# FIXED POINT THEORY ON EXTENSION-TYPE SPACES AND ESSENTIAL MAPS ON TOPOLOGICAL SPACES

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We present several new fixed point results for admissible self-maps in extension-type spaces. We also discuss a continuation-type theorem for maps between topological spaces.

## 1. Introduction

In Section 2, we begin by presenting most of the up-to-date results in the literature [3, 5, 6, 7, 8, 12] concerning fixed point theory in extension-type spaces. These results are then used to obtain a number of new fixed point theorems, one concerning approximate neighborhood extension spaces and another concerning inward-type maps in extension-type spaces. Our first result was motivated by ideas in [12] whereas the second result is based on an argument of Ben-El-Mechaiekh and Kryszewski [9]. Also in Section 2 we present a new continuation theorem for maps defined between Hausdorff topological spaces, and our theorem improves results in [3].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose $X$ and $Y$ are topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of $Y$) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \operatorname{Fix} F \neq \varnothing \ \forall F \in \mathcal{X}(Z,Z)\}, \tag{1.1}$$

where $\operatorname{Fix} F$ denotes the set of fixed points of $F$.

The class $\mathcal{A}$ of maps is defined by the following properties:

(i) $\mathcal{A}$ contains the class $\mathcal{C}$ of single-valued continuous functions;
(ii) each $F \in \mathcal{A}_c$ is upper semicontinuous and closed valued;
(iii) $B^n \in \mathcal{F}(\mathcal{A}_c)$ for all $n \in \{1,2,\ldots\}$; here $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

*Remark 1.1.* The class $\mathcal{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [7]. It includes the class of maps $\mathcal{U}$ of Park ($\mathcal{U}$ is the class of maps defined by (i), (iii), and (iv) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathcal{A}_c$ is compact


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valued, the classes $\mathcal{A}$ and $\mathcal{U}$ coincide and this is what occurs in Section 2 since our maps will be compact.

The following result can be found in [7, Proposition 2.2] (see also [11, page 286] for a special case).

THEOREM 1.2. *The Hilbert cube* $I^\infty$ *(subset of* $l^2$ *consisting of points* $(x_1,x_2,\dots)$ *with* $|x_i| \le 1/2^i$ *for all* $i$*) and the Tychonoff cube* $T$ *(Cartesian product of copies of the unit interval) are in* $\mathcal{F}(\mathcal{A}_c)$.

We next consider the class $\mathcal{U}_c^\kappa(X,Y)$ (resp., $\mathcal{A}_c^\kappa(X,Y)$) of maps $F : X \to 2^Y$ such that for each $F$ and each nonempty compact subset $K$ of $X$, there exists a map $G \in \mathcal{U}_c(K,Y)$ (resp., $G \in \mathcal{A}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

THEOREM 1.3. *The Hilbert cube* $I^\infty$ *and the Tychonoff cube* $T$ *are in* $\mathcal{F}(\mathcal{A}_c^\kappa)$ *(resp.,* $\mathcal{F}(\mathcal{U}_c^\kappa)$*).*

*Proof.* Let $F \in \mathcal{A}_c^\kappa(I^\infty,I^\infty)$. We must show that $\operatorname{Fix} F \neq \emptyset$. Now, by definition, there exists $G \in \mathcal{A}_c(I^\infty,I^\infty)$ with $G(x) \subseteq F(x)$ for all $x \in I^\infty$, so Theorem 1.2 guarantees that there exists $x \in I^\infty$ with $x \in Gx$. In particular, $x \in Fx$ so $\operatorname{Fix} F \neq \emptyset$. Thus $I^\infty \in \mathcal{F}(\mathcal{A}_c^\kappa)$. □

Notice that $\mathcal{U}_c^\kappa$ is closed under compositions. To see this, let $X$, $Y$, and $Z$ be topological spaces, $F_1 \in \mathcal{U}_c^\kappa(X,Y)$, $F_2 \in \mathcal{U}_c^\kappa(Y,Z)$, and $K$ a nonempty compact subset of $X$. Now there exists $G_1 \in \mathcal{U}_c(K,Y)$ with $G_1(x) \subseteq F_1(x)$ for all $x \in K$. Also [4, page 464] guarantees that $G_1(K)$ is compact so there exists $G_2 \in \mathcal{U}_c^\kappa(G_1(K),Z)$ with $G_2(y) \subseteq F_2(y)$ for all $y \in G_1(K)$. As a result,

$$G_2G_1(x) \subseteq F_2G_1(x) \subseteq F_2F_1(x) \quad \forall x \in K \tag{1.2}$$

and $G_2G_1 \in \mathcal{U}_c(X,Z)$.

