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Research Article

Existence of the Mild Solutions for Impulsive Fractional Equations with Infinite Delay

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This paper is concerned with the existence and uniqueness of a mild solution of a semilinear fractional-order functional evolution differential equation with the infinite delay and impulsive effects. The existence and uniqueness of a mild solution is established using a solution operator and the classical fixed-point theorems.

1. Introduction

This paper is concerned with the existence and uniqueness of a mild solution of an impulsive fractional-order functional differential equation with the infinite delay of the form

$$\begin{aligned} &D_t^{\alpha} x(t) = Ax(t) + f(t, x_t, Bx(t)), \quad t \in J = [0,T],\ t \neq t_k, \\ &\Delta x(t_k) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\ &x(t) = \phi(t), \quad \phi(t) \in \mathfrak{B}_h, \end{aligned} \tag{1.1}$$

where $T > 0$, $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t\geq 0}$, the solution operator $T_\alpha(t)_{t\geq 0}$ is defined on a complex Banach space X, D^α is the Caputo fractional derivative, $f : J \times \mathfrak{B}_h \times X \to X$ is a given function, and $\mathfrak{B}_h$ is a phase space defined in Section 2. Here, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(X, X)$, $(k = 1, 2, \ldots, m)$, are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

We assume that $x_t : (-\infty, 0] \to X$, $x_t(s) = x(t+s)$, $s \leq 0$, belongs to an abstract phase space $\mathfrak{B}_h$. The term $Bx(t)$ is given by $Bx(t) = \int_0^t K(t,s)x(s)ds$, where $K \in C(D, \mathbb{R}^+)$ is the set of all positive continuous functions on $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$.

Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering, and other areas of science. Many physical phenomena in evolution processes are modelled as impulsive differential equations and have been studied extensively by several authors, for instance, see [1–3], for more information on these topics. Impulsive integro-differential equations with delays represent mathematical models for problems in the areas such as population dynamics, biology, ecology, and epidemic and have been studied by many authors [2–7]. The study of fractional differential equations has emerged as a new branch of applied mathematics, which has been used for construction and analysis of mathematical models in science and engineering. In fact, the fractional differential equations are considered as models alternative to nonlinear differential equations. Many physical systems can be represented more accurately through fractional derivative formulation. For more detail, see, for instance, the papers [1, 3–5, 7–12] and references therein.

Recently, in [4], the author has established sufficient conditions for the existence of a mild solution for a fractional integro-differential equation with a state-dependent delay. Mophou and N'Guérékata [7] have investigated the existence and uniqueness of a mild solution for the fractional differential equation (1.1) without impulsive conditions. Authors of [7] have established the results assuming that A generates an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ on a complex Banach space X by means of classical fixed-point methods.

In [5], Benchohra et al. have considered the following nonlinear functional differential equation with infinite delay

$$D^q x(t) = f(t, x_t), \quad t \in [0,T], \ 0 < q < 1, \qquad x(t) = \phi(t), \quad t \in \left]-\infty, 0\right], \tag{1.2}$$

where D^q is Riemann-Liouville fractional derivative, $\phi \in \mathfrak{B}_h$, with $\phi(0) = 0$, and established the existence of a mild solution for the considered problem using the Banach fixed-point and the nonlinear alternative of Leray-Schauder theorems.

Motivated by the above-mentioned works, we consider the problem (1.1) to study the existence and uniqueness of a mild solution using the solution operator and fixed-point theorems. The paper is organized as follows: in Section 2, we introduce some function spaces and notations and present some necessary definitions and preliminary results that will be used to prove our main results. The proof of our main results is given in Section 3. In the last section one example is presented.

2. Preliminaries

In this section, we mention some definitions and properties required for establishing our results. Let X be a complex Banach space with its norm denoted as $\|\cdot\|_X$, and $L(X)$ represents the Banach space of all bounded linear operators from X into X, and the corresponding norm is denoted by $\|\cdot\|_{L(X)}$. Let $C(J,X)$ denote the space of all continuous functions from J into X with supremum norm denoted by $\|\cdot\|_{C(J,X)}$. In addition, $B_r(x,X)$ represents the closed ball in X with the center at x and the radius r.

To describe a fractional-order functional differential equation with the infinite delay, we need to discuss the abstract phase space $\mathfrak{B}_h$ in a convenient way (for details see [3]). Let

$h : (-\infty, 0] \to (0, \infty)$ be a continuous function with $l = \int_{-\infty}^{0} h(t)dt < \infty$. For any $a > 0$, we define

$\mathfrak{B} = \{\psi : [-a, 0] \to X$ such that $\psi(t)$ is bounded and measurable$\}$ and equip the space $\mathfrak{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} |\psi(s)|, \quad \forall\psi \in \mathfrak{B}. \tag{2.1}$$

Let us define by

$$\begin{aligned} \mathfrak{B}_h = \Big\{ &\psi : (-\infty, 0] \longrightarrow X, \text{ such that for any } c > 0, \ \psi|_{[-c,0]} \in \mathfrak{B} \text{ with} \\ &\psi(0) = 0 \text{ and } \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]} ds < \infty \Big\}. \end{aligned} \tag{2.2}$$

If $\mathfrak{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]} ds, \quad \forall\psi \in \mathfrak{B}_h, \tag{2.3}$$

then it is known that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space.

Now, we consider the space

$$\begin{aligned} \mathfrak{B}'_h = \Big\{ &x : (-\infty, T] \longrightarrow X \text{ such that } x|_{J_k} \in C(J_k, X) \text{ and there exist} \\ &x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), \ x_0 = \phi \in \mathfrak{B}_h, \ k = 1, \ldots, m \Big\}, \end{aligned} \tag{2.4}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}], \ k = 0, 1, 2, \ldots, m$. The function $\|\cdot\|_{\mathfrak{B}'_h}$ to be a seminorm in $\mathfrak{B}'_h$, it is defined by

$$\|x\|_{\mathfrak{B}'_h} = \sup\{|x(s)| : s \in [0, T]\} + \|\phi\|_{\mathfrak{B}_h}, \quad x \in \mathfrak{B}'_h. \tag{2.5}$$

If $x :]-\infty, T] \to X, \ T > 0$, is such that $x_0 \in \mathfrak{B}_h$, then for all $t \in J$, the following conditions hold:

(1) $x_t \in \mathfrak{B}_h$,

(2) $\|x_t\|_{\mathfrak{B}_h} \leq C_1(t)\sup_{0<s<t}\|x(s)\| + C_2(t)\|x_0\|_{\mathfrak{B}_h}$,

(3) $\|x(t)\| \leq H\|x_t\|_{\mathfrak{B}_h}$, where $H > 0$ is a constant and $C_1:[0, \infty) \to [0, \infty)$ is continuous, $C_2:[0, \infty) \to [0, \infty)$ is locally bounded, and $C_1, \ C_2$ are independent of $x(\cdot)$. For more details, see [6].

