SEVERAL FIXED POINT THEOREMS CONCERNING τ-DISTANCE

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Using the notion of τ-distance, we prove several fixed point theorems, which are generalizations of fixed point theorems by Kannan, Meir-Keeler, Edelstein, and Nadler. We also discuss the properties of τ-distance.

1. Introduction

In 1922, Banach proved the following famous fixed point theorem [1]. Let (X,d) be a complete metric space. Let T be a contractive mapping on X, that is, there exists $r \in [0,1)$ satisfying

$$d(Tx,Ty) \le rd(x,y) \tag{1.1}$$

for all $x,y \in X$. Then there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle, is a forceful tool in nonlinear analysis. This principle has many applications and is extended by several authors: Caristi [2], Edelstein [5], Ekeland [6, 7], Meir and Keeler [14], Nadler [15], and others. These theorems are also extended; see [4, 9, 10, 13, 23, 25, 26, 27] and others. In [20], the author introduced the notion of τ-distance and extended the Banach contraction principle, Caristi's fixed point theorem, and Ekeland's ε-variational principle. In 1969, Kannan proved the following fixed point theorem [12]. Let (X,d) be a complete metric space. Let T be a Kannan mapping on X, that is, there exists $\alpha \in [0,1/2)$ such that

$$d(Tx,Ty) \le \alpha\big(d(Tx,x)+d(Ty,y)\big) \tag{1.2}$$

for all $x,y \in X$. Then there exists a unique fixed point $x_0 \in X$ of T. We note that Kannan's fixed point theorem is not an extension of the Banach contraction principle. We also know that a metric space X is complete if and only if every Kannan mapping has a fixed point, while there exists a metric space X such that X is not complete and every contractive mapping on X has a fixed point; see [3, 17].

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In this paper, using the notion of τ-distance, we prove several fixed point theorems, which are generalizations of fixed point theorems by Kannan, Meir-Keeler, Edelstein, and Nadler. We also discuss the properties of τ-distance.

2. τ-distance

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers. In this section, we discuss some properties of τ-distance. Let (X,d) be a metric space. Then a function p from $X \times X$ into $[0,\infty)$ is called a *τ-distance* on X [20] if there exists a function η from $X \times [0,\infty)$ into $[0,\infty)$ and the following are satisfied:

(τ1) $p(x,z) \le p(x,y) + p(y,z)$ for all $x,y,z \in X$;
(τ2) $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $x \in X$ and $t \in [0,\infty)$, and η is concave and continuous in its second variable;
(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n,x_m)) : m \ge n\} = 0$ imply $p(w,x) \le \liminf_n p(w,x_n)$ for all $w \in X$;
(τ4) $\lim_n \sup\{p(x_n,y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$;
(τ5) $\lim_n \eta(z_n, p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$.

We may replace (τ2) by the following (τ2)′ (see [20]):

(τ2)′ $\inf\{\eta(x,t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

The metric d is a τ-distance on X. Many useful examples are stated in [11, 16, 18, 19, 20, 21, 22, 24]. It is very meaningful that one τ-distance generates other τ-distances. In the sequel, we discuss this fact.

Proposition 2.1. *Let (X,d) be a metric space. Let p be a τ-distance on X and let η be a function satisfying (τ2)′, (τ3), (τ4), and (τ5). Let q be a function from $X \times X$ into $[0,\infty)$ satisfying (τ1)$_q$. Suppose that*

(i) *there exists $c > 0$ such that $\min\{p(x,y),c\} \le q(x,y)$ for $x,y \in X$,*
(ii) *$\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n,q(z_n,x_m)) : m \ge n\} = 0$ imply $q(w,x) \le \liminf_n q(w,x_n)$ for $w \in X$.*

Then q is also a τ-distance on X.