For a subset $K$ of a topological space $X$, we denote by $\operatorname{Cov}_X(K)$ the set of all coverings of $K$ by open sets of $X$ (usually we write $\operatorname{Cov}(K) = \operatorname{Cov}_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in \operatorname{Cov}(X)$, a point $x \in X$ is said to be an $\alpha$-fixed point of $F$ if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps $F,G : X \to 2^Y$ and $\alpha \in \operatorname{Cov}(Y)$, $F$ and $G$ are said to be $\alpha$-close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$.

The following results can be found in [5, Lemmas 1.2 and 4.7].

THEOREM 1.4. *Let* $X$ *be a regular topological space and* $F : X \to 2^X$ *an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings* $\theta \subseteq \operatorname{Cov}_X(\overline{F(X)})$ *such that* $F$ *has an* $\alpha$*-fixed point for every* $\alpha \in \theta$*. Then* $F$ *has a fixed point.*

THEOREM 1.5. *Let* $T$ *be a Tychonoff cube contained in a Hausdorff topological vector space. Then* $T$ *is a retract of* $\operatorname{span}(T)$.

*Remark 1.6.* From Theorem 1.4 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values, it suffices [6, page 298] to prove the existence of approximate fixed points (since open covers of a compact set $A$

admit refinements of the form $\{U[x] : x \in A\}$ where $U$ is a member of the uniformity [14, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular) [10, page 431] (see also [10, page 434]). Note in Theorem 1.4 if $F$ is compact valued, then the assumption that $X$ is regular can be removed. For convenience in this paper we will apply Theorem 1.4 only when the space is uniform.

## 2. Extension-type spaces

We begin this section by recalling some results we established in [3]. By a space we mean a Hausdorff topological space. Let $Q$ be a class of topological spaces. A space $Y$ is an *extension space* for $Q$ (written $Y \in \mathrm{ES}(Q)$) if for all $X \in Q$ and all $K \subseteq X$ closed in $X$, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Using (i) the fact that every compact space is homeomorphic to a closed subset of the Tychonoff cube and (ii) Theorem 1.3, we established the following result in [3].

Theorem 2.1. *Let $X \in \mathrm{ES}(\text{compact})$ and $F \in \mathfrak{U}_c^{\kappa}(X,X)$ a compact map. Then $F$ has a fixed point.*

*Remark 2.2.* If $X \in \mathrm{AR}$ (an absolute retract as defined in [11]), then of course $X \in \mathrm{ES}(\text{compact})$.

A space $Y$ is an *approximate extension space* for $Q$ (written $Y \in \mathrm{AES}(Q)$) if for all $\alpha \in \mathrm{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in $X$, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is $\alpha$-close to $f_0$.

Theorem 2.3. *Let $X \in \mathrm{AES}(\text{compact})$ be a uniform space and $F \in \mathfrak{U}_c^{\kappa}(X,X)$ a compact upper semicontinuous map with closed values. Then $F$ has a fixed point.*

*Remark 2.4.* This result was established in [3]. However, we excluded some assumptions ($X$ uniform and $F$ upper semicontinuous with closed values) so the proof in [3] has to be adjusted slightly.

*Proof.* Let $\alpha \in \mathrm{Cov}_X(K)$ where $K = \overline{F(X)}$. From Theorem 1.4 (see Remark 1.6), it suffices to show that $F$ has an $\alpha$-fixed point. We know (see [13]) that $K$ can be embedded as a closed subset $K^*$ of $T$; let $s : K \to K^*$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^* \hookrightarrow T$ be inclusions. Next let $\alpha' = \alpha \cup \{X \backslash K\}$ and note that $\alpha'$ is an open covering of $X$. Let the continuous map $h : T \to X$ be such that $h|_{K^*}$ and $s^{-1}$ are $\alpha'$-close (guaranteed since $X \in \mathrm{AES}(\text{compact})$). Then it follows immediately from the definition (note that $\alpha' = \alpha \cup \{X \backslash K\}$) that $hs : K \to X$ and $i : K \to X$ are $\alpha$-close. Let $G = jsFh$ and notice that $G \in \mathfrak{U}_c^{\kappa}(T,T)$. Now Theorem 1.3 guarantees that there exists $x \in T$ with $x \in Gx$. Let $y = h(x)$, and so, from the above, we have $y \in hjsF(y)$, that is, $y = hjs(q)$ for some $q \in F(y)$. Now since $hs$ and $i$ are $\alpha$-close, there exists $U \in \alpha$ with $hs(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = hjs(q) = hs(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so $y \in U$ and $F(y) \cap U \neq \emptyset$ since $q \in F(y)$. As a result, $F$ has an $\alpha$-fixed point. □

