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*Research Article*

# Spatial Profile of the Dead Core for the Fast Diffusion Equation with Dependent Coefficient

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We consider the dead-core problem for the fast diffusion equation with spatially dependent coefficient and obtain precise estimates on the single-point final dead-core profile. The proofs rely on maximum principle and require much delicate computation.

## 1. Introduction

In this paper, we study the porous medium equation with the following initial boundary condition:

$$\begin{gathered} u_t = (u^m)_{xx} - x^q u^p, \quad (x,t) \in (0,1) \times (0,T), \\ u_x(0,t) = 0, \qquad u(1,t) = k, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in [0,1], \end{gathered} \tag{1.1}$$

where $0 < p < m < 1$ and $-2 < q < 0$. Assume $k > 0$ and that the initial data $u_0$ satisfies

$$u_0 \in C([0,1]), \quad u_0 > 0 \text{ in } [0,1], \qquad u_0(0) = 0, \qquad u_0(1) = k. \tag{1.2}$$

Moreover, we denote

$$\alpha = \frac{1}{1-p}. \tag{1.3}$$

Here we are mainly interested in the asymptotic behavior of nonnegative and global classical solutions. However, Problem (1.1) is singular at $x = 0$ for $-2 < q < 0$. In fact, the solutions can be approximated, if necessary, by the ones satisfying the following equation $u_t = (u^m)_{xx} - (x+\epsilon)^q u^p$ with the same initial-boundary value conditions and taking the limit $\epsilon \to 0$. We set

$$\theta(t) := \min_{0\le x\le 1} u(x,t) \tag{1.4}$$

and denote

$$T = T(u_0) := \inf\{t > 0;\ \theta(t) = 0\} > 0. \tag{1.5}$$

For suitable initial data, we will show that $T(u_0) < \infty$ (see Theorem 1.1). We say that the solution develops a dead core in finite time, and $T$ is called the dead-core time.

In the past few years, much attentions have been taken to the dead-core problems. For the semilinear case of $0 < p < m = 1$ and $q = 0$, the temporal dead-core profile was investigated in [1] by Guo and Souplet. For the quasilinear case of $0 < p < m < 1$ and $q = 0$, Guo et al. [2] firstly investigated the solution which develops a dead core in finite time; then they obtained the spatial profile of the dead core and also studied the non-self-similar dead-core rate of the solution. Numerous related works have been devoted to some of the regularity and the corresponding problems such as blowup, quenching, and gradient blowup; we refer the interested reader to [3–11] and the references therein.

Our aim of this paper is to study the dead-core problem for the fast diffusion with strong absorption. In view of the observation concerning the interaction of diffusion and absorption, this question is of interest since the effect of fast diffusion, as compared with linear diffusion, is much stronger near the level $u = 0$. Although our strategy of proof is close to that in [2], the proof is technically much more difficult due to the presence of a nonlinear operator and spatially dependent absorption coefficient.

The paper is organized as follows. In Section 2, we prove that the solution of the porous medium equation develops a dead core in finite time. In Section 3, firstly, we obtain the spatial profile of the dead-core upper bound estimate by the initial monotone assumption; then we construct auxiliary function and derive the lower bound estimate by maximum principle.

Our first result gives sufficient conditions under which the solution of Problem (1.1) develops a dead core in finite time. To formulate this, let us first recall some well-known facts: (1.1) admits a unique steady state $U_k \in C^2((0,1])$ under the condition $-2 < q < 0$ for each given $k > 0$. Moreover, $U_k$ is an even and nondecreasing function of $x$, and it is a nondecreasing function of $k$. Furthermore, there exists $k_0 = k_0(m,p) > 0$ such that if $k \in (0, k_0)$ then $U_k$ vanishes on an interval of positive length, if $k = k_0$ then $U_k$ vanishes only at $x = 0$, and if $k > k_0$ then $U_k$ is positive.

**Theorem 1.1.** *Assume* $0 < p < m < 1$, $-2 < q < 0$ *and* (1.2).

(i) *Let* $0 < k < k_0$. *Then* $T(u_0) < \infty$ *for any* $u_0$.