A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu, \quad \alpha, \beta > 0, \ z \in \mathbb{C}, \tag{2.6}$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise. For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cosh(z)$, $E_2(-z^2) = \cos(z)$, and plays an important role in the theory of fractional differential equations. The most interesting properties of the Mittag-Lefller functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \operatorname{Re}\lambda > \omega^{1/\alpha},\ \omega > 0, \tag{2.7}$$

see [12] for more details.

Definition 2.1. A closed and linear operator A is said to be sectorial if there are constants $\omega \in R, \theta \in [\pi/2, \pi]$, $M > 0$, such that the following two conditions are satisfied:

$$\begin{aligned}
&(1)\ \rho(A) \subset \sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}, \\
&(2)\ \|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \sum_{(\theta,\omega)}.
\end{aligned} \tag{2.8}$$

Sectorial operators are well studied in the literature. For details see [13].

Definition 2.2 (see Definition 2.3 in [10]). Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : R_+ \to L(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X, \tag{2.9}$$

in this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.3 (see Definition 2.1 in [4]). Let A be a closed linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : R_+ \to L(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X, \tag{2.10}$$

in this case, $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family (see [14] Chapter 1). For more details on α-resolvent family and solution operators, we refer to [14, 15] and the references therein.

Definition 2.4. The Riemann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}_+, X)$, is defined by

$$I^0 f(t) = f(t), \quad I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t > 0, \tag{2.11}$$

where $\Gamma(\cdot)$ is the Euler gamma function. The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega, \tag{2.12}$$

provided the integral is absolutely convergent for $\mathrm{Re}(\lambda) > \omega$.

Definition 2.5. Caputo's derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = I^{n-\alpha} f^{(n)}(t), \tag{2.13}$$

for $n-1 \leq \alpha < n,\ n \in N$. If $0 < \alpha \leq 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds. \tag{2.14}$$

Obviously, Caputo's derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^\alpha f(t); \lambda\} = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n. \tag{2.15}$$

Lemma 2.6. *If f satisfies the uniform Holder condition with the exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{gathered} D_t^\alpha x(t) = Ax(t) + f(t, x_t, Bx(t)), \quad t > t_0,\ t_0 \in R,\ 0 < \alpha < 1, \\ x(t) = \phi(t), \quad t \leq t_0, \end{gathered} \tag{2.16}$$

is given by

$$x(t) = T_\alpha(t-t_0)\left(x\left(t_0^+\right)\right) + \int_{t_0}^t S_\alpha(t-s) f(s, x_s, Bx(s))ds, \tag{2.17}$$

where

$$
\begin{aligned}
T_\alpha(t) &= E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda,\\
S_\alpha(t) &= t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda,
\end{aligned}
\tag{2.18}
$$

$\widehat{B}_r$ denotes the Bromwich path. $S_\alpha(t)$ is called the α-resolvent family, and $T_\alpha(t)$ is the solution operator, generated by A.

Proof. Let $t - t_0 = u$, then we get

$$
D_u^\alpha x(u+t_0) = Ax(u+t_0) + f(u+t_0, x_{u+t_0}, Bx(u+t_0)), \quad u > 0. \tag{2.19}
$$

Taking the Laplace transform of (2.19), we have

$$
\lambda^\alpha L\{x(u+t_0)\} - \lambda^{\alpha-1}x(t_0^+) = AL\{x(u+t_0)\} + L\{f(u+t_0, x_{u+t_0}, Bx(u+t_0))\}. \tag{2.20}
$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is, $\lambda^\alpha \in \rho(A)$, from (2.20), we obtain

$$
\begin{aligned}
&L\{x(u+t_0)\}\\
&\quad = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x(t_0^+) + (\lambda^\alpha I - A)^{-1}L\{f(u+t_0, x_{u+t_0}, Bx(u+t_0))\}.
\end{aligned}
\tag{2.21}
$$

By the inverse Laplace transform of (2.21), we get

$$
x(u+t_0) = E_{\alpha,1}(Au^\alpha)x(t_0^+) + \int_0^u (u-s)^{\alpha-1}E_{\alpha,\alpha}(A(u-s)^\alpha)f(s+t_0, x_{s+t_0}, Bx(s+t_0))ds. \tag{2.22}
$$

Set $u + t_0 = t$, in (2.22), we have

$$
\begin{aligned}
x(t) &= E_{\alpha,1}(A(t-t_0)^\alpha)x(t_0^+)\\
&\quad + \int_0^{t-t_0} (t-t_0-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-t_0-s)^\alpha)f(s+t_0, x_{s+t_0}, Bx(s+t_0))ds.
\end{aligned}
\tag{2.23}
$$

On simplification, we obtain

$$
\begin{aligned}
x(t) &= E_{\alpha,1}(A(t-t_0)^\alpha)x(t_0^+)\\
&\quad + \int_{t_0}^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^\alpha)f(\theta, x_\theta, Bx(\theta))d\theta.
\end{aligned}
\tag{2.24}
$$

Set $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$ and $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ in (2.24). We have

$$x(t) = T_\alpha(t-t_0)x\left(t_0^+\right) + \int_{t_0}^{t} S_\alpha(t-\theta)f(\theta, x_\theta, Bx(\theta))d\theta. \tag{2.25}$$

This completes the proof of the lemma. □

Now, we give the definition of a mild solution of the system (1.1) by investigating the classical solution of the system (1.1).