*Definition 2.5.* Let $V$ be a uniform space. Then $V$ is *Schauder admissible* if for every compact subset $K$ of $V$ and every covering $\alpha \in \mathrm{Cov}_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to V$ such that

(i) $\pi_\alpha$ and $i : K \hookrightarrow V$ are $\alpha$-close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in \text{AES(compact)}$.

Theorem 2.6. *Let $V$ be a uniform space and Schauder admissible and $F \in \mathfrak{U}_c^\kappa(V,V)$ a compact upper semicontinuous map with closed values. Then $F$ has a fixed point.*

*Proof.* Let $K = \overline{F(X)}$ and let $\alpha \in \text{Cov}_V(K)$. From Theorem 1.4 (see Remark 1.6), it suffices to show that $F$ has an $\alpha$-fixed point. There exists $\pi_\alpha : K \to V$ (as described in Definition 2.5) and a subset $C \subseteq V$ with $C \in \text{AES(compact)}$ such that (here $F_\alpha = \pi_\alpha F$)

$$F_\alpha(V) = \pi_\alpha F(V) \subseteq C. \tag{2.1}$$

Notice that $F_\alpha \in \mathfrak{U}_c^\kappa(C,C)$ is a compact upper semicontinuous map with closed (in fact compact) values. So Theorem 2.3 guarantees that there exists $x \in C$ with $x \in \pi_\alpha F(x)$, that is, $x = \pi_\alpha q$ for some $q \in F(x)$. Now Definition 2.5(i) guarantees that there exists $U \in \alpha$ with $\pi_\alpha(q) \in U$ and $i(q) \in U$, that is, $x \in U$ and $q \in U$. Thus $x \in U$ and $F(x) \cap U \neq \emptyset$ since $q \in F(x)$, so $F$ has an $\alpha$-fixed point. □

A space $Y$ is a *neighborhood extension space* for $Q$ (written $Y \in \text{NES}(Q)$) if for all $X \in Q$, all $K \subseteq X$ closed in $X$, and any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of $f_0$ over a neighborhood $U$ of $K$ in $X$.

Let $X \in \text{NES}(Q)$ and $F \in \mathfrak{U}_c^\kappa(X,X)$ a compact map. Now let $K$, $K^*$, $s$, and $i$ be as in the proof of Theorem 2.3. Let $U$ be an open neighborhood of $K^*$ in $T$ and let $h_U : U \to X$ be a continuous extension of $is^{-1} : K^* \to X$ on $U$ (guaranteed since $X \in \text{NES(compact)}$). Let $j_U : K^* \hookrightarrow U$ be the natural embedding, so $h_U j_U = is^{-1}$. Now consider $\text{span}(T)$ in a Hausdorff locally convex topological vector space containing $T$. Now Theorem 1.5 guarantees that there exists a retraction $r : \text{span}(T) \to T$. Let $i^* : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^* j_U s F h_U r$. Notice that $G \in \mathfrak{U}_c^\kappa(r^{-1}(U), r^{-1}(U))$. We now *assume* that

$$G \in \mathfrak{U}_c^\kappa(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.} \tag{2.2}$$

Now there exists $x \in r^{-1}(U)$ with $x \in Gx$. Let $y = h_U r(x)$, so $y \in h_U r i^* j_U s F(y)$, that is, $y = h_U r i^* j_U s(q)$ for some $q \in F(y)$. Since $h_U(z) = is^{-1}(z)$ for $z \in K^*$, we have

$$h_U r i^* j_U s(q) = (h_U r i^* j_U) s(q) = i(q), \tag{2.3}$$

so $y \in F(y)$.