(ii) *Let* $k \ge k_0$. *For any* $\eta, M > 0$ *there exists* $\delta = \delta(\eta, M) > 0$ *such that* $T(u_0) < \infty$ *whenever* $\|u_0\| \le M$ *and* $u_0 \le \delta$ *on a subinterval of* $(0,1]$ *of length.*

For our main results on the spatial profile of the dead-core problem, we will assume that $u_0$ satisfies the conditions

$$u_0 \in C^2([0,1]), \quad (u_0^m)'' \leq x^q u_0^p \quad \text{in } (0,1], \\ u_0 \text{ is even and nondecreasing in } x \text{ and } T(u_0) < \infty. \tag{1.6}$$

It then follows from the strong maximum principle that $u_t < 0$ in $Q_T := (0,1) \times (0,T)$, $u(-x,t) = u(x,t)$ for $(x,t) \in (-1,1) \times (0,T)$ and $u_x > 0$ in $(0,1) \times (0,T)$.

Our main goal in this paper is thus to obtain the following precise estimates on the single-point final dead-core profile near $x = 0$.

**Theorem 1.2.** *Let $k > 0$ and assume $0 < p < m < 1$, $p + m > 1$, $-1 < q < 0$, (1.2), and (1.6), then there exist $C_1, C_2 > 0$ such that*

$$C_1 x^{(q+2)/(m-p)} \leq u(x,T) \leq C_2 x^{(q+2)/(m-p)}, \quad 0 \leq x \leq 1, \tag{1.7}$$

*where $C_1 = [\varepsilon(m-p)/m(q+2)]^{1/(m-p)}$, $C_2 = [(m-p)/m(q+1)(q+2)]^{1/(m-p)}$, and $\varepsilon \leq (p+m-1)/(2p+m-1)(q+1)$ is an arbitrary positive constant.*

*Remark 1.3.* Due to the technical difficulty, we cannot prove that the coefficients of the upper and lower bounds in Theorem 1.2 are not identical. Also, it is very interesting whether Problem (1.1), even for the case $q > 0$, exists the non-self-similar dead-core rate similar to that in [1, 2]. We leave these open questions to the interested readers.

## 2. Quenching in Finite Time

*Proof of Theorem 1.1.*

*Step 1.* We look for a supersolution $\overline{u}$ of $u_t - (u^m)_{xx} + x^q u^p = 0$ in $Q_T := (0,1) \times (0,T)$, which develops a dead core at time $T$. For any $T \in (0, T_0)$, we will construct $\overline{u}$ under the following self-similar form:

$$\overline{u}(x,t) = \varepsilon (T-t)^{\alpha} V(y), \qquad y = x(T-t)^{-\beta}, \qquad V(y) = \left(1+y^2\right)^{\gamma}, \tag{2.1}$$

where

$$0 < \beta < \frac{\alpha(m-p)}{2} = \frac{m-p}{2(1-p)} \tag{2.2}$$

and $\gamma, \varepsilon, T_0 > 0$ will be determined. Note that $\overline{u}(0,T) = 0$. Computations yield

$$\begin{aligned} P\overline{u} &= \overline{u}_t - \left(\overline{u}^m\right)_{xx} + x^q \overline{u}^p \\ &= \varepsilon (T-t)^{\alpha-1}\left(-\alpha V + \beta y V'\right) - \varepsilon^m (T-t)^{\alpha m - 2\beta} (V^m)'' + \varepsilon^p x^q (T-t)^{\alpha p} V^p \\ &= \varepsilon (T-t)^{\alpha p}\left\{\varepsilon^{p-1} x^q V^p - \alpha V + \beta y V' - \varepsilon^{m-1} (T-t)^{\lambda} (V^m)''\right\} \end{aligned} \tag{2.3}$$

for $(x,t) \in Q_T$, where $\lambda = \alpha(m-p) - 2\beta > 0$. Assuming $T \le T_0(\varepsilon) := \varepsilon^{(1-m)/\lambda}$, we see that

$$P\overline{u} \ge \varepsilon(T-t)^{\alpha p}\left\{\varepsilon^{p-1}x^q - h(y)\right\}, \quad \text{where } h(y) = \alpha V - \beta y V' + \left|(V^m)''\right|. \tag{2.4}$$