Definition 2.7. A function $x : (-\infty, T] \to X$ is called a mild solution of (1.1) if the following holds: $x_0 = \phi \in \mathfrak{B}_h$ on $(-\infty, 0]$ with $\phi(0) = 0$; $\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, \dots, m$, the restriction of $x(\cdot)$ to the interval $[0, T) \setminus \{t_1, \dots, t_m\}$ is continuous and satisfies the following integral equation:

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \displaystyle\int_0^t S_\alpha(t-s)f(s, x_s, Bx(s))ds, & t \in [0, t_1], \\ T_\alpha(t-t_1)\left(x\left(t_1^-\right) + I_1\left(x\left(t_1^-\right)\right)\right) + \displaystyle\int_{t_1}^t S_\alpha(t-s)f(s, x_s, Bx(s))ds, & t \in (t_1, t_2], \\ \vdots & \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \displaystyle\int_{t_m}^t S_\alpha(t-s)f(s, x_s, Bx(s))ds, & t \in (t_m, T]. \end{cases} \tag{2.26}$$

Now, we introduce the following assumptions:

(H1) there exist $\mu_1, \mu_2 > 0$ such that

$$\|f(t, \varphi, x) - f(t, \psi, y)\|_X \le \mu_1\|\varphi - \psi\|_{\mathfrak{B}_h} + \mu_2\|x - y\|_X, \quad t \in I,\ (\varphi, \psi) \in \mathfrak{B}_h^2,\ x, y \in X. \tag{2.27}$$

(H2) for each $k = 1, \dots, m$, there exists $\rho_k > 0$ such that

$$\|I_k(x) - I_k(y)\|_X \le \rho_k\|x - y\|_X, \quad \forall x, y \in X. \tag{2.28}$$

(H3)

$$\max_{1\le i\le m}\left\{\widetilde{M}_T(1+\rho_i) + \frac{\widetilde{M}_S T^\alpha}{\alpha}\left(\mu_1 C_1^* + \mu_2 B^*\right)\right\} < 1, \tag{2.29}$$

where $C_1^* = \sup_{0<\tau<T} C_1(\tau)$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds < \infty$ and

$$\widetilde{M}_T = \sup_{0\le t\le T} \|T_\alpha(t)\|_{L(X)}, \qquad \widetilde{M}_S = \sup_{0\le t\le T} Ce^{\omega t}\left(1 + t^{1-\alpha}\right). \tag{2.30}$$

If $\alpha \in (0,1)$ and $A \in A^\alpha(\theta_0,\omega_0)$, then for any $x \in X$ and $t > 0$, we have $\|T_\alpha(t)\|_{L(X)} \le Me^{\omega t}$ and $\|S_\alpha(t)\|_{L(X)} \le Ce^{\omega t}(1 + t^{\alpha-1})$, $t > 0$, $\omega > \omega_0$. Hence, we have $\|T_\alpha(t)\|_{L(X)} \le \widetilde{M}_T$, $\|S_\alpha(t)\|_{L(X)} \le t^{\alpha-1}\widetilde{M}_S$. See [1] for details.

3. The Main Results

Our first result is based on the Banach contraction principle.

Theorem 3.1. *Assume that the assumptions (H1)–(H3) are satisfied. If $A \in \mathbb{A}^\alpha(\theta_0,\omega_0)$, then the system* (1.1) *has a unique mild solution.*

Proof. Consider the operator $N : \mathfrak{B}'_h \to \mathfrak{B}'_h$ defined by

$$(Nx)(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ \displaystyle\int_0^t S_\alpha(t-s) f(s,x_s,Bx(s))ds, & t \in [0,t_1], \\ T_\alpha(t-t_1)\left(x(t_1^-) + I_1\left(x(t_1^-)\right)\right) & \\ \quad + \displaystyle\int_{t_1}^t S_\alpha(t-s) f(s,x_s,Bx(s))ds, & t \in (t_1,t_2], \\ \vdots & \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) & \\ \quad + \displaystyle\int_{t_m}^t S_\alpha(t-s) f(s,x_s,Bx(s))ds, & t \in (t_m,T]. \end{cases} \tag{3.1}$$

Let $y(\cdot) : (-\infty,T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty,0] \\ 0, & t \in J, \end{cases} \tag{3.2}$$

then $y_0 = \phi$. For each $z \in C(J,\mathbb{R})$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty,0]; \\ z(t), & t \in J. \end{cases} \tag{3.3}$$

If $x(\cdot)$ satisfies (2.26), then we can decompose $x(\cdot)$ as $x(t) = y(t)+\overline{z}(t)$ for $t \in J$, which implies $x_t = y_t + \overline{z}_t$ for $t \in J$, and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in [0, t_1], \\ T_\alpha(t-t_1)\big[y(t_1^-) + \overline{z}(t_1^-) + I_1\big(\big(y(t_1^-)\big) + \overline{z}(t_1^-)\big)\big] & \\ \quad + \displaystyle\int_{t_1}^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in (t_1, t_2], \\ \vdots & \\ T_\alpha(t-t_m)\big[y(t_m^-) + \overline{z}(t_m^-) + I_m\big(\big(y(t_m^-)\big) + \overline{z}(t_m^-)\big)\big] & \\ \quad + \displaystyle\int_{t_m}^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in (t_m, T]. \end{cases} \tag{3.4}$$

Set $\mathfrak{B}_h'' = \{z \in \mathfrak{B}_h'$ such that $z_0 = 0\}$ and let $\|\cdot\|_{\mathfrak{B}_h''}$ be the seminorm in $\mathfrak{B}_h''$ defined by

$$\|z\|_{\mathfrak{B}_h''} = \sup_{t\in J} \|z(t)\|_X + \|z_0\|_{\mathfrak{B}_h} = \sup_{t\in J} \|z(t)\|_X, \quad z \in \mathfrak{B}_h'', \tag{3.5}$$

thus $(\mathfrak{B}_h'', \|\cdot\|_{\mathfrak{B}_h''})$ is a Banach space. We define the operator $P : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ by

$$(Pz)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in [0, t_1], \\ T_\alpha(t-t_1)\big[z(t_1^-) + I_1\big(z(t_1^-)\big)\big] & \\ \quad + \displaystyle\int_{t_1}^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in (t_1, t_2], \\ \vdots & \\ T_\alpha(t-t_m)[z(t_m^-) + I_m(z(t_m^-))] & \\ \quad + \displaystyle\int_{t_m}^t S_\alpha(t-s) f\big(s, y_s + \overline{z}_s, B\big(y(s) + \overline{z}(s)\big)\big) ds, & t \in (t_m, T]. \end{cases} \tag{3.6}$$

