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Research Article

A High Order Iterative Scheme for a Nonlinear Kirchhoff Wave Equation in the Unit Membrane

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A high-order iterative scheme is established in order to get a convergent sequence at a rate of order N ($N \geq 1$) to a local unique weak solution of a nonlinear Kirchhoff wave equation in the unit membrane. This extends a recent result in (EJDE, 2005, No. 138) where a recurrent sequence converges at a rate of order 2.

1. Introduction

In this paper we consider the initial and boundary value problem

$$\begin{gathered}
u_{tt} - B\left(\|u_r\|_0^2\right)\left(u_{rr} + \frac{1}{r}u_r\right) = f(r,t,u), \quad 0 < r < 1,\ 0 < t < T, \\
\left|\lim_{r \to 0_+} \sqrt{r}u_r(r,t)\right| < \infty, \\
u_r(1,t) + hu(1,t) = 0, \\
u(r,0) = \widetilde{u}_0(r), \qquad u_t(r,0) = \widetilde{u}_1(r),
\end{gathered} \tag{1.1}$$

where $B, f, \widetilde{u}_0, \widetilde{u}_1$ are given functions satisfying conditions specified later, $\|u_r\|_0^2 = \int_0^1 r|u_r(r,t)|^2\,dr$, and $h > 0$ is a given constant.

Equation $(1.1)_1$ herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of the unit membrane $\Omega_1 = \{(x, y) : x^2 + y^2 < 1\}$. In the vibration process, the area of the unit membrane and the tension at various points change in time. The condition on the boundary $\partial\Omega_1$ describes elastic constraints, where the constant h_1 has a mechanical signification. The boundary condition $|\lim_{r\to 0_+} \sqrt{r} u_r(r, t)| < \infty$ is satisfied automatically if u is a classical solution of the problem (1.1), for example, with $u \in C^1([0, 1] \times (0, T)) \cap C^2((0, 1) \times (0, T))$. This condition is also used in connection with Sobolev spaces with weight r (see [1–3]).

Equation $(1.1)_1$ is related to the Kirchhoff equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L} \int_0^L \left| \frac{\partial u}{\partial y}(y, t) \right|^2 dy \right) u_{xx} \tag{1.2}$$

presented by Kirchhoff in 1876 (see [4]). This equation is an extension of the classical D'Alembert wave equation which considers the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

The Kirchhoff wave equation of the form $(1.1)_1$ received much attention. Many interesting results about the existence, stability, regularity in time variable, asymptotic behavior, and asymptotic expansion of solutions can be found, for example, in [2, 3, 5–14] and references therein.

In [2], in a special case, sufficient conditions were established for a quadratic convergence to the solution of (1.1) with $f(r, t, u) = f(r, u)$ and $B(\|u_r\|_0^2) = b_0 + \|u_r\|_0^2$, $b_0 > 0$. Based on the ideas about recurrence relations for a third-order method for solving the nonlinear operator equation $F(u) = 0$ in [15], we extend the above result by the construction of a high-order iterative scheme for $(1.1)_1$, where f and B are more generalized.

In this paper, we associate with $(1.1)_1$ a recurrent sequence $\{u_m\}$ defined by

$$\frac{\partial^2 u_m}{\partial t^2} - B\left(\|u_{mr}\|_0^2\right)\left(\frac{\partial^2 u_m}{\partial r^2} + \frac{1}{r}\frac{\partial u_m}{\partial r}\right) = \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(r, t, u_{m-1})(u_m - u_{m-1})^i, \tag{1.3}$$

$0 < r < 1$, $0 < t < T$, with u_m satisfying $(1.1)_{2\text{-}3}$. The first term u_0 is chosen as $u_0 \equiv 0$. If $B \in C^1(\mathbb{R}_+)$ and $f \in C^N([0, 1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at a rate of order N to a unique weak solution of the problem (1.1). This result is a relative generalization of [2, 3, 8, 9, 14, 16].

2. Preliminary Results, Notations, Function Spaces

Put $\Omega = (0, 1)$. We omit the definitions of the usual function spaces $C^m(\overline{\Omega})$, $L^p(\Omega)$, $H^m(\Omega)$, and $W^{m,p}(\Omega)$. For any function $v \in C^0(\overline{\Omega})$ we define $\|v\|_0$ as $\|v\|_0 = \left(\int_0^1 r v^2(r) dr\right)^{1/2}$ and define the space V_0 as completion of the space $C^0(\overline{\Omega})$ with respect to the norm $\|\cdot\|_0$. Similarly, for any function $v \in C^1(\overline{\Omega})$ we define $\|v\|_1$ as $\|v\|_1 = \left(\|v\|_0^2 + \|v_r\|_0^2\right)^{1/2}$ and define the space V_1 as completion of the space $C^1(\overline{\Omega})$ with respect to the norm $\|\cdot\|_1$. Note that the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ can be defined, respectively, from the inner products

$$\langle u,v\rangle = \int_0^1 ru(r)v(r)dr, \qquad \langle u,v\rangle + \langle u_r, v_r\rangle. \tag{2.1}$$

Identifying V_0 with its dual V_0' we obtain the dense and continuous embedding $V_1 \hookrightarrow V_0 \equiv V_0' \hookrightarrow V_1'$. The inner product notation will be reutilized to denote the duality pairing between V_1 and V_1'.

We then have the following lemmas, the proofs of which can be found in [1].

Lemma 2.1. *There exist two constants $K_1 > 0$ and $K_2 > 0$ such that, for all $v \in C^1(\overline{\Omega})$, we have*

(i) $\|v_r\|_0^2 + v^2(1) \geq \|v\|_0^2$,

(ii) $|v(1)| \leq K_1\|v\|_1$,

(iii) $\sqrt{r}|v(r)| \leq K_2\|v\|_1$, *for all* $r \in \overline{\Omega}$.

Lemma 2.2. *The embedding $V_1 \hookrightarrow V_0$ is compact.*

Remark 2.3. In Lemma 2.1, the two constants K_1 and K_2 can be given explicitly as $K_1 = \sqrt{1+\sqrt{2}}$ and $K_2 = \sqrt{1+\sqrt{5}}$. We also note that $\lim_{r\to 0_+}\sqrt{r}v(r) = 0$ for all $v \in V_1$ (see [17, page 128/Lemma 5.40]). On the other hand, by $H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1])$, $0 < \varepsilon < 1$ and $\sqrt{\varepsilon}\|v\|_{H^1(\varepsilon,1)} \leq \|v\|_1$ for all $v \in V_1$, it follows that $v|_{[\varepsilon,1]} \in C^0([\varepsilon,1])$. From both relations we deduce that $\sqrt{r}v \in C^0(\overline{\Omega})$ for all $v \in V_1$.

Now, let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(u,v) = hu(1)v(1) + \int_0^1 ru_r(r)v_r(r)dr, \quad u,v \in V_1, \tag{2.2}$$

where h is a positive constant. Then, there exists a unique bounded linear operator $A : V_1 \to V_1'$ such that $a(u,v) = \langle Au, v\rangle$ for all $u,v \in V_1$. We then have the following lemma.

Lemma 2.4. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.2) *is continuous on $V_1 \times V_1$ and coercive on V_1, that is,*

(i) $|a(u,v)| \leq C_1\|u\|_1\|v\|_1$,

(ii) $a(v,v) \geq C_0\|v\|_1^2$,

for all $u,v \in V_1$, where $C_0 = (1/2)\min\{1,h\}$ and $C_1 = 1 + (1+\sqrt{2})h$.

The proof of Lemma 2.4 is straightforward and we omit it.

Lemma 2.5. *There exists an orthonormal Hilbert basis $\{w_j\}$ of the space V_0 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

(i) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \uparrow +\infty$ *as* $j \to \infty$,

(ii) $a(w_j, v) = \lambda_j\langle w_j, v\rangle$ *for all* $v \in V_1$ *and* $j \in \mathbb{N}$.

Note that it follows from (ii) that $\{w_j/\sqrt{\lambda_j}\}$ *is automatically an orthonormal set in* V_1 *with respect to* $a(\cdot,\cdot)$ *as inner product. The eigensolutions* w_j *are indeed eigensolutions for the boundary value problem*

$$Aw_j \equiv \frac{-1}{r}\frac{d}{dr}\left(r\frac{dw_j}{dr}\right) = \lambda_j w_j, \quad \textit{in } \Omega,$$
$$\left|\lim_{r\to 0_+}\sqrt{r}\frac{dw_j}{dr}(r)\right| < +\infty, \qquad \frac{dw_j}{dr}(1) + hw_j(1) = 0. \tag{2.3}$$

The proof of Lemma 2.5 can be found in ([18, page 87, Theorem 7.7]) with $V = V_1$, $H = V_0$ and $a(\cdot,\cdot)$ as defined by (2.2).

For any function $v \in C^2(\overline{\Omega})$ we define $\|v\|_2$ as

$$\|v\|_2 = \left(\|v\|_0^2 + \|v_r\|_0^2 + \|Av\|_0^2\right)^{1/2} \tag{2.4}$$

and define the space V_2 as completion of $C^2(\overline{\Omega})$ with respect to the norm $\|\cdot\|_2$. Note that V_2 is also a Hilbert space with respect to the scalar product

$$\langle u,v\rangle + \langle u_r,v_r\rangle + \langle Au,Av\rangle \tag{2.5}$$

and that V_2 can be defined also as $V_2 = \{v \in V_1 : Av \in V_0\}$.

We then have the following two lemmas the proof of which can be found in [1].