Theorem 2.7. *Let $X \in \text{NES(compact)}$ and $F \in \mathfrak{U}_c^\kappa(X,X)$ a compact map. Also assume that (2.2) holds with $K$, $K^*$, $s$, $i$, $i^*$, $j_U$, $h_U$, and $r$ as described above. Then $F$ has a fixed point.*

*Remark 2.8.* Theorem 2.7 was also established in [3]. Note that if $F$ is admissible in the sense of Gorniewicz and the Lefschetz set $\Lambda(F) \neq \{0\}$, then we know [11] that (2.2) holds. Note that if $X \in \text{ANR}$ (see [11]), then of course $X \in \text{NES(compact)}$.

A space $Y$ is an *approximate neighborhood extension space* for $Q$ (written $Y \in \text{ANES}(Q)$) if for all $\alpha \in \text{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in $X$, and any continuous function $f_0 : K \to Y$, there exists a neighborhood $U_\alpha$ of $K$ in $X$ and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and $f_0$ are $\alpha$.

Let $X \in \text{ANES(compact)}$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also let $\alpha \in \text{Cov}_X(K)$ where $K = \overline{F(X)}$. To show that $F$ has a fixed point, it suffices (Theorem 1.4 and Remark 1.6) to show that $F$ has an $\alpha$-fixed point. Let $\alpha' = \alpha \cup \{X \backslash K\}$ and let $K^*$, $s$, and $i$ be as in the proof of Theorem 2.3. Since $X \in \text{ANES(compact)}$, there exists an open neighborhood $U_\alpha$ of $K^*$ in $T$ and $f_\alpha : U_\alpha \to X$ a continuous function such that $f_\alpha|_{K^*}$ and $s^{-1}$ are $\alpha'$-close and as a result $f_\alpha s : K \to X$ and $i : K \to X$ are $\alpha$-close. Let $j_{U_\alpha} : K^* \hookrightarrow U_\alpha$ be the natural imbedding. We know (see [5, page 426]) that $U_\alpha \in \text{NES(compact)}$. Also notice that $G_\alpha = j_{U_\alpha} sFf_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha)$ is a compact upper semicontinuous map with closed values. We now *assume* that

$$G_\alpha = j_{U_\alpha} sFf_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha) \text{ has a fixed point for each } \alpha \in \text{Cov}_X\left(\overline{F(X)}\right). \tag{2.4}$$

We still have $\alpha \in \text{Cov}_X(K)$ fixed and we let $x$ be a fixed point of $G_\alpha$. Now let $y_\alpha = f_\alpha(x)$, so $y = f_\alpha j_{U_\alpha} sF(y)$, that is, $y = f_\alpha j_{U_\alpha} s(q)$ for some $q \in F(y)$. Now since $f_\alpha s$ and $i$ are $\alpha$-close, there exists $U \in \alpha$ with $f_\alpha s(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = f_\alpha j_{U_\alpha} s(q) = f_\alpha s(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so

$$y \in U, \quad F(y) \cap U \neq \emptyset \quad \text{since } q \in F(y). \tag{2.5}$$

Theorem 2.9. *Let $X \in \text{ANES(compact)}$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also assume that (2.4) holds with $K$, $s$, $U_\alpha$, $j_{U_\alpha}$, and $f_\alpha$ as described above. Then $F$ has a fixed point.*

Next we present continuation results for multimaps. Let $Y$ be a completely regular topological space and $U$ an open subset of $Y$. We consider a subclass $\mathfrak{D}$ of $\mathcal{U}_c^\kappa$. This subclass must have the following property: for subsets $X_1$, $X_2$, and $X_3$ of Hausdorff topological spaces, if $F \in \mathfrak{D}(X_2, X_3)$ is compact and $f \in \mathscr{C}(X_1, X_2)$, then $F \circ f \in \mathfrak{D}(X_1, X_3)$.

*Definition 2.10.* The map $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ if $F \in \mathfrak{D}(\overline{U}, Y)$ with $F$ compact and $x \notin Fx$ for $x \in \partial U$; here $\overline{U}$ (resp., $\partial U$) denotes the closure (resp., the boundary) of $U$ in $Y$.

*Definition 2.11.* A map $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$ if for every $G \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$ with $x \in Gx$.