It is clear that the operator N has a unique fixed-point if and only if P has a unique fixed-point. To prove that P has a unique fixed-point, let $z, z^* \in \mathfrak{B}''_h$, then for all $t \in [0, t_1]$. We have

$$\begin{aligned}
&\|P(z)(t) - P(z^*)(t)\|_X \\
&\quad \leq \int_0^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}^*_s, B(y(s) + \overline{z}^*(s))) \right\|_X ds \\
&\quad \leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \left[\mu_1 \left\| \overline{z}_s - \overline{z}^*_s \right\|_{\mathfrak{B}_h} + \mu_2 \left\| B\left(y(s) + \overline{z}(s)\right) - B\left(y(s) + \overline{z}^*\right) \right\|_X \right] ds \\
&\quad \leq \frac{\widetilde{M}_S}{\alpha} \left(\mu_1 C_1^* + \mu_2 B^*\right) T^\alpha \|z - z^*\|_{\mathfrak{B}''_h}.
\end{aligned} \tag{3.7}$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
&\|P(z)(t) - P(z^*)(t)\|_X \\
&\quad \leq \|T_\alpha(t-t_1)\|_{L(X)} \left[\left\| z(t_1^-) - z^*(t_1^-) \right\|_X + \left\| I_1(z(t_1^-)) - I_1(z^*(t_1^-)) \right\|_X \right] \\
&\qquad + \int_{t_1}^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}^*_s, B(y(s) + \overline{z}^*(s))) \right\|_X \\
&\quad \leq \widetilde{M}_T \left[\left\| z(t_1^-) - z^*(t_1^-) \right\|_X + \rho_1 \left\| z(t_1^-) - z^*(t_1^-) \right\|_X \right] \\
&\qquad + \widetilde{M}_S \int_{t_1}^t (t-s)^{\alpha-1} \left[\mu_1 \left\| \overline{z}_s - \overline{z}^*_s \right\|_{\mathfrak{B}_h} + \mu_2 \left\| B\left(y(s) + \overline{z}(s)\right) - B\left(y(s) + \overline{z}^*\right) \right\|_X \right] ds \\
&\quad \leq \widetilde{M}_T (1 + \rho_1) \|z - z^*\|_{\mathfrak{B}''_h} + \frac{\widetilde{M}_S}{\alpha} \left(\mu_1 C_1^* + \mu_2 B^*\right) T^\alpha \|z - z^*\|_{\mathfrak{B}''_h}.
\end{aligned} \tag{3.8}$$

Similarly, when $t \in (t_i, t_{i+1}]$, $i = 2, \ldots, m$, we get

$$\|P(z)(t) - P(z^*)(t)\|_X \leq \widetilde{M}_T (1 + \rho_i) \|z - z^*\|_{\mathfrak{B}''_h} + \frac{\widetilde{M}_S}{\alpha} \left(\mu_1 C_1^* + \mu_2 B^*\right) T^\alpha \|z - z^*\|_{\mathfrak{B}''_h}. \tag{3.9}$$

Thus, for all $t \in [0, T]$, we have

$$\|P(z) - P(z^*)\|_{\mathfrak{B}''_h} \leq \max_{1 \leq i \leq m} \left\{ \widetilde{M}_T (1 + \rho_i) + \frac{\widetilde{M}_S}{\alpha} \left(\mu_1 C_1^* + \mu_2 B^*\right) T^\alpha \right\} \|z - z^*\|_{\mathfrak{B}''_h}. \tag{3.10}$$

Hence, P is a contraction map, and therefore it has an unique fixed-point $z \in \mathfrak{B}''_h$, which is a mild solution of (1.1) on $(-\infty, T]$. This completes the proof of the theorem. □

The second result is established using the following Krasnoselkii's fixed-point theorem.

Theorem 3.2. *Let B be a closed-convex and nonempty subset of a Banach space X. Let P and Q be two operators such that (i) $Px + Qy \in B$ whenever $x, y \in B$, (ii) P is compact and continuous; (iii) Q is a contraction mapping, then there exists $z \in B$ such that $z = Pz + Qz$.*

Now, we make the following assumptions:

(H4) $f : J \times \mathfrak{B}_h \times X \to X$ is continuous, and there exist two continuous functions $\mu_1, \mu_2 : J \to (0, \infty)$ such that

$$\|f(t, \psi, x)\|_X \le \mu_1(t)\|\psi\|_{\mathfrak{B}_h} + \mu_2(t)\|x\|_X, \quad (t, \psi, x) \in J \times \mathfrak{B} \times X. \tag{3.11}$$

(H5) the function $I_k : X \to X$ is continuous, and there exists $\Omega > 0$ such that

$$\Omega = \max_{1\le k\le m, x\in B_r} \{\|I_k(x)\|_X\}. \tag{3.12}$$

Before going further, we need the following lemma.

Lemma 3.3 (see Lemma 3.2 in [7]). *Let*

$$C_1^* = \sup_{0<\tau<T} C_1(\tau), \qquad C_2^* = \sup_{0<\tau<T} C_2(\tau), \qquad \mu_1^* = \sup_{0<\tau<T} \mu_1(\tau), \qquad \mu_2^* = \sup_{0<\tau<T} \mu_2(\tau) \tag{3.13}$$

then for any $s \in J$,

$$\begin{aligned}\mu_1(s)\|y_s + \overline{z}_s\|_{\mathfrak{B}_h} + \mu_2(s)\|B(y(s) + \overline{z}(s))\|_X \le \mu_1^*\left[C_2^*\|\phi\|_{\mathfrak{B}_h} + C_1^* \sup_{0\le\tau\le s}\|z(\tau)\|_X\right] \\ + \mu_2^* \int_0^s K(s,\tau)\|z(\tau)\|_X d\tau.\end{aligned} \tag{3.14}$$

If $\|z\|_X < r$, $r > 0$, then

$$\mu_1(s)\|y_s + \overline{z}_s\|_{\mathfrak{B}_h} + \mu_2(s)\|B(y(s) + \overline{z}(s))\|_X \le \mu_1^*\left[C_2^*\|\phi\|_{\mathfrak{B}_h} + C_1^* r\right] + \mu_2^* r B^* = \lambda. \tag{3.15}$$

Theorem 3.4. *Suppose that the assumptions (H1), (H4), (H5) are satisfied with*

$$\left[\frac{\widetilde{M}_S}{\alpha}(\mu_1 C_1^* + \mu_2 B^*)T^\alpha\right] < 1, \tag{3.16}$$

then the impulsive problem (1.1) has at least one mild solution on $(-\infty, T]$.