Next taking $\gamma > \alpha/(2\beta)$ and using $|(V^m)''| \sim C|y|^{2m\gamma-2}$ as $|y| \to \infty$, we observe that

$$h(y) \sim (\alpha - 2\beta\gamma)|y|^{2\gamma} \longrightarrow -\infty, \quad \text{as } |y| \longrightarrow \infty. \tag{2.5}$$

It follows that $\sup_{y\in\mathbb{R}} h(y) < \infty$ and choosing $\varepsilon = \varepsilon(m,p,\beta,\gamma) > 0$ sufficiently small, we conclude that $P\overline{u} \ge 0$ in $Q_T$. For further reference we also note that

$$\overline{u}(x,t) \ge \varepsilon|x|^{2\gamma}T^{-\mu} \quad \text{in } Q_T, \quad \text{where } \mu = 2\beta\gamma - \alpha > 0. \tag{2.6}$$

*Step 2* (we prove assertion (ii)). Fix $\eta, M > 0$ and $x_0 \in [\eta/2, 1-\eta/2]$. Let $\overline{u}, T_0$ be as in Step 1 and set $\overline{v}(x,t) = \overline{u}(x-x_0,t)$. Taking $T \le \min(T_0, T_1)$, where $T_1 = T_1(\eta, M) > 0$ is sufficiently small, and using (2.6), we see that $\overline{v}(x,t) \ge M$ for $|x-x_0| \ge \eta/2$ and $t \in (0,T)$, hence in particular $\overline{v}(\pm 1, t) \ge k$ (here, we deal with the symmetry case in one dimension). Next put $\delta := \min_{|x-x_0|\ge\eta/2}\overline{v}(x,0)$. Then assuming $\|u_0\|_\infty \le M$ and $u_0 \le \delta$ for $|x-x_0| \ge \eta/2$, we get $u_0 \le \overline{v}(x,0)$, and it follows from the comparison principle that $u \le \overline{v}$ in $Q_T$; hence $T(u_0) \le T < \infty$. This proves conclusion (ii).

*Step 3* (we prove assertion (i)). First observe that assertion (ii) is actually true for any $k > 0$ in view of Step 2. On the other hand, by standard energy arguments, one can show that $u(x,t)$ converges to $U_k$ in $L^\infty(0,1)$ as $t \to \infty$. Since $U_k = 0$ on $[0, \eta/2]$ for some $\eta > 0$, it follows that for $t_0$ large, the new initial data $\tilde{u}_0 := u(x,t_0)$ satisfies the assumptions of part (ii) with $M = k+1$. The conclusion follows. □

## 3. Dead-Core Profile Upper and Lower Bound

In this section, we will derive some a prior estimates for solutions of (1.1). Since $u_t < 0$ in $Q_T$ and $u_x > 0$ in $(0,1)\times(0,T)$, we have $(u^m)_{xx} < x^q u^p$. Let $v = u^m$. Then from $v_x = mu^{m-1}u_x, 0 < u \le k$ in $Q_T$ and

$$v_{xx}(x,t) < x^q v^r(x,t), \quad (x,t) \in (0,1)\times(0,T), \tag{3.1}$$

it follows that $v_x$ and $u_x$ are bounded in $Q_T$.

Integrating the inequality $v_{xx}(x,t) < x^q v^r(x,t)$ using $v_x \ge 0$, we obtain

$$v_x(x,t) = \int_0^x v_{xx}(y,t)dy \le \int_0^x y^q v^r(y,t)dy \le v^r(x,t)\int_0^x y^q dy \le \frac{x^{q+1}}{q+1}v^r(x,t); \tag{3.2}$$

hence $v^{1-r}(x,t) - v^{1-r}(0,t) \le ((1-r)/(q+1)(q+2))x^{q+2}$. Consequently

$$u(x,T) = v^{1/m}(x,T) \le Cx^{(q+2)/m(1-r)} = Cx^{(q+2)/(m-p)}, \tag{3.3}$$

where $C = [(m-p)/m(q+1)(q+2)]^{1/(m-p)}$. Together with the following lower bound lemma, we obtain Theorem 1.2.