Theorem 2.12 (homotopy invariance). *Let $Y$ and $U$ be as above. Suppose $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$ and $H \in \mathfrak{D}(\overline{U} \times [0,1], Y)$ is a closed compact map with $H(x,0) = F(x)$ for $x \in \overline{U}$. Also assume that*

$$x \notin H_t(x) \quad \text{for any } x \in \partial U,\ t \in (0,1] \ \left(H_t(\cdot) = H(\cdot, t)\right). \tag{2.6}$$

*Then $H_1$ has a fixed point in $U$.*

*Proof.* Let

$$B = \left\{x \in \overline{U} : x \in H_t(x) \text{ for some } t \in [0,1]\right\}. \tag{2.7}$$

When $t = 0$, $H_t = F$, and since $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$, there exists $x \in U$ with $x \in Fx$. Thus $B \neq \emptyset$ and note that $B$ is closed, in fact compact (recall that $H$ is a closed, compact map). Notice also that (2.6) implies $B \cap \partial U = \emptyset$. Thus, since $Y$ is completely regular, there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R$ by $R(x) = H(x,\mu(x))$ for $x \in \overline{U}$. Let $j : \overline{U} \to \overline{U} \times [0,1]$ be given by $j(x) = (x,\mu(x))$. Note that $j$ is continuous, so $R = H \circ j \in \mathfrak{D}(\overline{U},Y)$ (see the description of the class $\mathfrak{D}$ before Definition 2.10). In addition, $R$ is compact, and for $x \in \partial U$, we have $R(x) = H_0(x) = F(x)$. As a result, $R \in \mathfrak{D}_{\partial U}(\overline{U},Y)$ with $R|_{\partial U} = F|_{\partial U}$. Now since $F$ is essential in $\mathfrak{D}_{\partial U}(\overline{U},Y)$, there exists $x \in U$ with $x \in R(x)$, that is, $x \in H_{\mu(x)}(x)$. Thus $x \in B$ and so $\mu(x) = 1$. Consequently, $x \in H_1(x)$. □

Next we give an example of an essential map.

THEOREM 2.13 (normalization). *Let $Y$ and $U$ be as above with $0 \in U$. Suppose the following conditions are satisfied:*

*for any map $\theta \in \mathfrak{D}_{\partial U}(\overline{U},Y)$ with $\theta|_{\partial U} = \{0\}$, the map $J$ is in $\mathfrak{U}_c^{\kappa}(Y,Y)$;*

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \in Y \setminus \overline{U}, \end{cases} \tag{2.8}$$

*and*

$$J \in \mathfrak{U}_c^{\kappa}(Y,Y) \text{ has a fixed point.} \tag{2.9}$$

*Then the zero map is essential in $\mathfrak{D}_{\partial U}(\overline{U},Y)$.*

*Remark 2.14.* Note that examples of spaces $Y$ for (2.9) to be true can be found in Theorems 2.1, 2.3, 2.6, 2.7, and 2.9 (notice that $J$ is compact).

*Proof of Theorem 2.13.* Let $\theta \in \mathfrak{D}_{\partial U}(\overline{U},Y)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Define a map $J$ as in (2.8). From (2.8) and (2.9), we know that there exists $x \in Y$ with $x \in J(x)$. Now if $x \notin U$, we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. □

*Remark 2.15.* Other homotopy and essential map results in a topological vector space setting can be found in [1, 2].

To conclude this paper, we discuss inward-type maps for a general class of admissible maps. The proof presented involves minor modifications of an argument due to Ben-El-Mechaiekh and Kryszewski [9]. Let $Y$ be a normed space and $X \subseteq Y$, and consider a subclass $\mathfrak{R}(X,Y)$ of $\mathfrak{U}_c^{\kappa}(X,Y)$. This subclass must have the following properties: (i) if $X \subseteq Z \subseteq Y$ and if $I : X \hookrightarrow Z$ is an inclusion, $t > 0$, and $F \in \mathfrak{R}(X,Y)$ with $(I + tF)(X) \subseteq Z$, then $I + tF \in \mathfrak{U}_c^{\kappa}(X,Z)$, and (ii) each $F \in \mathfrak{R}(X,Y)$ is upper semicontinuous and compact valued.

In our next result we assume that $\Omega$ is a compact $\mathcal{L}$-retract [9], that is,

(A) $\Omega$ is a compact neighborhood retract of a normed space $E = (E,\|\cdot\|)$ and there exist $\beta > 0$, $r : B(\Omega,\beta) \to \Omega$ a retraction, and $L > 0$ such that $\|r(x) - x\| \le Ld(x;\Omega)$ for $x \in B(\Omega,\beta)$.