Lemma 2.6. *The embedding* $V_2 \hookrightarrow V_1$ *is compact.*

Lemma 2.7. *For all* $v \in V_2$ *we have*

$$\text{(i)} \quad \|v_r\|_{L^\infty(\Omega)} \le \frac{1}{\sqrt{2}}\|Av\|_0,$$
$$\text{(ii)} \quad \|v_{rr}\|_0 \le \sqrt{\frac{3}{2}}\|Av\|_0, \tag{2.6}$$
$$\text{(iii)} \quad \|v\|_{L^\infty(\Omega)}^2 \le \left(2\|v\|_0 + \frac{1}{\sqrt{2}}\|Av\|_0\right)\|v\|_0.$$

For a Banach space X, we denote by $\|\cdot\|_X$ its norm, by X' its dual space, and by $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of all real measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty, \quad \text{for } 1 \le p < \infty,$$
$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T}\|u(t)\|_X \quad \text{for } p = \infty. \tag{2.7}$$

Let

$$u(t), \quad u'(t)=u_t(t)=\dot{u}(t), \quad u''(t)=u_{tt}(t)=\ddot{u}(t), \quad u_r(t)=\nabla u(t), \quad u_{rr}(t) \tag{2.8}$$

denote

$$u(r,t), \quad \frac{\partial u}{\partial t}(r,t), \quad \frac{\partial^2 u}{\partial t^2}(r,t), \quad \frac{\partial u}{\partial r}(r,t), \quad \frac{\partial^2 u}{\partial r^2}(r,t), \tag{2.9}$$

respectively.

With $f \in C^k(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(r,t,u)$, we put $D_1 f = \partial f/\partial r$, $D_2 f = \partial f/\partial t$, $D_3 f = \partial f/\partial u$, and $D^\gamma f = D_1^{\gamma_1} D_2^{\gamma_2} D_3^{\gamma_3} f$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}_+^3$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3 = k$.

3. The Hight Order Iterative Schemes

Fix $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0 \in V_2$ and $\tilde{u}_1 \in V_1$;

(H_2) $B \in C^1(\mathbb{R}_+)$ and there exist constants $b_* > 0$, $\alpha > 1$, $d_0, d_1 > 0$ such that

(i) $b_* \leq B(\eta) \leq d_0(1+\eta^\alpha)$, for all $\eta \geq 0$,

(ii) $|B'(\eta)| \leq d_1(1+\eta^{\alpha-1})$, for all $\eta \geq 0$;

(H_3) $f \in C^N(\overline{\Omega} \times [0,T^*] \times \mathbb{R})$ and satisfies the following condition : for all $M > 0$,

$$\begin{aligned} \widehat{K}_{*i}^{(1)}(M,f) &= \sup_{(r,t,u)\in \overline{A}_M} \left| (\sqrt{r})^{-i} \frac{\partial^i f}{\partial u^i}(r,t,u) \right| < +\infty, \quad i = 0,1,\ldots,N-1, \\ \widehat{K}_{*i}^{(2)}(M,f) &= \sup_{(r,t,u)\in \overline{A}_M} \left| (\sqrt{r})^{-i} \frac{\partial^i f}{\partial r \partial u^{i-1}}(r,t,u) \right| < +\infty, \quad i = 1,\ldots,N-1, \end{aligned} \tag{3.1}$$

where $\overline{A}_M = \{(r,t,u) \in [0,1] \times [0,T^*] \times \mathbb{R} : |u| \leq M\sqrt{2 + 1/\sqrt{2}}\}$. We put

$$\widehat{K}_{*i}(M,f) = \begin{cases} \widehat{K}_{*0}^{(1)}(M,f), & i = 0, \\ \max\left\{ \widehat{K}_{*i}^{(1)}(M,f),\ \widehat{K}_{*i}^{(2)}(M,f) \right\}, & i = 1,\ldots,N-1. \end{cases} \tag{3.2}$$

With B and f satisfying assumptions (H_2) and (H_3), respectively, for each $M > 0$ given, we introduce the following constants:

$$\widetilde{K}_M(B) = \sup_{0\leq\eta\leq M^2} \left(B(\eta) + \left|B'(\eta)\right|\right),$$
$$\overline{K}_0(M, f) = \sup_{(r,t,u)\in\overline{A}_M} |f(r,t,u)|, \tag{3.3}$$
$$\overline{K}_N(M, f) = \sum_{|\gamma|\leq N} \overline{K}_0(M, D^\gamma f).$$

For each $T \in (0, T^*]$ and $M > 0$ we get

$$W(M,T) = \Big\{v \in L^\infty(0,T;V_2) : v' \in L^\infty(0,T;V_1),\ v'' \in L^2(0,T;V_0),$$
$$\text{with } \|v\|_{L^\infty(0,T;V_2)}, \left\|v'\right\|_{L^\infty(0,T;V_1)}, \left\|v''\right\|_{L^2(0,T;V_0)} \leq M\Big\}, \tag{3.4}$$
$$W_1(M,T) = \left\{v \in W(M,T) : v'' \in L^\infty(0,T;V_0)\right\}.$$

We will choose as first initial term $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.5}$$

and associate with the problem (1.1) the following variational problem.

Find $u_m \in W_1(M,T)(m \geq 1)$ so that

$$\left\langle u_m''(t), v\right\rangle + b_m(t)a(u_m(t), v) = \langle F_m(t), v\rangle, \quad \forall v \in V_1,$$
$$u_m(0) = \widetilde{u}_0, \qquad u_m'(0) = \widetilde{u}_1, \tag{3.6}$$

where

$$b_m(t) = B\left(\|\nabla u_m(t)\|_0^2\right), \quad F_m(r,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(r,t,u_{m-1})(u_m - u_{m-1})^i. \tag{3.7}$$

Then, we have the following theorem.

Theorem 3.1. *Let assumptions (H_1)–(H_3) hold. Then there exist a constant $M > 0$ depending on $T^*, \widetilde{u}_0, \widetilde{u}_1, B, h$ and a constant $T > 0$ depending on $T^*, \widetilde{u}_0, \widetilde{u}_1, B, h, f$ such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.6), (3.7).

Proof. The proof consists of several steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [19]). Consider as in Lemma 2.5 the basis $\{w_j\}$ for V_1 and put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.8}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of the following nonlinear differential equations:

$$\begin{gathered}\left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + b_m^{(k)}(t) a\left(u_m^{(k)}(t), w_j\right) = \left\langle F_m^{(k)}(t), w_j \right\rangle, \quad 1 \le j \le k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k}, \qquad \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k},\end{gathered} \tag{3.9}$$

where

$$\begin{gathered}\widetilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \widetilde{u}_0 \text{ strongly in } V_2, \\ \widetilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \widetilde{u}_1 \text{ strongly in } V_1.\end{gathered} \tag{3.10}$$

$$b_m^{(k)}(t) = B\left(\left\| \nabla u_m^{(k)}(t) \right\|_0^2\right),$$

$$F_m^{(k)}(r,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(r,t,u_{m-1}) \left(u_m^{(k)} - u_{m-1}\right)^i = \sum_{j=0}^{N} \Psi_j(r,t,u_{m-1}) \left(u_m^{(k)}\right)^j, \tag{3.11}$$

$$\Psi_j(r,t,u_{m-1}) = \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} (-1)^{i-j} D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j}, \quad 0 \le j \le N-1.$$

Let us suppose that u_{m-1} satisfies (3.5). Then we have the following lemma.

Lemma 3.2. *Let assumptions* (H_1)–(H_3) *hold. For fixed* $M > 0$ *and* $T > 0$*, then, the system* (3.8)–(3.11) *has a unique solution* $u_m^{(k)}(t)$ *on an interval* $[0, T_m^{(k)}] \subset [0, T]$.

Proof of Lemma 3.2. The system of (3.8)–(3.11) is rewritten in the form

$$\begin{gathered}\ddot{c}_{mj}^{(k)}(t) = -\lambda_j b_m^{(k)}(t) c_{mj}^{(k)}(t) + \left\langle F_m^{(k)}(t), w_j \right\rangle, \quad 1 \le j \le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)},\end{gathered} \tag{3.12}$$

and it is equivalent to the system of integral equations

$$c_{mj}^{(k)}(t) = \alpha_j^{(k)} + \beta_j^{(k)} t - \lambda_j \int_0^t d\tau \int_0^\tau b_m^{(k)}(s) c_{mj}^{(k)}(s) ds + \int_0^t d\tau \int_0^\tau \left\langle F_m^{(k)}(s), w_j \right\rangle ds, \tag{3.13}$$

for $1 \le j \le k$. Omitting the index m, it is written as follows:

$$c = \mathcal{F}[c], \tag{3.14}$$

where $\mathcal{F}[c] = (\mathcal{F}_1[c], \ldots, \mathcal{F}_k[c])$, $c = (c_1, \ldots, c_k)$,

$$\begin{aligned}
\mathcal{F}_j[c](t) &= q_j(t) - \lambda_j \int_0^t d\tau \int_0^\tau \widetilde{b}(s) c_j(s) ds \\
&\quad + \sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \left\langle \Psi_i(\cdot, s, u_{m-1}) u^i(s), w_j \right\rangle ds, \quad 1 \le j \le k, \\
q_j(t) &= \alpha_j^{(k)} + \beta_j^{(k)} t + \int_0^t d\tau \int_0^\tau \langle \Psi_0(\cdot, s, u_{m-1}), w_j \rangle ds, \quad 1 \le j \le k, \\
\widetilde{b}(t) &= \widetilde{b}[c](t) = B\left(\|\nabla u(t)\|_0^2\right), \quad u(t) = \sum_{j=1}^k c_j(t) w_j.
\end{aligned} \tag{3.15}$$

For every $T_m^{(k)} \in (0, T]$ and $\rho > 0$ that will be chosen later, we put $X = C^0([0, T_m^{(k)}]; \mathbb{R}^k)$, $S = \{c \in Y : \|c\|_X \le \rho\}$, where $\|c\|_X = \sup_{0 \le t \le T_m^{(k)}} |c(t)|_1$, $|c(t)|_1 = \sum_{j=1}^k |c_j(t)|$, for each $c = (c_1, \ldots, c_k) \in X$. Clearly S is a closed nonempty subset in X, and we have the operator $\mathcal{F} : X \to X$. In what follows, we will choose $\rho > 0$ and $T_m^{(k)} > 0$ such that

(i) S is mapped into itself by $\mathcal{F}$,

(ii) $\mathcal{F} : S \to S$ is contractive.