Proof. Choose $r \geq [\widetilde{M}_T(r+\Omega) + (\widetilde{M}_S T^\alpha \lambda/\alpha)]$ and consider $B_r = \{z \in \mathfrak{B}''_h : \|z\|_{\mathfrak{B}''_h} \leq r\}$, then B_r is a bounded, closed-convex subset in $\mathfrak{B}''_h$.

Let $\Gamma_1 : B_r \to B_r$ and $\Gamma_2 : B_r \to B_r$ be defined as

$$(\Gamma_1 z)(t) = \begin{cases} 0, & t \in [0, t_1], \\ T_\alpha(t-t_1)\left[z\left(t_1^-\right) + I_1\left(z\left(t_1^-\right)\right)\right], & t \in (t_1, t_2], \\ \vdots & \\ T_\alpha(t-t_m)[z(t_m^-) + I_m(z(t_m^-))], & t \in (t_m, T], \end{cases} \tag{3.17}$$

$$(\Gamma_2 z)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s) f\left(s, y_s + \overline{z}_s, B\left(y(s) + \overline{z}(s)\right)\right) ds, & t \in [0, t_1], \\ \displaystyle\int_{t_1}^t S_\alpha(t-s) f\left(s, y_s + \overline{z}_s, B\left(y(s) + \overline{z}(s)\right)\right) ds, & t \in (t_1, t_2], \\ \vdots & \\ \displaystyle\int_{t_m}^t S_\alpha(t-s) f\left(s, y_s + \overline{z}_s, B\left(y(s) + \overline{z}(s)\right)\right) ds, & t \in (t_m, T]. \end{cases} \tag{3.18}$$

Step 1. Let $z, z^* \in B_r$, then show that $\Gamma_1 z + \Gamma_2 z^* \in B_r$, for $t \in [0, t_1]$, we have

$$\begin{aligned} &\|(\Gamma_1 z)(t) + (\Gamma_2 z^*)(t)\|_X \\ &\quad \leq \int_0^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s^*, B(y(s) + \overline{z}^*(s))) \right\|_X ds \\ &\quad \leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \left[\mu_1(s) \left\| y_s + \overline{z}_s^* \right\|_{\mathfrak{B}''_h} + \mu_2(s) \left\| B(y(s) + \overline{z}^*(s)) \right\|_X \right] ds, \end{aligned} \tag{3.19}$$

and by using Lemma 3.3, we conclude that

$$\|(\Gamma_1 z) + (\Gamma_2 z^*)\|_{\mathfrak{B}''_h} \leq \frac{\widetilde{M}_S \lambda T^\alpha}{\alpha} < r. \tag{3.20}$$

Similarly, when $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have the estimate

$$\begin{aligned} &\|(\Gamma_1 z)(t) + (\Gamma_2 z^*)(t)\|_X \\ &\quad \leq \left\| T_\alpha(t-t_i)[z(t_i^-) + I_i(z(t_i^-))] \right\|_X \\ &\qquad + \int_{t_i}^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s^*, B(y(s) + \overline{z}^*(s))) \right\|_X ds \end{aligned}$$

$$
\begin{aligned}
&\leq \widetilde{M}_T\left(\|z\|_{\mathfrak{B}_h''} + \left\|I_i(z(t_i^-))\right\|_X\right)\\
&\quad + \int_{t_i}^{t} \|S_\alpha(t-s)\|_{L(X)} \left[\mu_1(s)\left\|y_s + \overline{z}_s^*\right\|_{\mathfrak{B}_h''} + \mu_2(s)\left\|B(y(s) + \overline{z}^*(s))\right\|_X\right] ds\\
&\leq \widetilde{M}_T(r+\Omega) + \frac{\widetilde{M}_S T^\alpha \lambda}{\alpha} < r,
\end{aligned}
\tag{3.21}
$$

which implies that $\|\Gamma_1 z + \Gamma_2 z\|_{\mathfrak{B}_h''} \leq r$.

Step 2. We will show that the mapping $(\Gamma_1 z)(t)$ is continuous on B_r. For this purpose, let $\{z^n\}_{n=1}^{\infty}$ be a sequence in B_r with $\lim z^n \to z \in B_r$, then for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$
\begin{aligned}
&\|(\Gamma_1 z^n)(t) - (\Gamma_1 z)(t)\|_X\\
&\quad \leq \|T_\alpha(t-t_i)\|_{L(X)} \left[\left\|z^n(t_i^-) - z(t_i^-)\right\|_X + \left\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\right\|_X\right].
\end{aligned}
\tag{3.22}
$$

Since the functions I_i, $i = 1, 2, \ldots, m$ are continuous, hence $\lim_{n\to\infty}\Gamma_1 z^n = \Gamma_1 z$ in B_r which implies that the mapping Γ_1 is continuous on B_r.