As a result,

$$\exists \eta > 0, \quad \eta < \frac{\beta}{2} \quad \text{with } \|r(x) - x\| < \eta \ \forall x \in B(\Omega, \eta). \tag{2.10}$$

Theorem 2.16. *Let $E = (E, \|\cdot\|)$ be a normed space and $\Omega$ as in assumption (A), and assume either (i) $\Omega$ is Schauder admissible or (ii) (2.2) holds with $X = \Omega$. In addition, suppose $F \in \mathcal{R}(\Omega, E)$ with*

$$F(x) \subseteq C_\Omega(x) \quad \forall x \in \Omega. \tag{2.11}$$

*Then there exists $x \in \Omega$ with $0 \in Fx$.*

*Remark 2.17.* Here $C_\Omega$ is the Clarke tangent cone, that is,

$$C_\Omega(x) = \{v \in E : c(x, v) = 0\}, \tag{2.12}$$

where

$$c(x, y) = \limsup_{\substack{y \to x,\, y \in \Omega \\ t \downarrow 0}} \frac{d(x + tv; \Omega)}{t}. \tag{2.13}$$

*Remark 2.18.* If $\Omega$ is a compact neighborhood retract, then of course $\Omega \in \text{NES(compact)}$.

*Remark 2.19.* The proof is basically due to Ben-El-Mechaiekh and Kryszewski [9] and is based on [9, Lemma 5.1] (this lemma is a modification of a standard argument in the literature using partitions of unity).

*Proof.* Now [9, Lemma 5.1] (choose $\Psi(x) = \{x \in E : c(x, v) < \delta\}$ ($\delta > 0$ appropriately chosen), $\Phi(x) = co(F(x))$ and apply the argument in [9, page 4176]) implies that there exists $M > 0$ such that for each $x \in K$ and each $y \in Fx$, we have $\|y\| \leq M$. Choose $\tau > 0$ with $M\tau < \eta$ (here $\eta$ is as in (2.10)) and a sequence $(t_n)_{n \in N}$ in $(0, \tau]$ with $t_n \downarrow 0$; here $N = \{1, 2, \ldots\}$. Define a sequence of maps $\psi_n$, $n \in N$, by

$$\psi_n(x) = r(x + t_n F(x)) \quad \text{for } x \in \Omega; \tag{2.14}$$

note that $d(x + t_n y; \Omega) < \eta$ for $x \in \Omega$ and $y \in F(x)$ since $M\tau < \eta$. Fix $n \in N$ and notice that $\psi_n \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ is a compact map (note that $\Omega$ is compact and $\psi_n$ is upper semicontinuous with compact values). Now Theorem 2.6 or Theorem 2.7 guarantees that there exists $x_n \in \Omega$ and $y_n \in Fx_n$ with

$$x_n = r(x_n + t_n y_n). \tag{2.15}$$

Also notice from (2.15) and assumption (A) (note that $M\tau < \eta < \beta/2 < \beta$) that

$$t_n \|y_n\| = \|x_n + t_n y_n - r(x_n + t_n y_n)\| \leq L d(x_n + t_n y_n; \Omega). \tag{2.16}$$

Now $\Omega$ is compact so $F(\Omega)$ is compact, and as a result, there exists a subsequence $S$ of $N$ with $(x_n, y_n) \in \text{Graph}\, F$ and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ in $S$. Of course, since $F$ is upper semicontinuous, we have $\overline{y} \in F(\overline{x})$. Also from (2.11), we have $F(\overline{x}) \subseteq C_\Omega(\overline{x})$ and as a result, $\overline{y} \in F(\overline{x}) \subseteq C_\Omega(\overline{x})$, so $c(\overline{x},\overline{y}) = 0$. Note also that

$$d(x_n + t_n y_n;\Omega) \leq d(x_n + t_n\overline{y};\Omega) + t_n||y_n - \overline{y}|| \tag{2.17}$$

and this together with (2.16) yields

$$||\overline{y}|| = \limsup_{n\to\infty}||y_n|| \leq \limsup\left(\frac{Ld(x_n + t_n\overline{y};\Omega)}{t_n} + ||y_n - \overline{y}||\right) = c(\overline{x},\overline{y}) = 0, \tag{2.18}$$

so $0 \in F(\overline{x})$. □

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