**Lemma 3.1.** *Let* $0 < p < m < 1$, $p + m > 1$ *and* $-1 < q < 0$. *Let* (1.2), *and* (1.6) *be in force and fix* $t_0 \in (0, T)$. *Then there exists* $\varepsilon > 0$ *such that the auxiliary function*

$$J := (u^m)_x - \varepsilon x^{q+1} u^p \tag{3.4}$$

*satisfies* $J \geq 0$ *in* $[0,1] \times (t_0, T)$. *In particular, there exists* $C_\varepsilon > 0$ *such that*

$$u(x,t) \geq C_\varepsilon x^{(q+2)/(m-p)}, \quad (x,t) \in (0,1) \times (t_0, T), \tag{3.5}$$

*where* $C_\varepsilon = [\varepsilon(m-p)/m(q+2)]^{1/(m-p)}$ *and* $0 < \varepsilon \leq (p+m-1)/(2p+m-1)(q+1)$.

*Proof.* The equation in (1.1) can be written under the form

$$u_t - a u_{xx} = m(m-1)u^{m-2}(u_x)^2 - x^q u^p, \tag{3.6}$$

with $a = m u^{m-1}$. For $(x,t) \in (0,1) \times (0,T)$, we compute

$$\begin{aligned}
\left(x^{q+1}u^p\right)_t &= p x^{q+1} u^{p-1} u_t, \\
\left(x^{q+1}u^p\right)_x &= (q+1)x^q u^p + p x^{q+1} u^{p-1} u_x \\
\left(x^{q+1}u^p\right)_{xx} &= (q+1)q x^{q-1} u^p + 2(q+1)p x^q u^{p-1} u_x + p(p-1) x^{q+1} u^{p-2} (u_x)^2 + p x^{q+1} u^{p-1} u_{xx}.
\end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned}
\left(x^{q+1}u^p\right)_t - a\left(x^{q+1}u^p\right)_{xx} &= p x^{q+1} u^{p-1}(u_t - a u_{xx}) \\
&\quad - a\left((q+1)q x^{q-1} u^p + 2(q+1)p x^q u^{p-1} u_x + p(p-1) x^{q+1} u^{p-2} (u_x)^2\right) \\
&= pm(m-1) x^{q+1} u^{p+m-3} u_x^2 - p x^{2q+1} u^{2p-1} - 2m(q+1)p x^q u^{m+p-2} u_x \\
&\quad - mq(q+1) x^{q-1} u^{m+p-1} - mp(p-1) x^{q+1} u^{m+p-3} u_x^2 \\
&= -pm^{-1}(-m+p) x^{q+1} u^{p-m-1} (u^m)_x^2 - 2p(q+1) x^q u^{p-1} (u^m)_x \\
&\quad - p x^{2q+1} u^{2p-1} - mq(q+1) x^{q-1} u^{m+p-1}.
\end{aligned} \tag{3.8}$$

Using

$$(u^m)_x = J + \varepsilon x^{q+1} u^p, \tag{3.9}$$

we deduce that

$$\left(x^{q+1}u^p\right)_t - a\left(x^{q+1}u^p\right)_{xx} = b_1 J - px^{2q+1}u^{2p-1}\left\{1 + 2\varepsilon(q+1) + \varepsilon^2 m^{-1}(p-m)x^{q+2}u^{p-m}\right\} - mq(q+1)x^{q-1}u^{m+p-1} \tag{3.10}$$

with $b_1 = pm^{-1}(m-p)x^{q+1}u^{p-m-1}(u^m)_x^2(J + 2\varepsilon x^{q+1}u^p) - 2p(q+1)x^q u^{p-1}$.

On the other hand, we have

$$\begin{aligned}(u^m)_{xt} - a(u^m)_{xxx} &= (au_x)_t - a(u^m)_{xxx} = a(u_t - (u^m)_{xx})_x + a_t u_x \\ &= -aqx^{q-1}u^p - apx^q u^{p-1}u_x + m(m-1)u^{m-2}u_t u_x \\ &= -mqx^{q-1}u^{m+p-1} - px^q u^{p-1}(u^m)_x + (m-1)u^{-1}u_t(u^m)_x \\ &= -mqx^{q-1}u^{m+p-1} - px^q u^{p-1}\left(J + \varepsilon x^{q+1}u^p\right) + (m-1)u^{-1}u_t\left(J + \varepsilon x^{q+1}u^p\right) \\ &= b_2 J + \varepsilon x^{q+1}u^{2p-1}\{-px^q + (m-1)u^{-p}u_t\} - mqx^{q-1}u^{m+p-1} \\ &= b_2 J + \varepsilon x^{q+1}u^{2p-1}\{(1-m-p)x^q + (m-1)u^{-p}(u^m)_{xx}\} - mqx^{q-1}u^{m+p-1}\end{aligned} \tag{3.11}$$

where $b_2 = -px^q u^{p-1} + (m-1)u^{-1}u_t$.