Step 3. Uniform boundedness of the map $(\Gamma_1 z)(t)$ is an implication of the following inequality: for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$
\begin{aligned}
\|(\Gamma_1 z)(t)\|_X &\leq \|T_\alpha(t-t_i)\|_{L(X)} \left[\left\|z(t_i^-)\right\|_X + \left\|I_i(z(t_i^-))\right\|_X\right]\\
&\leq \widetilde{M}_T(r+\Omega).
\end{aligned}
\tag{3.23}
$$

Step 4. To show that the map (3.17) is equicontinuous, we proceed as follows. Let $u, v \in (t_i, t_{i+1}]$, $t_i \leq u < v \leq t_{i+1}$, $i = 0, 1, \ldots, m$, $z \in B_r$, then we obtain

$$
\begin{aligned}
\|(\Gamma_1 z)(v) - (\Gamma_1 z)(u)\|_X &\leq \|T_\alpha(v-t_i) - T_\alpha(u-t_i)\|_{L(X)} \left\|z(t_i^-) + I_i(z(t_i^-))\right\|_X\\
&\leq (r+\Omega)\|T_\alpha(v-t_i) - T_\alpha(u-t_i)\|_{L(X)}.
\end{aligned}
\tag{3.24}
$$

Since T_α is strongly continuous, the continuity of the function $t \mapsto \|T(t)\|$ allows us to conclude that $\lim_{u\to v}\|T_\alpha(v-t_i) - T_\alpha(u-t_i)\|_{L(X)} = 0$, which implies that $\Gamma_1(B_r)$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator Γ_1 is compact.

Now, it only remains to show that the map Γ_2 is a contraction mapping. Let $z, z^* \in B_r$ and $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, then we have

$$\begin{aligned}
&\|(\Gamma_2 z)(t) - (\Gamma_2 z^*)(t)\|_X \\
&\quad \le \int_{t_i}^{t} \|S_\alpha(t-s)\|_{L(X)} \left\| f\left(s, y_s + \overline{z}_s, B\left(y(s) + \overline{z}(s)\right)\right) - f\left(s, y_s + \overline{z}_s^*, B\left(y(s) + \overline{z}^*(s)\right)\right) \right\|_X ds \\
&\quad \le \widetilde{M}_S \int_{t_i}^{t} (t-s)^{\alpha-1} \left[\mu_1 \|\overline{z}_s - \overline{z}_s^*\|_{\mathfrak{B}_h''} + \mu_2 \left\| B\left(y(s) + \overline{z}(s)\right) - B\left(y(s) + \overline{z}^*(s)\right) \right\|_X \right] ds \\
&\quad \le \frac{\widetilde{M}_S}{\alpha} \left(\mu_1 C_1^* + \mu_2 B^*\right) T^\alpha \|z - z^*\|_{\mathfrak{B}_h''},
\end{aligned} \tag{3.25}$$

since $(\widetilde{M}_S/\alpha)(\mu_1 C_1^* + \mu_2 B^*)T^\alpha < 1$, which implies that Γ_2 is a contraction mapping. Hence, by the Krasnoselkii fixed-point theorem, we can conclude that the problem (1.1) has at least one solution on $(-\infty, T]$. This completes the proof of the theorem. □

Our last result is based on the following Schaefer's fixed-point theorem.

Theorem 3.5. *Let P be a continuous and compact mapping on a Banach space X into itself, such that the set $\{x \in X : x = \nu P x \ for\ some\ 0 \le \nu \le 1\}$ is bounded, then P has a fixed-point.*

Lemma 3.6 (see [5]). *Let $v : [0, T] \to [0, \infty)$ be a real function, $w(\cdot)$ is nonnegative and locally integrable function on $[0, T]$, and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds. \tag{3.26}$$

Then there exists a constant $K(\alpha)$ such that

$$v(t) \le w(t) + aK(\alpha) \int_0^t \frac{w(s)}{(t-s)^\alpha} ds, \quad \textit{for every } t \in [0, T]. \tag{3.27}$$

Theorem 3.7. *Assume that the assumptions (H4)-(H5) are satisfied, and if $A \in A^\alpha(\theta_0, \omega_0)$ and $\widetilde{M}_T < 1$, then the impulsive problem (1.1) has at least one mild solution on $(-\infty, T]$.*

Proof. We define the operator $P : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ as in Theorem 3.3. Note that P is well defined in $\mathfrak{B}_h''$. We complete the proof in the following steps.

Step 1. For the continuity of the map P, let $\{z^n\}$ be a sequence in $\mathfrak{B}_h''$ such that $z^n \to z$ in $\mathfrak{B}_h''$. Since the function f is continuous on $J \times \mathfrak{B}_h \times X$, This implies that

$$\begin{aligned}
&f\left(s, y_s + \overline{z}_s^n, B\left(y(s) + \overline{z}^n(s)\right)\right) \\
&\qquad \longrightarrow f\left(s, y_s + \overline{z}_s, B\left(y(s) + \overline{z}(s)\right)\right) \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.28}$$

Now, for every $t \in [0, t_1]$, we get

$$\begin{aligned}
&\|Pz^n(t) - Pz(t)\|_X \\
&\quad \leq \int_0^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s^n, B(y(s) + \overline{z}^n(s))) - f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\
&\quad \leq \widetilde{M}_S \frac{T^\alpha}{\alpha} \varepsilon,
\end{aligned} \tag{3.29}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$. Moreover, we have

$$\begin{aligned}
&\|Pz^n(t) - Pz(t)\|_X \\
&\quad \leq \widetilde{M}_T \left[\left\| z^n(t_i^-) - z(t_i^-) \right\|_X + \left\| I_i(z^n(t_i^-)) - I_i(z(t_i^-)) \right\|_X \right] \\
&\qquad + \int_{t_i}^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s^n, B(y(s) + \overline{z}^n(s))) - f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\
&\quad \leq \widetilde{M}_T \left[\left\| z^n(t_i^-) - z(t_i^-) \right\|_X + \left\| I_i(z^n(t_i^-)) - I_i(z(t_i^-)) \right\|_X \right] + \widetilde{M}_S \frac{T^\alpha}{\alpha} \varepsilon,
\end{aligned} \tag{3.30}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$, for all $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$. The impulsive functions I_k, $k = 1, \ldots, m$ are continuous, then we get

$$\lim_{n \to \infty} \|\mathrm{P}z^n - Pz\|_{\mathfrak{B}_h''} = 0. \tag{3.31}$$

This implies that P is continuous.