Proof (i). First we note that, for all $c = (c_1, \ldots, c_k) \in S$,

$$\begin{aligned}
u(t) &= \sum_{j=1}^k c_j(t) w_j, \\
\|u(t)\|_0 &= \sqrt{\sum_{j=1}^k c_j^2(t)} \le |c(t)|_1, \\
\|\nabla u(t)\|_0^2 &\le a(u(t), u(t)) = \sum_{i,j=1}^k c_i(t) c_j(t) a(w_i, w_j) = \sum_{j=1}^k \lambda_j c_j^2(t) \le \lambda_k \|u(t)\|_0^2, \\
\|u(t)\|_0 &\le |c(t)|_1 \le \|c\|_X \le \rho, \qquad \|\nabla u(t)\|_0 \le \sqrt{\lambda_k} |c(t)|_1 \le \sqrt{\lambda_k} \rho, \\
\|u(t)\|_1 &\le \sqrt{\frac{1}{C_0}} \sqrt{a(u(t), u(t))} \le \sqrt{\frac{1}{C_0}} \sqrt{\lambda_k} \|u(t)\|_0 \le \sqrt{\frac{\lambda_k}{C_0}} |c(t)|_1 \le \sqrt{\frac{\lambda_k}{C_0}} \rho,
\end{aligned} \tag{3.16}$$

so

$$\widetilde{b}(t) = B\left(\|\nabla u(t)\|_0^2\right) \le d_0 \left(1 + \|\nabla u(t)\|_0^{2\alpha}\right) \le d_0 \left(1 + \lambda_k^{2\alpha} \rho^{2\alpha}\right). \tag{3.17}$$

On the other hand, by

$$
\begin{aligned}
|u_{m-1}| &\le M\sqrt{2+\frac{1}{\sqrt{2}}} \equiv \theta, \\
|\Psi_0(r,t,u_{m-1})| &= \left|\sum_{i=0}^{N-1}\frac{1}{i!}(-1)^i D_3^i f(r,t,u_{m-1})u_{m-1}^i\right| \le \sum_{i=0}^{N-1}\left|\frac{1}{i!}D_3^i f(r,t,u_{m-1})u_{m-1}^i\right| \\
&\le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{1}{i!}\left(M\sqrt{2+\frac{1}{\sqrt{2}}}\right)^i = \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!},
\end{aligned}
\tag{3.18}
$$

we have

$$
\left|\left\langle \Psi_0(t,u_{m-1}),w_j\right\rangle\right| \le \|\Psi_0(t,u_{m-1})\|_0\left\|w_j\right\|_0 = \|\Psi_0(t,u_{m-1})\|_0 \le \frac{1}{\sqrt{2}}\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}. \tag{3.19}
$$

By Lemma 2.1, (iii), and the assumption (H_3), we deduce from (3.16) that

$$
\begin{aligned}
&\left|\left\langle \Psi_i(s,u_{m-1})u^i(s),w_j\right\rangle\right| \\
&\quad= \left|\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}(-1)^{l-i}\left\langle D_3^l f(r,t,u_{m-1})u_{m-1}^{l-i}u^i(s),w_j\right\rangle\right| \\
&\quad= \left|\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}(-1)^{l-i}\left\langle (\sqrt{r})^{-l}D_3^l f(r,s,u_{m-1})(\sqrt{r})^{l-i}u_{m-1}^{l-i}(\sqrt{r})^i u^i(s),w_j\right\rangle\right| \\
&\quad\le \sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\|u(s)\|_1^i\left\langle (\sqrt{r})^{l-i},|w_j|\right\rangle \\
&\quad\le \sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^i\frac{1}{\sqrt{2+l-i}} \\
&\quad\le \sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\frac{1}{\sqrt{2+l-i}}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^i, \quad 1\le i\le N-1.
\end{aligned}
\tag{3.20}
$$

It follows that

$$
\begin{aligned}
\left|\mathcal{F}_j[c](t)\right| &\le \left|q_j(t)\right| + \lambda_k d_0\left(1+\lambda_k^{2\alpha}\rho^{2\alpha}\right)\int_0^t d\tau\int_0^\tau\left|c_j(s)\right|ds \\
&\quad+\frac{1}{2}t^2\sum_{i=1}^{N-1}\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\frac{1}{\sqrt{2+l-i}}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^i.
\end{aligned}
\tag{3.21}
$$

Thus

$$\begin{aligned}|\mathcal{F}[c](t)|_1 &\le |q(t)|_1 + \lambda_k d_0\left(1+\lambda_k^{2\alpha}\rho^{2\alpha}\right)\int_0^t d\tau \int_0^\tau |c(s)|_1 ds \\ &\quad + \frac{1}{2}kt^2\sum_{i=1}^{N-1}\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\frac{1}{\sqrt{2+l-i}}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^i \\ &\le \|q\|_T + \frac{1}{2}\overline{D}_\rho\left(T_m^{(k)}\right)^2,\end{aligned} \tag{3.22}$$

where $\|q\|_T = \sup_{0\le t\le T}|q(t)|_1$ and

$$\begin{aligned}\overline{D}_\rho &= \overline{D}_\rho(f,\rho,k,M,m,N) \\ &= \lambda_k d_0\left(1+\lambda_k^{2\alpha}\rho^{2\alpha}\right)\rho + k\sum_{i=1}^{N-1}\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\frac{1}{\sqrt{2+l-i}}\widehat{K}_{*l}(M,f)\theta^{l-i}K_2^i\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^i.\end{aligned} \tag{3.23}$$

Hence, we obtain

$$\|\mathcal{F}[c]\|_X \le \|q\|_T + \frac{1}{2}\overline{D}_\rho\left(T_m^{(k)}\right)^2, \tag{3.24}$$

choosing $\rho > \|q\|_T$ and $T_m^{(k)} \in (0,T]$, such that

$$\frac{1}{2}\overline{D}_\rho\left(T_m^{(k)}\right)^2 \le \rho - \|q\|_T, \qquad \frac{1}{2}\widetilde{D}_\rho\left(T_m^{(k)}\right)^2 < 1, \tag{3.25}$$

where

$$\begin{aligned}\widetilde{D}_\rho &= \widetilde{D}_\rho(\rho,k,M,T,m,N,f) \\ &\equiv d_0\lambda_k\left(1+\lambda_k^{2\alpha}\rho^{2\alpha}\right) + 2d_1\lambda_k^2\rho^2\left(1+\lambda_k^{\alpha-1}\rho^{2\alpha-2}\right) \\ &\quad + k\sum_{i=1}^{N-1} i\left(\sqrt{\frac{\lambda_k}{C_0}}K_2\right)^i\rho^{i-1}\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\frac{1}{\sqrt{2+l-i}}\widehat{K}_{*l}(M,f)\theta^{l-i}.\end{aligned} \tag{3.26}$$

Then

$$\|\mathcal{F}[c]\|_X \le \|q\|_T + \frac{1}{2}\overline{D}_\rho\left(T_m^{(k)}\right)^2 \le \rho, \quad \forall c \in S, \tag{3.27}$$

which means that $\mathcal{F}$ maps S into itself. □

Proof (ii). We now prove that, for all $c, d \in S$, for all $t \in [0, T_m^{(k)}]$,

$$|\mathcal{F}[c](t) - \mathcal{F}[d](t)|_1 \leq \frac{1}{2}\widetilde{D}_\rho t^2 \|c-d\|_X, \quad \forall n \in \mathbb{N}, \tag{3.28}$$

where $\widetilde{D}_\rho$ is defined as (3.26).

Proof of (3.28) is as follows.

For all $j = 1, 2, \ldots, k$, for all $t \in [0, T_m^{(k)}]$, we have

$$\begin{aligned} \left|\mathcal{F}_j[c](t) - \mathcal{F}_j[d](t)\right| &\leq \lambda_j \int_0^t d\tau \int_0^\tau \left|\widetilde{b}[c](s)\left(c_j(s) - d_j(s)\right)\right| ds \\ &\quad + \lambda_j \int_0^t d\tau \int_0^\tau \left|\left(\widetilde{b}[c](s) - \widetilde{b}[d](s)\right) d_j(s)\right| ds \\ &\quad + \sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \left|\left\langle \Psi_i(s, u_{m-1})\left(u^i(s) - v^i(s)\right), w_j \right\rangle\right| ds, \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} \widetilde{b}[c](t) &= B\left(\|\nabla u(t)\|_0^2\right), \qquad \widetilde{b}[d](t) = B\left(\|\nabla v(t)\|_0^2\right), \\ u(t) &= \sum_{j=1}^k c_j(t) w_j, \qquad v(t) = \sum_{j=1}^k d_j(t) w_j, \end{aligned} \tag{3.30}$$

so

$$\begin{aligned} |\mathcal{F}[c](t) - \mathcal{F}[d](t)|_1 &\leq \lambda_k \int_0^t d\tau \int_0^\tau \widetilde{b}[c](s)|c(s) - d(s)|_1 ds \\ &\quad + \lambda_k \int_0^t d\tau \int_0^\tau \left|\widetilde{b}[c](s) - \widetilde{b}[d](s)\right| |d(s)|_1 ds \\ &\quad + \sum_{j=1}^k \sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \left|\left\langle \Psi_i(s, u_{m-1})\left(u^i(s) - v^i(s)\right), w_j \right\rangle\right| ds \\ &\leq \lambda_k \int_0^t d\tau \int_0^\tau \widetilde{b}[c](s)|c(s) - d(s)|_1 ds + \lambda_k \rho \int_0^t d\tau \int_0^\tau \left|\widetilde{b}[c](s) - \widetilde{b}[d](s)\right| ds \\ &\quad + \sum_{j=1}^k \sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \left|\left\langle \Psi_i(s, u_{m-1})\left(u^i(s) - v^i(s)\right), w_j \right\rangle\right| ds \equiv J_1 + J_2 + J_3, \end{aligned} \tag{3.31}$$

in which

$$J_1 = \lambda_k \int_0^t d\tau \int_0^\tau \widetilde{b}[c](s)|c(s) - d(s)|_1 ds$$
$$\leq \lambda_k d_0 \left(1 + \lambda_k^{2\alpha}\rho^{2\alpha}\right) \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds \tag{3.32}$$
$$\equiv \zeta_1 \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds.$$