Since

$$\begin{aligned}(u^m)_{xx} &= \left(J + \varepsilon x^{q+1}u^p\right)_x = J_x + \varepsilon\left[(q+1)x^q u^p + px^{q+1}u^{p-1}u_x\right] \\ &= J_x + \varepsilon x^q u^p\left[q + 1 + pm^{-1}xu^{-m}(u^m)_x\right] \\ &= J_x + \varepsilon x^q u^p\left[q + 1 + pm^{-1}xu^{-m}\left(J + \varepsilon x^{q+1}u^p\right)\right] \\ &= J_x + b_3 J + \varepsilon x^q u^p\left[q + 1 + \varepsilon pm^{-1}x^{q+2}u^{p-m}\right]\end{aligned} \tag{3.12}$$

with $b_3 = \varepsilon pm^{-1}x^{q+1}u^{p-m}$ being a smooth function on $[0,1] \times (0,T)$, it follows that

$$\begin{aligned}(u^m)_{xt} - a(u^m)_{xxx} &= b_2 J + \varepsilon x^{q+1}u^{2p-1}\Big\{(1-m-p)x^q + (m-1)u^{-p} \\ &\qquad \times\left(J_x + b_3 J + \varepsilon x^q u^p\left[q + 1 + \varepsilon pm^{-1}x^{q+2}u^{p-m}\right]\right)\Big\} \\ &\quad - mqx^{q-1}u^{m+p-1} \\ &= b_4 J + b_5 J_x + \varepsilon x^{q+1}u^{2p-1}\Big\{(1-m-p)x^q + \varepsilon(m-1)x^q \\ &\qquad \times\left[q + 1 + \varepsilon pm^{-1}x^{q+2}u^{p-m}\right]\Big\} - mqx^{q-1}u^{m+p-1},\end{aligned} \tag{3.13}$$

where

$$b_4 = b_2 + \varepsilon(m-1)x^{q+1}u^{p-1}b_3,$$
$$b_5 = \varepsilon(m-1)x^{q+1}u^{p-1} \text{ is a smooth function on } [0,1]\times(0,T). \tag{3.14}$$

Combining (3.10) and (3.13), we obtain

$$\begin{aligned} &b_4 J + b_5 J_x - mqx^{q-1}u^{m+p-1} \\ &\quad + \varepsilon x^{q+1}u^{2p-1}\Big\{(1-m-p)x^q + \varepsilon(m-1)x^q\Big[q+1+\varepsilon p m^{-1}x^{q+2}u^{p-m}\Big]\Big\} \\ &= J_t - aJ_{xx} + \varepsilon b_1 J - \varepsilon mqx^{q-1}u^{m+p-1} \\ &\quad - \varepsilon x^{2q+1}u^{2p-1}\Big\{1 + 2\varepsilon(q+1) + \varepsilon^2 m^{-1}(p-m)x^{q+2}u^{p-m}\Big\}. \end{aligned} \tag{3.15}$$

Namely

$$\begin{aligned} &J_t - aJ_{xx} - \Big(b_5 + (m-1)u^{-1}\Big)J_x - b_7 J \\ &\quad = \varepsilon x^{2q+1}u^{2p-1}\Big\{(1-m-p) + \varepsilon(m-1)\Big[q+1+\varepsilon p m^{-1}x^{q+2}u^{p-m}\Big]\Big\} \\ &\qquad + p\varepsilon x^{2q+1}u^{2p-1}\Big\{1 + 2\varepsilon(q+1) + \varepsilon^2 m^{-1}(p-m)x^{q+2}u^{p-m}\Big\} \\ &\qquad - mq(1-\varepsilon(q+1))x^{q-1}u^{m+p-1} \\ &\quad = \varepsilon x^{2q+1}u^{2p-1}\Big\{1-m+\varepsilon(m-1)\Big[q+1+\varepsilon p m^{-1}x^{q+2}u^{p-m}\Big]\Big\} \\ &\qquad + p\varepsilon x^{2q+1}u^{2p-1}\Big\{2\varepsilon(q+1) + \varepsilon^2 m^{-1}(p-m)x^{q+2}u^{p-m}\Big\} \\ &\qquad - mq(1-\varepsilon(q+1))x^{q-1}u^{m+p-1} \end{aligned} \tag{3.16}$$