Step 2. P maps bounded sets into bounded sets in $\mathfrak{B}_h''$. To prove that for any $r > 0$, there exists a $\gamma > 0$ such that for each $z \in B_r = \{z \in \mathfrak{B}_h'' : \|z\|_{\mathfrak{B}_h''} \leq r\}$, then we have $\|Pz\|_{\mathfrak{B}_h''} \leq \gamma$, then for any $z \in B_r$, $t \in [0, t_1]$, we have

$$\begin{aligned}
\|Pz(t)\|_X &\leq \int_0^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\
&\leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \left[\mu_1(s) \left\| y_s + \overline{z}_s \right\|_{\mathfrak{B}_h} + \mu_2(s) \left\| B(y(s) + \overline{z}(s)) \right\|_X \right] ds.
\end{aligned} \tag{3.32}$$

Using Lemma 3.3, we obtain $\|Pz(t)\|_X \leq \widetilde{M}_S (T^\alpha / \alpha) \lambda$. Similarly, we have

$$\|Pz(t)\|_X \leq \widetilde{M}_T (r + \Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \lambda, \quad t \in (t_i, t_{i+1}],\ i = 1, \ldots, m. \tag{3.33}$$

This implies that

$$\|Pz\|_{\mathfrak{B}_h''} \leq \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha}\lambda = \gamma, \quad t \in [0,T]. \tag{3.34}$$

Step 3. We will prove that $P(B_r)$ is equicontinuous. Let $u, v \in [0, t_1]$, with $u < v$, we have

$$\begin{aligned}\|Pz(v) - Pz(u)\|_X \leq{}& \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\ &+ \int_u^v \|S_\alpha(v-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \leq Q_1 + Q_2,\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}Q_1 &= \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\ &\leq \lambda \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(X)} ds.\end{aligned} \tag{3.36}$$

Since $\|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(X)} \leq 2\widetilde{M}_S(t_1 - s)^{\alpha-1} \in L^1(I, \mathbb{R}_+)$ for $s \in [0, t_1]$ and $S_\alpha(v-s) - S_\alpha(u-s) \to 0$ as $u \to v$, S_α is strongly continuous. This implies that $\lim_{u\to v} Q_1 = 0$,

$$Q_2 = \int_u^v \|S_\alpha(v-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \leq \lambda \frac{\widetilde{M}_S (v-u)^\alpha}{\alpha}. \tag{3.37}$$

Hence, $\lim_{u\to v} Q_2 = 0$. Similarly, for $u, v \in (t_i, t_{i+1}]$, with $u < v$, $i = 1, \ldots, m$, we have

$$\|Pz(v) - Pz(u)\|_X \leq \|T_\alpha(v-t_i) - T_\alpha(u-t_i)\|_{L(X)} \left[\left\| z(t_i^-) \right\|_X + \left\| I_i(z(t_i^-)) \right\|_X \right] + Q_1 + Q_2. \tag{3.38}$$

Since T_α is also strongly continuous, so $T_\alpha(v-t_i) - T_\alpha(u-t_i) \to 0$ as $u \to v$. Thus, from the above inequalities, we have $\lim_{u\to v}\|Pz(v) - Pz(u)\|_X = 0$. So, $P(B_r)$ is equicontinuous. Finally, combining Step 1 to Step 3 with Ascoli's theorem, we conclude that the operator P is compact.

Step 4. We show that the set

$$E = \left\{ z \in \mathfrak{B}_h'' \text{ such that } z = \nu P z \text{ for some } 0 < \nu < 1 \right\} \tag{3.39}$$

is bounded. Let $z \in E$, then $z(t) = \nu P z(t)$ for some $0 < \nu < 1$. Then for each $t \in [0, t_1]$, we have

$$\begin{aligned}\|z(t)\|_X &\le \nu \int_0^t \|S_\alpha(t-s)\|_{L(X)} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \\ &\le \nu \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds,\end{aligned} \tag{3.40}$$

for $t \in (t_i, t_{i+1}],\ i = 1, \dots, m$, we get

$$\begin{aligned}\|z(t)\|_X &\le \nu \Bigg[\|T_\alpha(t-t_i)\|_{L(X)} \left(\left\| z(t_i^-) \right\|_X + \left\| I_i(z(t_i^-)) \right\|_X \right) + \int_{t_i}^t \|S_\alpha(t-s)\|_{L(X)} \\ &\qquad \times \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \Bigg] \\ &\le \left[\widetilde{M}_T \left\| z(t_i^-) \right\|_X + \widetilde{M}_T \Omega + \widetilde{M}_S \int_{t_i}^t (t-s)^{\alpha-1} \left\| f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) \right\|_X ds \right].\end{aligned} \tag{3.41}$$

then for all $t \in [0, T]$, we have

$$\begin{aligned}\|z(t)\|_X &\le \frac{\widetilde{M}_T \Omega}{1 - \widetilde{M}_T} \\ &\quad + \frac{\widetilde{M}_S}{1 - \widetilde{M}_T} \int_0^t (t-s)^{\alpha-1} \left[\mu_1(s) \left\| y_s + \overline{z}_s \right\|_{\mathfrak{B}_h} + \mu_2(s) \left\| B(y(s) + \overline{z}(s)) \right\|_X \right] ds \\ &\le \frac{\widetilde{M}_T \Omega}{1 - \widetilde{M}_T} + \frac{\widetilde{M}_S \mu_1^* C_2^* \|\phi\|_{\mathfrak{B}_h} T^\alpha}{\alpha \left(1 - \widetilde{M}_T \right)} \\ &\quad + \frac{\widetilde{M}_S}{1 - \widetilde{M}_T} \left(\mu_1^* C_1^* + \mu_2^* B^* \right) \int_0^t (t-s)^{\alpha-1} \sup_{0 \le \tau \le s} \|z(\tau)\|_X ds \\ &\le \omega_1 + \omega_2 \int_0^t (t-s)^{\alpha-1} \sup_{0 \le \tau \le s} \|z(\tau)\|_X ds,\end{aligned} \tag{3.42}$$

where $\omega_1 = \widetilde{M}_T \Omega / (1 - \widetilde{M}_T) + \widetilde{M}_S \mu_1^* C_2^* \|\phi\|_{\mathfrak{B}_h} T^\alpha / \alpha (1 - \widetilde{M}_T)$ and $\omega_2 = (\widetilde{M}_S / 1 - \widetilde{M}_T)(\mu_1^* C_1^* + \mu_2^* B^*)$. Let $\tau^* \in [0, s]$ be such that $\sup_{0 \le \tau \le s} \|z(\tau)\|_X = \|z(\tau^*)\|_X,\ 0 \le s \le t$. If $\tau^* \in [0, t]$, then (3.43) can be written as

$$\|z(t)\|_X \le \omega_1 + \omega_2 \int_0^t (t-s)^{\alpha-1} \|z(s)\|_X ds. \tag{3.43}$$