In order to consider J_2, we also note that

$$\widetilde{b}[c](s) - \widetilde{b}[d](s) = B'(\xi)\left(\|\nabla u(s)\|_0^2 - \|\nabla v(s)\|_0^2\right), \tag{3.33}$$

where

$$\xi = \theta\|\nabla u(s)\|_0^2 + (1-\theta)\|\nabla v(s)\|_0^2, \qquad 0 \leq \xi \leq \lambda_k \rho^2, \quad 0 < \theta < 1, \tag{3.34}$$

and $B'(\xi)$ satisfy the following inequality:

$$\left|B'(\xi)\right| \leq d_1\left(1 + \xi^{\alpha-1}\right) \leq d_1\left(1 + \lambda_k^{\alpha-1}\rho^{2\alpha-2}\right). \tag{3.35}$$

It implies that

$$\left|\widetilde{b}[c](s) - \widetilde{b}[d](s)\right| = \left|B'(\xi)\left(\|\nabla u(s)\|_0^2 - \|\nabla v(s)\|_0^2\right)\right|$$
$$\leq d_1\left(1 + \lambda_k^{\alpha-1}\rho^{2\alpha-2}\right) 2\lambda_k \rho |c(s) - d(s)|_1 \tag{3.36}$$
$$= 2\lambda_k \rho d_1 \left(1 + \lambda_k^{\alpha-1}\rho^{2\alpha-2}\right)|c(s) - d(s)|_1,$$

and then

$$J_2 = \lambda_k \rho \int_0^t d\tau \int_0^\tau \left|\widetilde{b}[c](s) - \widetilde{b}[d](s)\right| ds$$
$$\leq 2\lambda_k^2 \rho^2 d_1 \left(1 + \lambda_k^{\alpha-1}\rho^{2\alpha-2}\right) \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds \tag{3.37}$$
$$\equiv \zeta_2 \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds.$$

It remains to estimate J_3. By

$$\left(\sqrt{r}u(s)\right)^i - \left(\sqrt{r}v(s)\right)^i = \sum_{j=0}^{i-1}\left(\sqrt{r}u(s)\right)^j\left(\sqrt{r}v(s)\right)^{i-j-1}\sqrt{r}(u(s)-v(s)), \tag{3.38}$$

we obtain

$$\begin{aligned}
\left|\left(\sqrt{r}u(s)\right)^i - \left(\sqrt{r}v(s)\right)^i\right| &\le K_2^i\sum_{j=0}^{i-1}\|u(s)\|_1^j\|v(s)\|_1^{i-j-1}\|u(s)-v(s)\|_1 \\
&\le K_2^i\sum_{j=0}^{i-1}\|u(s)\|_1^j\|v(s)\|_1^{i-j-1}\|u(s)-v(s)\|_1 \\
&\le K_2^i\sum_{j=0}^{i-1}\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^j\left(\sqrt{\frac{\lambda_k}{C_0}}\rho\right)^{i-j-1}\sqrt{\frac{\lambda_k}{C_0}}|c(s)-d(s)|_1 \\
&= K_2^i\sum_{j=0}^{i-1}\left(\sqrt{\frac{\lambda_k}{C_0}}\right)^i\rho^{i-1}|c(s)-d(s)|_1 \\
&= i\left(\sqrt{\frac{\lambda_k}{C_0}}K_2\right)^i\rho^{i-1}|c(s)-d(s)|_1.
\end{aligned} \tag{3.39}$$

On the other hand,

$$\begin{aligned}
\Psi_i(r,t,u_{m-1}) &= \sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}(-1)^{l-i}D_3^l f(r,t,u_{m-1})u_{m-1}^{l-i}, \\
\left(\sqrt{r}\right)^{-i}|\Psi_i(r,t,u_{m-1})| &= \left|\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}(-1)^{l-i}\left(\sqrt{r}\right)^{-l}D_3^l f(r,t,u_{m-1})\left(\sqrt{r}\right)^{l-i}u_{m-1}^{l-i}\right| \\
&\le \left|\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\left(\sqrt{r}\right)^{-l}\left|D_3^l f(r,t,u_{m-1})\right|\left(\sqrt{r}\right)^{l-i}\left|u_{m-1}^{l-i}\right|\right| \\
&\le \sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\widehat{K}_{*l}(M,f)\theta^{l-i}\left(\sqrt{r}\right)^{l-i}.
\end{aligned} \tag{3.40}$$

Hence, we deduce from (3.39), (3.40) that

$$\begin{aligned}
J_3 &= \sum_{j=1}^{k}\sum_{i=1}^{N-1}\int_0^t d\tau\int_0^\tau\left|\left\langle\left(\sqrt{r}\right)^{-i}\Psi_i(s,u_{m-1})\left(\left(\sqrt{r}u(s)\right)^i-\left(\sqrt{r}v(s)\right)^i\right),w_j\right\rangle\right|ds \\
&\le \sum_{j=1}^{k}\sum_{i=1}^{N-1}\sum_{l=i}^{N-1}\frac{1}{i!(l-i)!}\widehat{K}_{*l}(M,f)\theta^{l-i}i\left(\sqrt{\frac{\lambda_k}{C_0}}K_2\right)^i\rho^{i-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t d\tau \int_0^\tau \left| \left\langle \left(\sqrt{r}\right)^{l-i}, |w_j| \right\rangle \right| |c(s) - d(s)|_1 ds \\
&= \sum_{j=1}^{k} \sum_{i=1}^{N-1} \sum_{l=i}^{N-1} \frac{1}{i!(l-i)!} \widehat{K}_{*l}(M, f) \theta^{l-i} i \left(\sqrt{\frac{\lambda_k}{C_0}} K_2 \right)^i \rho^{i-1} \frac{1}{\sqrt{2+l-i}} \\
&\times \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds \\
&= k \sum_{i=1}^{N-1} i \left(\sqrt{\frac{\lambda_k}{C_0}} K_2 \right)^i \rho^{i-1} \sum\nolimits_{l=i}^{N-1} \frac{1}{i!(l-i)!} \widehat{K}_{*l}(M, f) \theta^{l-i} \frac{1}{\sqrt{2+l-i}} \\
&\times \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds \\
&\equiv \zeta_3 \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds.
\end{aligned}
\tag{3.41}
$$

We deduce that

$$
|\mathcal{F}[c](t) - \mathcal{F}[d](t)|_1 \leq (\zeta_1 + \zeta_2 + \zeta_3) \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 ds \leq \frac{1}{2} \widetilde{D}_\rho t^2 \|c - d\|_X. \tag{3.42}
$$

We note that

$$
\zeta_1 + \zeta_2 + \zeta_3 = \widetilde{D}_\rho(\rho, k, M, T, m, N, f) = \widetilde{D}_\rho. \tag{3.43}
$$

It follows from (3.28) that

$$
\|\mathcal{F}[c] - \mathcal{F}[d]\|_X \leq \frac{1}{2} \widetilde{D}_\rho \left(T_m^{(k)}\right)^2 \|c - d\|_X, \quad \forall c, d \in S. \tag{3.44}
$$

By (3.25), it follows that $\mathcal{F} : S \rightarrow S$ is contractive. We deduce that $\mathcal{F}$ has a unique fixed point in S; that is, the system (3.8)–(3.11) has a unique solution $u_m^{(k)}(t)$ on an interval $[0, T_m^{(k)}]$. The proof of Lemma 3.2 is complete. □ □

The following estimates allow one to take constant $T_m^{(k)} = T$ for all m and k.

Step 2. A priori estimates. Put

$$
S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 ds, \tag{3.45}
$$

where

$$X_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + b_m^{(k)}(t)a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right),$$
$$Y_m^{(k)}(t) = a\left(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) + b_m^{(k)}(t)\left\|Au_m^{(k)}(t)\right\|_0^2, \tag{3.46}$$

with A is defined by (2.2). Then it follows that

$$\begin{aligned} S_m^{(k)}(t) = S_m^{(k)}(0) &+ \int_0^t \dot{b}_m^{(k)}(s)\left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\|Au_m^{(k)}(s)\right\|_0^2\right]ds \\ &+ 2\int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right\rangle ds + 2\int_0^t a\left(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right)ds \\ &+ \int_0^t \left\langle F_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\right\rangle ds - \int_0^t b_m^{(k)}(s)\left\langle Au_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\right\rangle ds \\ \equiv S_m^{(k)}(0) &+ \sum_{j=1}^{5} I_j. \end{aligned} \tag{3.47}$$

We will now require the following lemma.

Lemma 3.3. *We have*

$$\begin{aligned} &\text{(i)} \quad 0 < b_* \le b_m^{(k)}(t) \le d_0\left(1 + \left\|\nabla u_m^{(k)}(t)\right\|_0^{2\alpha}\right), \\ &\text{(ii)} \quad \left|\dot{b}_m^{(k)}(t)\right| \le \frac{2d_1}{\sqrt{b_*}}\left[S_m^{(k)}(t) + b_*^{1-\alpha}\left(S_m^{(k)}(t)\right)^{\alpha}\right], \\ &\text{(iii)} \quad \left\|F_m^{(k)}(t)\right\|_0 \le \sum_{j=0}^{N-1} \widetilde{a}_j^{(0)}\left(\sqrt{S_m^{(k)}(t)}\right)^j, \\ &\text{(iv)} \quad \left\|\frac{\partial}{\partial r}F_m^{(k)}(t)\right\|_0 \le \sum_{j=0}^{N-1} \widetilde{a}_j^{(1)}\left(\sqrt{S_m^{(k)}(t)}\right)^j, \end{aligned} \tag{3.48}$$

where $\widetilde{a}_j^{(0)}, \widetilde{a}_j^{(1)},\ j = 0, 1, \ldots, N-1$ *are defined as follows:*

$$\widetilde{a}_j^{(0)} = \begin{cases} \widetilde{a}_0^{(0)} = \dfrac{1}{\sqrt{2}}\overline{K}_N(M, f)\displaystyle\sum_{i=0}^{N-1}\frac{\theta^i}{i!}, & j = 0, \\ \widetilde{a}_j^{(0)} = \dfrac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j}\displaystyle\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\frac{1}{\sqrt{2+i-j}}\widehat{K}_{*i}(M, f), & 1 \le j \le N-1, \end{cases}$$

$$\widetilde{a}_j^{(1)} = \begin{cases} \left(\dfrac{1}{\sqrt{2}} + 2M\right)\overline{K}_N(M,f)\displaystyle\sum_{i=0}^{N-1}\frac{\theta^i}{i!}, & j = 0, \\ \dfrac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j}\displaystyle\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\Bigg[jK_2^{-1}\widehat{K}_{*i}(M,f) \\ \qquad\qquad + \left(\dfrac{1}{\sqrt{i-j+3}} + 2M\right)\widehat{K}_{*i+1}(M,f)\Bigg], & 1 \le j \le N-1, \end{cases}$$

$$\theta = M\sqrt{2 + \frac{1}{\sqrt{2}}}. \tag{3.49}$$

Proof of Lemma 3.3. Proof (i), (ii). Note that

$$\begin{gathered} S_m^{(k)}(t) \ge X_m^{(k)}(t) \ge b_* a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right) \ge b_*\left\|\nabla u_m^{(k)}(t)\right\|_0^2, \\ S_m^{(k)}(t) \ge Y_m^{(k)}(t) \ge a\left(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) \ge \left\|\nabla \dot{u}_m^{(k)}(t)\right\|_0^2. \end{gathered} \tag{3.50}$$