with $b_6 = b_5 + (m-1)u^{-1}$ being a smooth function on $[0,1]\times(0,T)$.

In order to make $b_7 \le 0$ in force, we require $\varepsilon \le (p+m-1)/(2p+m-1)(q+1)$ and $p+m>1$.

Since $m<1$, by choosing $0<\varepsilon\le\varepsilon_0$ with $\varepsilon_0=\varepsilon_0(m,p)>0$ small enough, it follows that

$$\begin{aligned} J_t - aJ_{xx} - b_6 J_x - b_7 J &\ge \varepsilon x^{2q+1}u^{2p-1}\left\{\frac{1-m}{2} - \frac{p(1-p)}{m}\varepsilon^2 x^{q+2}u^{p-m}\right\} \\ &= \frac{1-m}{2}\varepsilon x^{2q+1}u^{3p-m-1}\Big\{u^{m-p} - \kappa\varepsilon^2 x^{q+2}\Big\}, \end{aligned} \tag{3.17}$$

where $\kappa := 2p(1-p)/[m(1-m)] > 0$. Now observe that

$$\begin{aligned}\left[u^{m-p} - \frac{m-p}{(q+2)m}\varepsilon x^{q+2}\right]_x &= (m-p)\left[u^{m-p-1}u_x - m^{-1}\varepsilon x^{q+1}\right]\\ &= \frac{m-p}{m}u^{-p}\left[(u^m)_x - \varepsilon x^{q+1}u^p\right] = \frac{m-p}{m}u^{-p}J,\end{aligned} \tag{3.18}$$

hence

$$u^{m-p} - \frac{m-p}{(q+2)m}\varepsilon x^{q+2} \geq \frac{m-p}{m}u^{-p}\int_0^x u^{-p}J(y,t)\,dy. \tag{3.19}$$

Thus, taking $\varepsilon_0$ possible smaller, we get

$$J_t - aJ_{xx} - b_6 J_x - b_7 J \geq \frac{1-m}{2}\varepsilon x^{2q+1}u^{3p-m-1}\left\{u^{m-p} - \frac{m-p}{2m}\varepsilon x^{q+2}\right\}, \tag{3.20}$$

hence

$$J_t - aJ_{xx} - b_6 J_x - b_7 J \geq C\varepsilon x^{2q+1}u^{3p-m-1}\int_0^x u^{-p}J(y,t)\,dy, \tag{3.21}$$

with $C = (1-m)(m-p)/2m > 0$. Now for any $0 < t_0 < t_1 < T$, it follows from the maximum principle that $J$ attains its minimum in $Q = [0,1] \times [t_0, t_1]$ on the parabolic boundary of $Q$ (see [1, 2]).

It is thus sufficient to check that $J \geq 0$ on the parabolic boundary of $Q$ for $\varepsilon$ small. Clearly $J = 0$ for $x = 0$. Since $u_x$ is bounded on $Q_T$, $u(x,t) \geq \eta > 0$ in $[1-\eta, 1] \times (t_0, T)$ for some small constant $\delta > 0$. Therefore $u$ extends to a classical solution on $[1-\eta, 1] \times (t_0, T]$, and Hopf's Lemma implies that $u_x(1,t) \geq \widetilde{\delta} > 0$ for $t_0 < t < T$; hence $J(1,t) \geq 0$ for $t_0 < t < T$ if $\varepsilon$ is chosen small enough. Moreover, also as a consequence of Hopf's Lemma, we have $u_x(x, t_0) \geq cx^{q+1}$ in $[0,1]$ for some $c > 0$. Again decreasing $\varepsilon$ if necessary, we deduce that $J(x, t_0) \geq 0$ in $[0,1]$. The lemma follows. □

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