Using Lemma 3.6, there exists a constant $K(\alpha)$, and (3.43) becomes

$$\|z(t)\|_X \leq \omega_1 \left[1 + \int_0^t K(\alpha)(t-s)^{\alpha-1} ds\right] \leq \omega_1 \left[1 + \frac{K(\alpha)T^\alpha}{\alpha}\right]. \tag{3.44}$$

As a consequence of Schaefer's fixed-point theorem, we deduce that P has a fixed-point on $(-\infty, T]$. This completes the proof of the theorem. □

4. Applications

To illustrate the application of the theory, we consider the following partial integro-differential equation with fractional derivative of the form

$$\begin{aligned} D_t^q u(t,x) &= \frac{\partial^2}{\partial x^2} u(t,x) + \int_{-\infty}^t H(t,x,s-t) Q(u(s,x)) ds \\ &\quad + \int_0^t k(s,t) e^{-u(s,x)} ds, \quad x \in [0,\pi],\ t \in [0,b],\ t \neq t_k, \\ u(t,0) &= 0 = u(t,\pi), \quad t \geq 0, \\ u(t,x) &= \phi(t,x), \quad t \in (-\infty, 0],\ x \in [0,\pi], \\ \Delta u(t_i)(x) &= \int_{-\infty}^{t_i} q_i(t_i - s) u(s,x) ds, \quad x \in [0,\pi], \end{aligned} \tag{4.1}$$

where D_t^q is Caputo's fractional derivative of order $0 < q < 1$, $0 < t_1 < t_2 < \cdots < t_n < b$ are prefixed numbers, and $\phi \in \mathfrak{B}_h$. Let $X = L^2[0,\pi]$, and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : w, w'$ are absolutely continuous, $w'' \in X, w(0) = 0 = w(\pi)\}$, then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \tag{4.2}$$

where $w_n(x) = \sqrt{2/\pi} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ in X and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \quad \forall w \in X, \text{ and every } t > 0. \tag{4.3}$$

From these expressions, it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$, that is, $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$. Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = 1/2$, and define

$$\|\phi\|_h = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\|_{L^2} ds. \tag{4.4}$$

Hence, for $(t, \phi) \in [0, b] \times \mathfrak{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set $u(t)(x) = u(t, x)$,

$$f(t, \phi, Bu(t))(x) = \int_{-\infty}^{0} H(t, x, \theta) Q(\phi(\theta)(x)) d\theta + Bu(t)(x), \tag{4.5}$$

where $Bu(t)(x) = \int_0^t k(s,t)e^{-u(s,x)}ds$. Then with these settings, the above equation (4.1) can be written in the abstract form of the equations (1.1). The functions H, k, and Q are satisfying some conditions, and $q_i : \mathbb{R} \to \mathbb{R}$ are continuous and $d_i = \int_{-\infty}^{0} h(s)q_i^2(s)ds < \infty$ for $i = 1, 2, \ldots, n$. Suppose further that

(1) the function $H(t, x, \theta)$ is continuous in $[0, b] \times [0, \pi] \times (-\infty, 0]$ and $H(t, x, \theta) \geq 0$, $\int_{-\infty}^{0} H(t, x, \theta)d\theta = p_1(t, x) < \infty$,

(2) the function $Q(\cdot)$ is continuous and for each $(\theta, y) \in (-\infty, 0] \times [0, \pi]$, $0 \leq Q(u(\theta)(x)) \leq (\int_{-\infty}^{0} e^{2s}\|u(s,\cdot)\|_{L^2}ds)$.

Now, we can see that

$$\begin{aligned}
\|f(t, \phi, Bu(t))\|_{L^2} &= \left[\int_0^{\pi}\left(\int_{-\infty}^{0} h(t, x, \theta)Q(\phi(\theta)(x))d\theta + Bu(t)(x)dx\right)^2\right]^{1/2} \\
&\leq \left[\int_0^{\pi}\left(\int_{-\infty}^{0} h(t, x, \theta)\left(\int_{-\infty}^{0} e^{2s}\|\phi(s)(\cdot)\|_{L^2}ds\right)d\theta\right)^2 dx\right]^{1/2} \\
&\quad + \left[\int_0^{\pi}\left(\int_0^t k(s,t)e^{-u(s,x)}ds\right)^2 dx\right]^{1/2} \\
&\leq \left[\int_0^{\pi}\left(\int_{-\infty}^{0} h(t, x, \theta)\left(\int_{-\infty}^{0} e^{2s}\sup_{s\in[\theta,0]}\|\phi(s)\|_{L^2}ds\right)d\theta\right)^2 dx\right]^{1/2} \\
&\quad + \left[\int_0^{\pi}\left(\int_0^t k(s,t)e^{-u(s,x)}ds\right)^2 dx\right]^{1/2} \\
&\leq \left[\int_0^{\pi}\left(\int_{-\infty}^{0} h(t, x, \theta)d\theta\right)^2 dx\right]^{1/2} \|\phi\|_{\mathfrak{B}_h} + \|Bu(t)\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_0^{\pi} (p(t,x))^2 dx\right]^{1/2} \|\phi\|_{\mathfrak{B}_h} + \|Bu(t)\|_{L^2} \\
&\leq \overline{p}(t)\|\phi\|_{\mathfrak{B}_h} + \|Bu(t)\|_{L^2}.
\end{aligned} \tag{4.6}$$

If we take $\mu_1(t) = \overline{p}(t)$ and $\mu_2(t) = 1$, hence f satisfies (H4), and similarly we can show that I_k satisfy (H5). All conditions of Theorem 3.7 are now fulfilled, so we deduce that the system (4.1) has a mild solution on $(-\infty, T]$.

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