Proof. We put

$$\theta(x,t) = t + \eta(x,t) \tag{2.1}$$

for $x \in X$ and $t \in [0,\infty)$. Note that $\eta(x,t) \le \theta(x,t)$ for all $x \in X$ and $t \in [0,\infty)$. Then, by the assumption, (τ1)$_q$, (τ2)'$_\theta$, and (τ3)$_{q,\theta}$ hold. We assume that $\lim_n \sup\{q(x_n,y_m) : m \ge n\} = 0$ and $\lim_n \theta(x_n,t_n) = 0$. Then $\lim_n \sup\{p(x_n,y_m) : m \ge n\} = 0$ and $\lim_n t_n = \lim_n \eta(x_n,t_n) = 0$ clearly hold. From (τ4), we have $\lim_n \eta(y_n,t_n) = 0$ and hence $\lim_n \theta(y_n,t_n) = 0$. Therefore, we have shown (τ4)$_{q,\theta}$. We assume that $\lim_n \theta(z_n,q(z_n,x_n)) = 0$ and $\lim_n \theta(z_n,q(z_n,y_n)) = 0$. By the definition of θ, we have $\lim_n \eta(z_n,q(z_n,x_n)) = 0$ and $\lim_n q(z_n,x_n) = 0$. So, by the assumption, $\lim_n \eta(z_n,p(z_n,x_n)) = 0$ holds. We can similarly prove $\lim_n \eta(z_n,p(z_n,y_n)) = 0$. Therefore, from ($\tau$5), $\lim_n d(x_n,y_n) = 0$. Hence, we have shown (τ5)$_{q,\theta}$. This completes the proof. □

As a direct consequence of Proposition 2.1, we obtain the following proposition.

Proposition 2.2. *Let p be a τ-distance on a metric space X. Let q be a function from $X \times X$ into $[0,\infty)$ satisfying $(\tau 1)_q$. Suppose that*

(i) *there exists $c > 0$ such that $\min\{p(x,y),c\} \le q(x,y)$ for $x,y \in X$,*

(ii) *for every convergent sequence $\{x_n\}$ with limit x satisfying $p(w,x) \le \liminf_n p(w,x_n)$ for all $w \in X$, $q(w,x) \le \liminf_n q(w,x_n)$ holds for all $w \in X$.*

Then q is also a τ-distance on X.

Using the above proposition, we obtain the following one which is used in the proof of generalized Kannan's fixed point theorem.

Proposition 2.3. *Let p be a τ-distance on a metric space X and let α be a function from X into $[0,\infty)$. Then two functions q_1 and q_2 from $X \times X$ into $[0,\infty)$, defined by*

(i) *$q_1(x,y) = \max\{\alpha(x), p(x,y)\}$ for $x,y \in X$,*

(ii) *$q_2(x,y) = \alpha(x) + p(x,y)$ for $x,y \in X$,*

are τ-distances on X.

Proof. We have

$$\begin{aligned} q_1(x,z) &= \max\{\alpha(x), p(x,z)\} \\ &\le \max\{\alpha(x)+\alpha(y), p(x,y)+p(y,z)\} \\ &\le q_1(x,y) + q_1(y,z), \\ q_2(x,z) &= \alpha(x) + p(x,z) \\ &\le \alpha(x)+\alpha(y)+p(x,y)+p(y,z) \\ &= q_2(x,y)+q_2(y,z), \end{aligned} \tag{2.2}$$

for all $x,y,z \in X$. We note that

$$p(x,y) \le q_1(x,y) \le q_2(x,y) \tag{2.3}$$

for all $x,y \in X$. We assume that a sequence $\{x_n\}$ satisfies $\lim_n x_n = x$ and $p(w,x) \le \liminf_n p(w,x_n)$ for all $w \in X$. Then it is clear that $q_1(w,x) \le \liminf_n q_1(w,x_n)$ and $q_2(w,x) \le \liminf_n q_2(w,x_n)$ for all $w \in X$. By Proposition 2.2, q_1 and q_2 are τ-distances on X. This completes the proof. □

Let (X,d) be a metric space and let p be a τ-distance on X. Then a sequence $\{x_n\}$ in X is called *p-Cauchy* [20] if there exist a function η from $X \times [0,\infty)$ into $[0,\infty)$ satisfying $(\tau 2)$–$(\tau 5)$ and a sequence $\{z_n\}$ in X such that $\lim_n \sup\{\eta(z_n, p(z_n,x_m)) : m \ge n\} = 0$. The following lemmas are very useful in the proofs of fixed point theorems in Section 3.