We deduce that

$$\begin{aligned} b_m^{(k)}(t) &= B\left(\left\|\nabla u_m^{(k)}(t)\right\|_0^2\right) \le d_0\left(1 + \left\|\nabla u_m^{(k)}(t)\right\|_0^{2\alpha}\right), \\ \left|\dot{b}_m^{(k)}(t)\right| &= 2\left|B'\left(\left\|\nabla u_m^{(k)}(t)\right\|_0^2\right)\right|\left|\left\langle \nabla u_m^{(k)}(t), \nabla \dot{u}_m^{(k)}(t)\right\rangle\right| \\ &\le 2d_1\left(1 + \left\|\nabla u_m^{(k)}(t)\right\|_0^{2\alpha-2}\right)\left\|\nabla u_m^{(k)}(t)\right\|_0\left\|\nabla \dot{u}_m^{(k)}(t)\right\|_0 \\ &\le 2d_1\left(1 + b_*^{1-\alpha}\left(S_m^{(k)}(t)\right)^{\alpha-1}\right)\frac{1}{\sqrt{b_*}}S_m^{(k)}(t) \\ &= \frac{2d_1}{\sqrt{b_*}}\left[S_m^{(k)}(t) + b_*^{1-\alpha}\left(S_m^{(k)}(t)\right)^{\alpha}\right]. \end{aligned} \tag{3.51}$$

□

Proof (iii). We have

$$\left\|F_m^{(k)}(t)\right\|_0 \le \|\Psi_0(r,t,u_{m-1})\|_0 + \sum_{j=1}^{N-1}\left\|\Psi_j(r,t,u_{m-1})\left(u_m^{(k)}\right)^j\right\|_0. \tag{3.52}$$

By $(3.18)_3$, we have

$$\|\Psi_0(r,t,u_{m-1})\|_0 \le \frac{1}{\sqrt{2}}\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!} \equiv \widetilde{a}_0^{(0)}. \tag{3.53}$$

On the other hand, it follows from (3.49) and $u_{m-1} \in W_1(M,T)$ that

$$\begin{aligned}
\left\| \Psi_j(r,t,u_{m-1})\left(u_m^{(k)}\right)^j \right\|_0 &\leq \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} \left\| D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j} \left(u_m^{(k)}\right)^j \right\|_0 \\
&= \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} \left\| (\sqrt{r})^{i-j} (\sqrt{r})^{-i} D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j} (\sqrt{r})^j \left(u_m^{(k)}\right)^j \right\|_0 \\
&\leq \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} \left\| (\sqrt{r})^{i-j} \right\|_0 \widehat{K}_{*i}(M,f) \theta^{i-j} K_2^j \left\| u_m^{(k)}(t) \right\|_1^j \\
&= \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} \frac{1}{\sqrt{2+i-j}} \widehat{K}_{*i}(M,f) \theta^{i-j} K_2^j \left\| u_m^{(k)}(t) \right\|_1^j \\
&\leq \frac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j} \sum_{i=j}^{N-1} \frac{\theta^{i-j}}{j!(i-j)!} \frac{1}{\sqrt{2+i-j}} \widehat{K}_{*i}(M,f) \left(\sqrt{S_m^{(k)}(t)}\right)^j \\
&\equiv \widetilde{a}_j^{(0)} \left(\sqrt{S_m^{(k)}(t)}\right)^j.
\end{aligned} \tag{3.54}$$

It follows from (3.52)–(3.54) that

$$\left\| F_m^{(k)}(t) \right\|_0 \leq \sum_{j=0}^{N-1} \widetilde{a}_j^{(0)} \left(\sqrt{S_m^{(k)}(t)}\right)^j, \tag{3.55}$$

where $\widetilde{a}_j^{(0)},\ 0 \leq j \leq N-1$ are defined by $(3.49)_1$. □

Proof (iv). We have

$$\begin{aligned}
\frac{\partial}{\partial r} F_m^{(k)}(r,t) &= \frac{\partial}{\partial r} \Psi_0(r,t,u_{m-1}) + \sum_{j=1}^{N-1} j \Psi_j(r,t,u_{m-1}) \left(u_m^{(k)}\right)^{j-1} \nabla u_m^{(k)} \\
&\quad + \sum_{j=1}^{N-1} \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \left(u_m^{(k)}\right)^j.
\end{aligned} \tag{3.56}$$

Hence

$$\begin{aligned}
\left\| \frac{\partial}{\partial r} F_m^{(k)}(t) \right\|_0 &\leq \left\| \frac{\partial}{\partial r} \Psi_0(r,t,u_{m-1}) \right\|_0 + \sum_{j=1}^{N-1} j \left\| \Psi_j(r,t,u_{m-1}) \left(u_m^{(k)}\right)^{j-1} \nabla u_m^{(k)} \right\|_0 \\
&\quad + \sum_{j=1}^{N-1} \left\| \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \left(u_m^{(k)}\right)^j \right\|_0 \equiv L_1 + L_2 + L_3.
\end{aligned} \tag{3.57}$$

We shall estimate step by step the terms on the right-hand side of (3.57) as follows.

(iv.1) Estimating $L_1 = \|(\partial/\partial r)\Psi_0(r,t,u_{m-1})\|_0$. We have

$$\begin{aligned}\frac{\partial}{\partial r}\Psi_0(r,t,u_{m-1}) &= \sum_{i=1}^{N-1}\frac{1}{i!}(-1)^i D_3^i f(r,t,u_{m-1}) i u_{m-1}^{i-1}\nabla u_{m-1} \\ &\quad + \sum_{i=0}^{N-1}\frac{1}{i!}(-1)^i D_1 D_3^i f(r,t,u_{m-1}) u_{m-1}^{i} \\ &\quad + \sum_{i=0}^{N-1}\frac{1}{i!}(-1)^i D_3^{i+1} f(r,t,u_{m-1}) u_{m-1}^{i}\nabla u_{m-1} \\ &= a(1)+a(2)+a(3).\end{aligned} \tag{3.58}$$

We will estimate step by step the terms $a(1), a(2), a(3)$ as follows.

(iv.1.1) Estimating $\|a(1)\|_0$. We have

$$\begin{aligned}|a(1)| &= \left|\sum_{i=1}^{N-1}\frac{1}{i!}(-1)^i D_3^i f(r,t,u_{m-1}) i u_{m-1}^{i-1}\nabla u_{m-1}\right| \\ &\le \overline{K}_N(M,f)\sum_{i=1}^{N-1}\frac{1}{i!} i\theta^{i-1}|\nabla u_{m-1}| \\ &= \overline{K}_N(M,f)\sum_{i=0}^{N-2}\frac{\theta^i}{i!}|\nabla u_{m-1}| \\ &\le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}|\nabla u_{m-1}|.\end{aligned} \tag{3.59}$$

Hence

$$\|a(1)\|_0 \le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}\|\nabla u_{m-1}\|_0 \le M\overline{K}_N(M,f)\sum\nolimits_{i=0}^{N-1}\frac{\theta^i}{i!}. \tag{3.60}$$

(iv.1.2) Estimating $\|a(2)\|_0$. It follows from

$$|a(2)| = \left|\sum_{i=0}^{N-1}\frac{1}{i!}(-1)^i D_1 D_3^i f(r,t,u_{m-1}) u_{m-1}^{i}\right| \le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!} \tag{3.61}$$

that

$$\|a(2)\|_0 \le \frac{1}{\sqrt{2}}\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}. \tag{3.62}$$

(iv.1.3) Estimating $\|a(3)\|_0$. Similarly, with

$$|a(3)| = \left|\sum_{i=0}^{N-1}\frac{1}{i!}(-1)^i D_3^{i+1} f(r,t,u_{m-1})u_{m-1}^i \nabla u_{m-1}\right| \le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}|\nabla u_{m-1}|, \tag{3.63}$$

we obtain

$$\|a(3)\|_0 \le \overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}\|\nabla u_{m-1}\|_0 \le M\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}. \tag{3.64}$$

It follows from (3.58), (3.60), (3.62), (3.64) that

$$\begin{aligned} L_1 &= \left\|\frac{\partial}{\partial r}\Psi_0(r,t,u_{m-1})\right\|_0 \le \|a(1)\|_0 + \|a(2)\|_0 + \|a(3)\|_0 \\ &\le \frac{1}{\sqrt{2}}\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!} + 2M\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!} \\ &\le \left(\frac{1}{\sqrt{2}} + 2M\right)\overline{K}_N(M,f)\sum_{i=0}^{N-1}\frac{\theta^i}{i!}. \end{aligned} \tag{3.65}$$

