tau \leq 0$, then system* (4.1) *is mean squarely stable over any impulsive time sequence, where*

$$\begin{aligned} \lambda &= \lambda_{\max}\left(-2C + \varepsilon_1 AA^TP + \varepsilon_2 BB^TP + P^{-1}\left(\varepsilon_1^{-1}F^TF + \lambda_{\max}(P)K^TK\right)\right), \\ \lambda_\tau &= \lambda_{\max}\left(P^{-1}\left(\varepsilon_2^{-1}G^TG + \lambda_{\max}(P)K_\tau^TK_\tau\right)\right). \end{aligned} \tag{4.4}$$

Remark 4.2. Obviously, for this application, we extended and improved the according results in Yang et al. [9].

By Corollary (4.1), we consider the numerical example in Yang et al. [9].

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} -10.5 & 0 \\ 0 & -12.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.2 & -0.2 \\ 0.6 & 2.4 \end{bmatrix} \begin{bmatrix} \sin x_1(t) \\ \arctan x_2(t) \end{bmatrix} + \begin{bmatrix} 1.6 & 0.3 \\ -0.5 & 1.8 \end{bmatrix} \right.$$
$$\left. \times \begin{bmatrix} \sin x_1\left(t-\frac{1}{2}\right) \\ \arctan x_2\left(t-\frac{1}{2}\right) \end{bmatrix} \right\} dt + \begin{bmatrix} 2x_1(t) & x_2\left(t-\frac{1}{3}\right) \\ x_1\left(t-\frac{1}{2}\right) & -x_2(t) \end{bmatrix} \begin{bmatrix} d\omega_1(t) \\ d\omega_2(t) \end{bmatrix}, \quad t \geq t_0, \ t \neq t_k,$$
$$\begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix} = e^{-0.1k} \begin{bmatrix} 0.5 & -0.15 \\ 0.12 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(t_k^-) \\ x_2(t_k^-) \end{bmatrix}, \quad k = 1, 2, \ldots, \tag{4.5}$$

where $t_0 = 0$.

Similar to the result, we can verify that the point $(0,0)^T$ is an equilibrium point and can obtain by calculation that

$$P = \begin{pmatrix} 0.56 & 0 \\ 0 & 0.68 \end{pmatrix}, \quad K^T K = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.6}$$

and $\varepsilon_i = 1$ $(i = 1, 2)$, $\lambda_{\max}(P) = 0.68$, $\lambda_{\min}(P) = 0.56$, $\gamma_k = 0.620 \exp(-0.1k)$, $K_\tau^T K_\tau = I$, $F = G = I$, $\mu = 0.4668 < 1$, $\lambda = -12.0443$, $\lambda_\tau = 3$, and, hence, we have $\mu\lambda + \lambda_\tau = -2.6223$, which implies by (iv) in Corollary (4.1) that the above system is mean squarely stable over any impulsive time sequence.

5. Conclusion

In this paper, mean square stability of a class of impulsive stochastic differential equations with time delay has been considered. By Lyapunov-Krasovakii function and stochastic analysis, we obtain some new criteria ensuring mean square stability of the system (2.1). Some related results in Chen and Zheng [3] and Yang et al. [9] have been improved.

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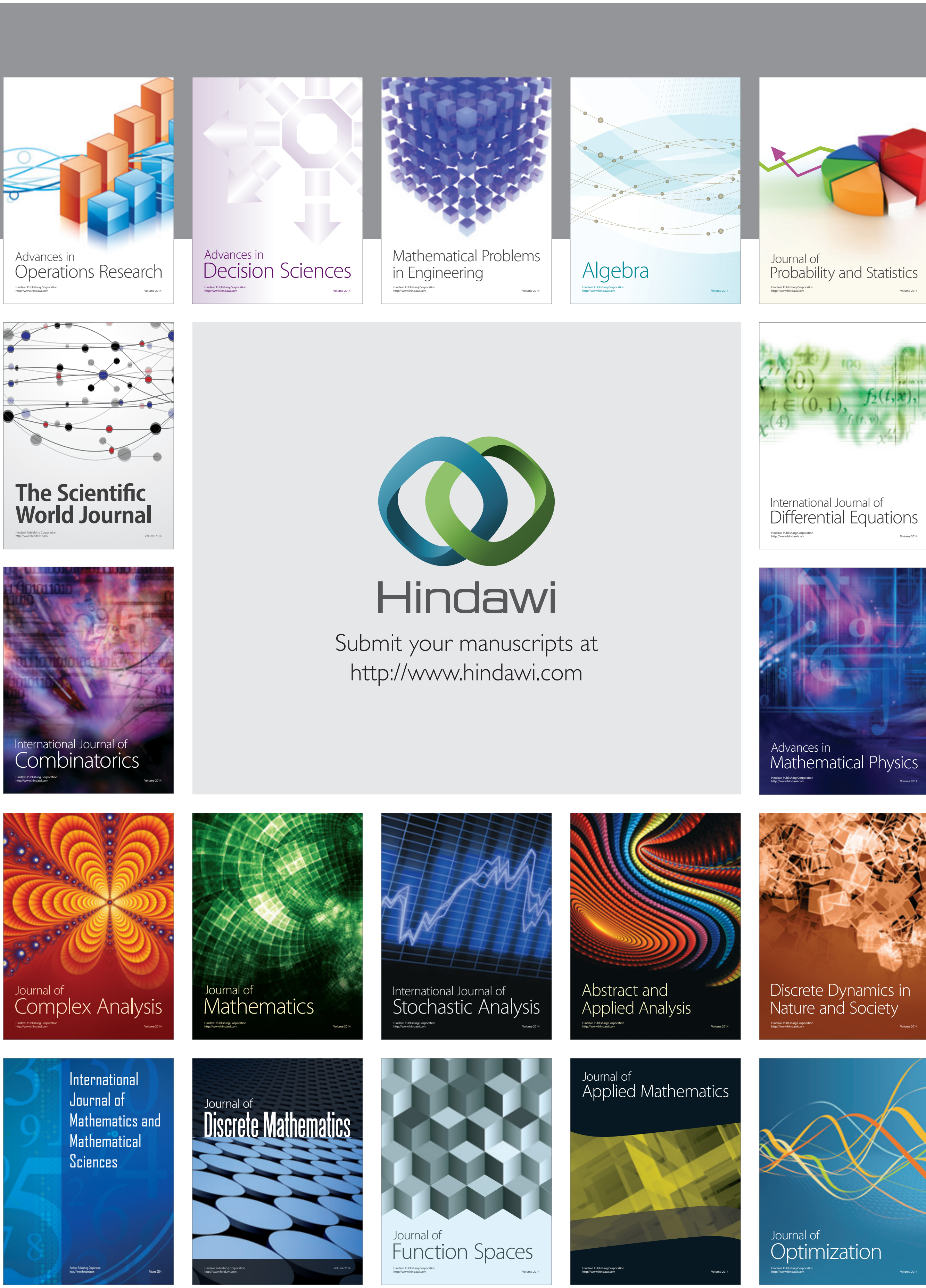
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