(iv.2) Estimating $L_2 = \sum_{j=1}^{N-1} j\|\Psi_j(r,t,u_{m-1})(u_m^{(k)})^{j-1}\nabla u_m^{(k)}\|_0$. By the assumption (H_3), we deduce that

$$\begin{aligned} L_2 &= \sum_{j=1}^{N-1} j\left\|\Psi_j(r,t,u_{m-1})\left(u_m^{(k)}\right)^{j-1}\nabla u_m^{(k)}\right\|_0 \\ &\le \sum_{j=1}^{N-1} j\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left\|D_3^i f(r,t,u_{m-1})u_{m-1}^{i-j}\left(u_m^{(k)}\right)^{j-1}\nabla u_m^{(k)}\right\|_0 \\ &= \sum_{j=1}^{N-1} j\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left\|(\sqrt{r})^{-i}D_3^i f(r,t,u_{m-1})(\sqrt{r})^{i-j}u_{m-1}^{i-j}\left(\sqrt{r}u_m^{(k)}\right)^{j-1}\sqrt{r}\nabla u_m^{(k)}\right\|_0 \\ &= \sum_{j=1}^{N-1} j\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\widehat{K}_{*i}(M,f)\left\|(\sqrt{r})^{i-j}\sqrt{r}\nabla u_m^{(k)}\right\|_0\theta^{i-j}K_2^{j-1}\left\|u_m^{(k)}(t)\right\|_1^{j-1} \\ &\le \sum_{j=1}^{N-1} j\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\widehat{K}_{*i}(M,f)K_2^{j-1}\left\|u_m^{(k)}(t)\right\|_1^j \\ &\le \sum_{j=1}^{N-1}\frac{jK_2^{j-1}}{\left(\sqrt{b_*C_0}\right)^j}\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\widehat{K}_{*i}(M,f)\left(\sqrt{S_m^{(k)}(t)}\right)^j. \end{aligned} \tag{3.66}$$

(iv.3) Estimating $L_3 = \sum_{j=1}^{N-1} \|((\partial/\partial r)\Psi_j(r,t,u_{m-1}))(u_m^{(k)})^j\|_0$. We have

$$\begin{aligned}
L_3 &= \sum_{j=1}^{N-1} \left\| \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \left(u_m^{(k)} \right)^j \right\|_0 \\
&= \sum_{j=1}^{N-1} \left\| (\sqrt{r})^{-j} \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) (\sqrt{r})^j \left(u_m^{(k)} \right)^j \right\|_0 \\
&\leq \sum_{j=1}^{N-1} \left\| (\sqrt{r})^{-j} \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \right\|_0 K_2^j \left\| u_m^{(k)}(t) \right\|_1^j \\
&\leq \sum_{j=1}^{N-1} \left\| (\sqrt{r})^{-j} \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \right\|_0 \frac{1}{\left(\sqrt{b_* C_0}\right)^j} K_2^j \left(\sqrt{S_m^{(k)}(t)} \right)^j \\
&\equiv \sum_{j=1}^{N-1} \widetilde{L}_3(m,j,t) \frac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j} \left(\sqrt{S_m^{(k)}(t)} \right)^j.
\end{aligned} \tag{3.67}$$

Now we need an estimation of the term $\widetilde{L}_3(m,j,t) = \|(\sqrt{r})^{-j}((\partial/\partial r)\Psi_j(r,t,u_{m-1}))\|_0$. We have

$$\begin{aligned}
\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) &= \sum_{i=j+1}^{N-1} \frac{1}{j!(i-j)!} (-1)^{i-j} (i-j) D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j-1} \nabla u_{m-1} \\
&\quad + \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} (-1)^{i-j} D_1 D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j} \\
&\quad + \sum_{i=j}^{N-1} \frac{1}{j!(i-j)!} (-1)^{i-j} D_3^{i+1} f(r,t,u_{m-1}) u_{m-1}^{i-j} \nabla u_{m-1} \\
&= b(1) + b(2) + b(3).
\end{aligned} \tag{3.68}$$

It follows from (3.68) that

$$\begin{aligned}
\widetilde{L}_3(m,j,t) &= \left\| (\sqrt{r})^{-j} \left(\frac{\partial}{\partial r} \Psi_j(r,t,u_{m-1}) \right) \right\|_0 \\
&\leq \left\| (\sqrt{r})^{-j} b(1) \right\|_0 + \left\| (\sqrt{r})^{-j} b(2) \right\|_0 + \left\| (\sqrt{r})^{-j} b(3) \right\|_0.
\end{aligned} \tag{3.69}$$

(iv.3.1) Estimating $\|(\sqrt{r})^{-j} b(1)\|_0$. We have

$$\begin{aligned}
\left\| (\sqrt{r})^{-j} b(1) \right\|_0 &= \left\| (\sqrt{r})^{-j} \sum_{i=j+1}^{N-1} \frac{1}{j!(i-j)!} (-1)^{i-j} (i-j) D_3^i f(r,t,u_{m-1}) u_{m-1}^{i-j-1} \nabla u_{m-1} \right\|_0 \\
&\leq \sum_{i=j+1}^{N-1} \frac{1}{j!(i-j)!} (i-j) \left\| (\sqrt{r})^{-i} D_3^i f(r,t,u_{m-1}) (\sqrt{r})^{i-j-1} u_{m-1}^{i-j-1} \sqrt{r} \nabla u_{m-1} \right\|_0
\end{aligned}$$

$$\leq \sum_{i=j+1}^{N-1} \frac{1}{j!(i-j)!}(i-j)\widehat{K}_{*i}(M,f)\theta^{i-j-1}M$$
$$= M\sum_{i=j}^{N-2} \frac{\theta^{i-j}}{j!(i-j)!}\widehat{K}_{*i+1}(M,f) \leq M\sum_{i=j}^{N-1} \frac{\theta^{i-j}}{j!(i-j)!}M\widehat{K}_{*i+1}(M,f). \tag{3.70}$$

(iv.3.2) Estimating $\|(\sqrt{r})^{-j}b(2)\|_0$. We have

$$\begin{aligned}\left\|(\sqrt{r})^{-j}b(2)\right\|_0 &= \left\|(\sqrt{r})^{-j}\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}(-1)^{i-j}D_1D_3^i f(r,t,u_{m-1})u_{m-1}^{i-j}\right\|_0 \\ &\leq \sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left\|(\sqrt{r})^{-i-1}D_1D_3^i f(r,t,u_{m-1})(\sqrt{r})^{i-j+1}u_{m-1}^{i-j}\right\|_0 \\ &\leq \sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\widehat{K}_{*i+1}(M,f)\theta^{i-j}\left\|(\sqrt{r})^{i-j+1}\right\|_0 \\ &= \sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\frac{1}{\sqrt{i-j+3}}\widehat{K}_{*i+1}(M,f).\end{aligned} \tag{3.71}$$

(iv.3.3) Estimating $\|(\sqrt{r})^{-j}b(3)\|_0$. We have

$$\begin{aligned}\left\|(\sqrt{r})^{-j}b(3)\right\|_0 &= \left\|(\sqrt{r})^{-j}\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}(-1)^{i-j}D_3^{i+1} f(r,t,u_{m-1})u_{m-1}^{i-j}\nabla u_{m-1}\right\|_0 \\ &\leq \sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left\|(\sqrt{r})^{-i-1}D_3^{i+1} f(r,t,u_{m-1})(\sqrt{r})^{i-j}u_{m-1}^{i-j}\sqrt{r}\nabla u_{m-1}\right\|_0 \\ &\leq \sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\widehat{K}_{*i+1}(M,f)\theta^{i-j}M = M\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\widehat{K}_{*i+1}(M,f).\end{aligned} \tag{3.72}$$

It follows from (3.69)–(3.72) that

$$\begin{aligned}\widetilde{L}_3(m,j,t) &= \left\|(\sqrt{r})^{-j}\left(\frac{\partial}{\partial r}\Psi_j(r,t,u_{m-1})\right)\right\|_0 \\ &\leq 2M\sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\widehat{K}_{*i+1}(M,f) + \sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\frac{1}{\sqrt{i-j+3}}\widehat{K}_{*i+1}(M,f) \\ &= \sum_{i=j}^{N-1}\frac{\theta^{i-j}}{j!(i-j)!}\left(\frac{1}{\sqrt{i-j+3}}+2M\right)\widehat{K}_{*i+1}(M,f).\end{aligned} \tag{3.73}$$

It follows from (3.67) – (3.73) that

$$\begin{aligned} L_3 &\le \sum_{j=1}^{N-1} \widetilde{L}_3(m,j,t) \frac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j} \left(\sqrt{S_m^{(k)}(t)}\right)^j \\ &\le \sum_{j=1}^{N-1}\sum_{i=j}^{N-1} \frac{\theta^{i-j}}{j!(i-j)!} \left(\frac{1}{\sqrt{i-j+3}} + 2M\right) \widehat{K}_{*i+1}(M,f) \frac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j} \left(\sqrt{S_m^{(k)}(t)}\right)^j. \end{aligned} \tag{3.74}$$

We deduce from (3.57), (3.65), (3.66), (3.74) that

$$\begin{aligned} \left\| \frac{\partial}{\partial r} F_m^{(k)}(t) \right\|_0 &\le \left(\frac{1}{\sqrt{2}} + 2M\right) \overline{K}_N(M,f) \sum_{i=0}^{N-1} \frac{\theta^i}{i!} \\ &\quad + \sum_{j=1}^{N-1} \frac{K_2^j}{\left(\sqrt{b_* C_0}\right)^j} \sum_{i=j}^{N-1} \frac{\theta^{i-j}}{j!(i-j)!} \\ &\quad \times \left[j K_2^{-1} \widehat{K}_{*i}(M,f) + \left(\frac{1}{\sqrt{i-j+3}} + 2M\right) \widehat{K}_{*i+1}(M,f) \right] \left(\sqrt{S_m^{(k)}(t)}\right)^j \\ &\equiv \sum_{j=1}^{N-1} \widetilde{a}_j^{(1)} \left(\sqrt{S_m^{(k)}(t)}\right)^j, \end{aligned} \tag{3.75}$$

where $\widetilde{a}_j^{(1)}$, $0 \le j \le N-1$ are defined by $(3.49)_2$.

Next, we will estimate step by step all integrals $I_1, \ldots, I_5$.

Integral $I_1 = \int_0^t \dot{b}_m^{(k)}(s)[a(u_m^{(k)}(s), u_m^{(k)}(s)) + \|Au_m^{(k)}(s)\|_0^2] ds$.

Now, using the inequalities (3.48) and

$$s^q \le 1 + s^{N_0}, \quad \forall s \ge 0,\ \forall q,\ 0 < q \le N_0 = \max\{1+\alpha, N\}, \tag{3.76}$$

we estimate without difficulty the following integrals in the right-hand side of (3.47) as follows.

The integral I_1:

$$\begin{aligned} I_1 &= \int_0^t \dot{b}_m^{(k)}(s) \left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\| Au_m^{(k)}(s) \right\|_0^2 \right] ds \\ &\le 2 \frac{d_1}{b_*^{3/2}} \int_0^t \left[\left(S_m^{(k)}(s)\right)^2 + b_*^{1-\alpha} \left(S_m^{(k)}(s)\right)^{\alpha+1} \right] ds \end{aligned}$$

$$\leq 2\frac{d_1}{b_*^{3/2}}\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}+b_*^{1-\alpha}\left(1+\left(S_m^{(k)}(s)\right)^{N_0}\right)\right]ds$$
$$\leq 2\frac{d_1}{b_*^{3/2}}\left(1+b_*^{1-\alpha}\right)\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds. \tag{3.77}$$

The integral $I_2 = 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds$:

$$\begin{aligned} I_2 &= 2\int_0^t\left\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right\rangle ds \leq 2\int_0^t\left\|F_m^{(k)}(s)\right\|_0\left\|\dot{u}_m^{(k)}(s)\right\|_0 ds \\ &\leq 2\sum_{j=0}^{N-1}\tilde{a}_j^{(0)}\int_0^t\left(\sqrt{S_m^{(k)}(s)}\right)^{j+1}ds \leq 2\sum_{j=0}^{N-1}\tilde{a}_j^{(0)}\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds. \end{aligned} \tag{3.78}$$

The integral $I_3 = 2\int_0^t a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s))ds$:

$$\begin{aligned} I_3 &= 2\int_0^t a\left(F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right)ds \\ &\leq 2\int_0^t\sqrt{a\left(F_m^{(k)}(s),F_m^{(k)}(s)\right)}\sqrt{a\left(\dot{u}_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right)}ds \\ &\leq 2\sqrt{C_1}\int_0^t\left\|F_m^{(k)}(s)\right\|_1\sqrt{a\left(\dot{u}_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right)}ds \\ &\leq 2\sqrt{C_1}\sum_{j=0}^{N-1}\tilde{a}_j^{(1)}\int_0^t\left(\sqrt{S_m^{(k)}(s)}\right)^{j+1}ds \\ &\leq 2\sqrt{C_1}\sum_{j=0}^{N-1}\tilde{a}_j^{(1)}\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds. \end{aligned} \tag{3.79}$$

The integral $I_4 = \int_0^t \langle F_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle ds$:

$$\begin{aligned} I_4 &= \int_0^t\left\langle F_m^{(k)}(s),\ddot{u}_m^{(k)}(s)\right\rangle ds \leq \int_0^t\left\|F_m^{(k)}(s)\right\|_0\left\|\ddot{u}_m^{(k)}(s)\right\|_0 ds \\ &\leq \int_0^t\left\|F_m^{(k)}(s)\right\|_0^2 ds + \frac{1}{4}\int_0^t\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \\ &\leq \int_0^t\left(\sum_{j=0}^{N-1}\tilde{a}_j^{(0)}\left(\sqrt{S_m^{(k)}(s)}\right)^j\right)^2 ds + \frac{1}{4}S_m^{(k)}(t) \end{aligned}$$

$$\begin{aligned}
&\leq N\sum_{j=0}^{N-1}\left|\tilde{a}_j^{(0)}\right|^2\int_0^t\left(S_m^{(k)}(s)\right)^j ds+\frac{1}{4}S_m^{(k)}(t)\\
&\leq N\sum_{j=0}^{N-1}\left|\tilde{a}_j^{(0)}\right|^2\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds+\frac{1}{4}S_m^{(k)}(t).
\end{aligned} \tag{3.80}$$

The integral $I_5=-\int_0^t b_m^{(k)}(s)\langle Au_m^{(k)}(s),\ddot{u}_m^{(k)}(s)\rangle ds$:

$$\begin{aligned}
I_5&=-\int_0^t b_m^{(k)}(s)\left\langle Au_m^{(k)}(s),\ddot{u}_m^{(k)}(s)\right\rangle ds\leq\int_0^t b_m^{(k)}(s)\left\|Au_m^{(k)}(s)\right\|_0\left\|\ddot{u}_m^{(k)}(s)\right\|_0 ds\\
&\leq\int_0^t b_m^{(k)}(s)b_m^{(k)}(s)\left\|Au_m^{(k)}(s)\right\|_0^2 ds+\frac{1}{4}\int_0^t\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds\\
&\leq d_0\int_0^t\left[1+b_*^{-\alpha}\left(S_m^{(k)}(s)\right)^{\alpha}\right]S_m^{(k)}(s)ds+\frac{1}{4}S_m^{(k)}(t)\\
&\leq d_0\int_0^t\left[S_m^{(k)}(s)+b_*^{-\alpha}\left(S_m^{(k)}(s)\right)^{\alpha+1}\right]ds+\frac{1}{4}S_m^{(k)}(t)\\
&\leq d_0\int_0^t\left\{\left(1+\left(S_m^{(k)}(s)\right)^{N_0}\right)+b_*^{-\alpha}\left(1+\left(S_m^{(k)}(s)\right)^{N_0}\right)\right\}ds+\frac{1}{4}S_m^{(k)}(t)\\
&\leq d_0(1+b_*^{-\alpha})\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds+\frac{1}{4}S_m^{(k)}(t).
\end{aligned} \tag{3.81}$$

From the convergences in (3.10), we can deduce the existence of a constant $M\geq 2$ independent of k and m such that

$$S_m^{(k)}(0)\leq\frac{M^2}{4}. \tag{3.82}$$

Combining (3.47), (3.77)–(3.82), we then have

$$S_m^{(k)}(t)\leq\frac{M^2}{2}+TD_0(M)+D_0(M)\int_0^t\left(S_m^{(k)}(s)\right)^{N_0}ds, \tag{3.83}$$

where

$$D_0(M)=2d_0(1+b_*^{-\alpha})+\frac{4d_1}{b_*^{3/2}}\left(1+b_*^{1-\alpha}\right)+4\sum_{j=0}^{N-1}\left(\tilde{a}_j^{(0)}+\sqrt{C_1}\tilde{a}_j^{(1)}+\frac{1}{2}N\left|\tilde{a}_j^{(0)}\right|^2\right). \tag{3.84}$$

□

Then, we have the following Lemma.

Lemma 3.4. *There exists a constant $T > 0$ independent of k and m such that*

$$S_m^{(k)}(t) \leq M^2 \quad \forall t \in [0,T],\ \forall k, m. \tag{3.85}$$

Proof of Lemma 3.4. Put

$$Y(t) = \frac{M^2}{2} + TD_0(M) + D_0(M)\int_0^t \left(S_m^{(k)}(s)\right)^{N_0} ds, \quad 0 \leq t \leq T. \tag{3.86}$$

Clearly

$$\begin{gathered} Y(t) > 0, \quad 0 \leq S_m^{(k)}(t) \leq Y(t), \quad 0 \leq t \leq T, \\ Y'(t) \leq D_0(M) Y^{N_0}(t), \quad 0 \leq t \leq T, \\ Y(0) = \frac{M^2}{2} + TD_0(M). \end{gathered} \tag{3.87}$$

Put $Z(t) = Y^{-N_0+1}(t)$, after integrating of (3.87)

$$\begin{aligned} Z(t) &\geq \left(\frac{M^2}{2} + TD_0(M)\right)^{-N_0+1} - (N_0 - 1)D_0(M)t \\ &\geq \left(\frac{M^2}{2} + TD_0(M)\right)^{-N_0+1} - (N_0 - 1)TD_0(M), \quad \forall t \in [0,T]. \end{aligned} \tag{3.88}$$

Then, by

$$\lim_{T \to 0_+} \left[\left(\frac{M^2}{2} + TD_0(M)\right)^{-N_0+1} - (N_0 - 1)TD_0(M)\right] = \left(\frac{M^2}{2}\right)^{-N_0+1} > M^{-2N_0+2}, \tag{3.89}$$

we can always choose the constant $T \in (0, T^*]$ such that

$$\left(\frac{M^2}{2} + TD_0(M)\right)^{-N_0+1} - (N_0 - 1)TD_0(M) \geq M^{-2N_0+2}. \tag{3.90}$$

Finally, it follows from (3.87), (3.88) and (3.90), that

$$0 \le S_m^{(k)}(t) \le Y(t) = \frac{1}{\sqrt[N_0-1]{Z(t)}} \le \frac{1}{\sqrt[N_0-1]{\left(M^2/2 + TD_0(M)\right)^{-N_0+1} - (N_0-1)TD_0(M)}} \le M^2, \quad \forall t \in [0,T]. \tag{3.91}$$

The proof of Lemma 3.4 is complete. □

Remark 3.5. The function $Y(t) = \left[(M^2/2 + TD_0(M))^{1-N_0} - (N_0-1)D_0(M)t\right]^{1/(1-N_0)}$, $0 \le t \le T$, is the maximal solution of the following Volterra integral equation with non-decreasing kernel [20].

$$Y(t) = \frac{M^2}{2} + TD_0(M) + D_0(M)\int_0^t Y^{N_0}(s)ds, \quad 0 \le t \le T. \tag{3.92}$$

By Lemma 3.4, we can take constant $T_m^{(k)} = T$ for all k and m. Therefore, we have

$$u_m^{(k)} \in W_1(M,T) \quad \forall m,\ k. \tag{3.93}$$

From (3.92), we can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_i)}\}$ such that

$$\begin{aligned} u_m^{(k_i)} &\longrightarrow u_m \quad \text{in } L^\infty(0,T;V_2) \text{ weak}^*, \\ \dot{u}_m^{(k_i)} &\longrightarrow u_m' \quad \text{in } L^\infty(0,T;V_1) \text{ weak}^*, \\ \ddot{u}_m^{(k_i)} &\longrightarrow u_m'' \quad \text{in } L^2(0,T;V_0) \text{ weak}, \end{aligned} \tag{3.94}$$

$$u_m \in W(M,T). \tag{3.95}$$

We can easily check from (3.9), (3.10), (3.94) that u_m satisfies (3.6), (3.7) in $L^2(0,T)$, weak. On the other hand, it follows from $(3.6)_1$ and $u_m \in W(M,T)$ that $u_m'' = -b_m(t)Au_m + F_m \in L^\infty(0,T;V_0)$, hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

The following result gives a convergence at a rate of order N of $\{u_m\}$ to a weak solution of (1.1).

First, we note that $W_1(T) = \{v \in L^\infty(0,T;V_1) : v' \in L^\infty(0,T;V_0)\}$ is a Banach space with respect to the norm (see [19]):

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V_1)} + \|v'\|_{L^\infty(0,T;V_0)}. \tag{3.96}$$

Then, we have the following theorem.

Theorem 3.6. *Let* (H_1)*–*(H_3) *hold. Then, there exist constants* $M > 0$ *and* $T > 0$ *such that*

(i) *the problem* (1.1) *has a unique weak solution* $u \in W_1(M,T)$*;*

(ii) *the recurrent sequence* $\{u_m\}$ *defined by* (3.6), (3.7) *converges at a rate of order* N *to the solution* u *strongly in the space* $W_1(T)$ *in the sense*

$$\|u_m - u\|_{W_1(T)} \le C\|u_{m-1} - u\|_{W_1(T)}^N, \tag{3.97}$$

for all $m \ge 1$*, where C is suitable constant.*

Furthermore, we have the estimation

$$\|u_m - u\|_{W_1(T)} \le C_T \beta^{N^m}, \tag{3.98}$$

for all $m \ge 1$*, where* C_T *and* $\beta < 1$ *are positive constants depending only on* T*.*

Proof. (a) Existence of the solution. First, we will prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{aligned} &\left\langle v_m''(t), w\right\rangle + b_{m+1}(t) a(v_m(t), w) + (b_{m+1}(t) - b_m(t))\langle Au_m(t), w\rangle \\ &\quad = \langle F_{m+1}(t) - F_m(t), w\rangle, \quad \forall w \in V_1, \\ &v_m(0) = v_m'(0) = 0, \end{aligned} \tag{3.99}$$

with

$$\begin{gathered} f(r,t,u_m) - (r,t,u_{m-1}) = \sum_{i=1}^{N-1} \frac{1}{i!} D_3^i f(r,t,u_{m-1}) v_{m-1}^i + \frac{1}{N!} D_3^N f(r,t,\lambda_m) v_{m-1}^N, \\ F_{m+1}(t) - F_m(t) = \sum_{i=1}^{N-1} \frac{1}{i!} D_3^i f(r,t,u_{m-1}) v_m^i + \frac{1}{N!} D_3^N f(r,t,\lambda_m) v_{m-1}^N, \\ \lambda_m = u_{m-1} + \theta_1 v_{m-1}, \quad (0 < \theta_1 < 1), \\ b_{m+1}(t) - b_m(t) = B\left(\|\nabla u_{m+1}(t)\|_0^2\right) - B\left(\|\nabla u_m(t)\|_0^2\right). \end{gathered} \tag{3.100}$$

Taking $w = v_m'$ in (3.99) and integrating in t we get

$$\begin{aligned} \gamma_m(t) = &\int_0^t b_{m+1}'(s) a(v_m(s), v_m(s)) ds \\ &- 2\int_0^t \left[B\left(\|\nabla u_{m+1}(s)\|_0^2\right) - B\left(\|\nabla u_m(s)\|_0^2\right)\right] \langle Au_m(s), v_m'(s)\rangle ds \end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=1}^{N-1}\frac{1}{i!}\int_0^t\left\langle D_3^i f(r,s,u_{m-1})v_m^i, v_m'(s)\right\rangle ds+\frac{1}{N!}\int_0^t\left\langle D_3^N f(r,s,\lambda_m)v_{m-1}^N, v_m'(s)\right\rangle ds\\
&=J_1+\cdots+J_4,
\end{aligned}
\tag{3.101}
$$

where

$$
\gamma_m(t)=\left\|v_m'(t)\right\|_0^2+b_{m+1}(t)a(v_m(t),v_m(t))\geq\left\|v_m'(t)\right\|_0^2+b_*C_0\|v_m(t)\|_1^2\equiv E_m(t). \tag{3.102}
$$

We estimate without difficulty the following integrals $J_1,\ldots,J_4$ in the right-hand side of (3.101) as follows.

The integral $J_1=\int_0^t b_{m+1}'(s)a(v_m(s),v_m(s))ds$

$$
\begin{aligned}
J_1&\leq 2C_1d_1\left(M^2+M^{2\alpha}\right)\int_0^t\|v_m(s)\|_1^2ds\\
&\leq 2C_1d_1\left(M^2+M^{2\alpha}\right)\frac{1}{b_*C_0}\int_0^t E_m(s)ds\equiv\xi_1\int_0^t E_m(s)ds.
\end{aligned}
\tag{3.103}
$$

The integral $J_2=-2\int_0^t[B(\|\nabla u_{m+1}(s)\|_0^2)-B(\|\nabla u_m(s)\|_0^2)]\langle Au_m(s),v_m'(s)\rangle ds$

$$
\begin{aligned}
J_2&\leq 4d_1\left(M^2+M^{2\alpha}\right)\int_0^t\|v_m(s)\|_1\left\|v_m'(s)\right\|_0ds\\
&\leq 4d_1\left(M^2+M^{2\alpha}\right)\frac{1}{2\sqrt{b_*C_0}}\int_0^t E_m(s)ds\equiv\xi_2\int_0^t E_m(s)ds.
\end{aligned}
\tag{3.104}
$$

The integral $J_3=2\sum_{i=1}^{N-1}(1/i!)\int_0^t\langle D_3^i f(r,s,u_{m-1})v_m^i, v_m'(s)\rangle ds$.

$$
\begin{aligned}
J_3&=2\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\int_0^t\left\langle\left(\sqrt{r}\right)^{-i}D_3^i f(r,s,u_{m-1})\left(\sqrt{r}\right)^i v_m^i, v_m'(s)\right\rangle ds\\
&\leq 2\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\widehat{K}_{*i}(M,f)K_2^i\int_0^t\|v_m(s)\|_1^i\langle 1,\left|v_m'(s)\right|\rangle ds\\
&\leq\sqrt{2}\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\widehat{K}_{*i}(M,f)K_2^i\int_0^t\|v_m(s)\|_1^i\left\|v_m'(s)\right\|_0ds\\
&\leq\sqrt{2}\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\widehat{K}_{*i}(M,f)K_2^iM^{i-1}\int_0^t\|v_m(s)\|_1\left\|v_m'(s)\right\|_0ds\\
&\leq\sqrt{2}\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\widehat{K}_{*i}(M,f)K_2^iM^{i-1}\frac{1}{2\sqrt{b_*C_0}}\int_0^t E_m(s)ds\equiv\xi_3\int_0^t E_m(s)ds.
\end{aligned}
\tag{3.105}
$$

The integral $J_4 = (1/N!)\int_0^t \langle D_3^N f(r,s,\lambda_m) v_{m-1}^N, v_m'(s)\rangle ds$

$$\begin{aligned}
J_4 &= \frac{1}{N!}\int_0^t \Big\langle (\sqrt{r})^{-N} D_3^N f(r,s,\lambda_m)(\sqrt{r})^N v_{m-1}^N, v_m'(s)\Big\rangle ds \\
&\le \frac{1}{N!}\widehat{K}_{*N}(M,f) K_2^N \int_0^t \|v_{m-1}(s)\|_1^N \langle 1, |v_m'(s)|\rangle ds \\
&\le \frac{1}{N!}\widehat{K}_{*N}(M,f) K_2^N \frac{1}{\sqrt{2}}\int_0^t \|v_{m-1}(s)\|_1^N \|v_m'(s)\|_0 ds \\
&\le \frac{1}{N!}\widehat{K}_{*N}(M,f) K_2^N \frac{1}{2\sqrt{2}} 2\|v_{m-1}\|_{W_1(T)}^N \int_0^t \|v_m'(s)\|_0 ds \\
&\le \frac{1}{N!}\widehat{K}_{*N}(M,f) K_2^N \frac{1}{2\sqrt{2}}\left[T\|v_{m-1}\|_{W_1(T)}^{2N} + \int_0^t \|v_m'(s)\|_0^2 ds\right] \\
&= \xi_4\left[T\|v_{m-1}\|_{W_1(T)}^{2N} + \int_0^t E_m(s)ds\right].
\end{aligned} \tag{3.106}$$

Combining (3.101), (3.103)–(3.106), we then obtain

$$\begin{aligned}
E_m(t) &\le T\xi_4\|v_{m-1}\|_{W_1(T)}^{2N} + (\xi_1+\xi_2+\xi_3+\xi_4)\int_0^t E_m(s)ds \\
&\equiv T\xi_4\|v_{m-1}\|_{W_1(T)}^{2N} + 2\zeta_M^{(1)}\int_0^t E_m(s)ds.
\end{aligned} \tag{3.107}$$

By using Gronwall's lemma, we obtain from (3.107), that

$$\|v_m\|_{W_1(T)} \le \mu_T \|v_{m-1}\|_{W_1(T)}^N, \tag{3.108}$$

where μ_T is the constant given by

$$\mu_T = \left(1 + \frac{1}{\sqrt{b_* C_0}}\right)\sqrt{T\xi_4}\exp\left(T\zeta_M^{(1)}\right). \tag{3.109}$$

Hence, we obtain from (3.108) that

$$\left\|u_m - u_{m+p}\right\|_{W_1(T)} \le (1-k_T)^{-1}(\mu_T)^{-1/(N-1)}(k_T)^{N^m}, \tag{3.110}$$

for all m and p. We take $T>0$ small enough, such that $k_T = (\mu_T)^{1/(N-1)}M < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that $u_m \to u$ strongly in $W_1(T)$. By the same argument used in the proof of Theorem 3.1, $u \in W_1(M,T)$ is a unique weak solution of the problem (1.1). Passing to the limit as $p \to +\infty$ for m fixed, we obtain the estimate (3.98) from (3.110) and Theorem 3.6 follows. □

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