Lemma 2.4 [20]. *Let (X,d) be a metric space and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_n \sup\{p(x_n,y_m) : m \ge n\} = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n,y_n) = 0$.*

Lemma 2.5 [20]. *Let (X,d) be a metric space and let p be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z,x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a p-Cauchy sequence.*

Moreover, if a sequence $\{y_n\}$ *in* X *also satisfies* $\lim_n p(z,y_n)=0$*, then* $\lim_n d(x_n,y_n)=0$*. In particular, for* $x,y,z\in X$*,* $p(z,x)=0$ *and* $p(z,y)=0$ *imply* $x=y$*.*

Lemma 2.6 [20]. *Let* (X,d) *be a metric space and let* p *be a* τ*-distance on* X*. If a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n \sup\{p(x_n,x_m) : m>n\}=0$*, then* $\{x_n\}$ *is a* p*-Cauchy sequence. Moreover, if a sequence* $\{y_n\}$ *in* X *satisfies* $\lim_n p(x_n,y_n)=0$*, then* $\{y_n\}$ *is also a* p*-Cauchy sequence and* $\lim_n d(x_n,y_n)=0$*.*

3. Fixed point theorems

In this section, we prove several fixed point theorems in complete metric spaces. In [20], the following theorem connected with Hicks-Rhoades theorem [8] was proved and used in the proofs of generalizations of the Banach contraction principle, Caristi's fixed point theorem, and so on. In this paper, this theorem is used in the proof of a generalization of Kannan's fixed point theorem.

Theorem 3.1 [20]. *Let* X *be a complete metric space and let* T *be a mapping on* X*. Suppose that there exist a* τ*-distance* p *on* X *and* $r\in[0,1)$ *such that* $p(Tx,T^2x)\le r\,p(x,Tx)$ *for all* $x\in X$*. Assume that either of the following holds:*

(i) *if* $\lim_n \sup\{p(x_n,x_m) : m>n\}=0$, $\lim_n p(x_n,Tx_n)=0$*, and* $\lim_n p(x_n,y)=0$*, then* $Ty=y$*;*
(ii) *if* $\{x_n\}$ *and* $\{Tx_n\}$ *converge to* y*, then* $Ty=y$*;*
(iii) T *is continuous.*

Then there exists $x_0\in X$ *such that* $Tx_0=x_0$*. Moreover, if* $Tz=z$*, then* $p(z,z)=0$*.*

As a direct consequence, we obtain the following theorem.

Theorem 3.2. *Let* X *be a complete metric space and let* p *be a* τ*-distance on* X*. Let* T *be a mapping on* X*. Suppose that there exists* $r\in[0,1)$ *such that either (a) or (b) holds:*

(a) $\max\{p(T^2x,Tx),p(Tx,T^2x)\}\le r\max\{p(Tx,x),p(x,Tx)\}$ *for all* $x\in X$*;*
(b) $p(T^2x,Tx)+p(Tx,T^2x)\le r\,p(Tx,x)+r\,p(x,Tx)$ *for all* $x\in X$*.*

Further, assume that either of the following holds:

(i) *if* $\lim_n \sup\{p(x_n,x_m) : m>n\}=0$, $\lim_n p(Tx_n,x_n)=0$, $\lim_n p(x_n,Tx_n)=0$*, and* $\lim_n p(x_n,y)=0$*, then* $Ty=y$*;*
(ii) *if* $\{x_n\}$ *and* $\{Tx_n\}$ *converge to* y*, then* $Ty=y$*;*
(iii) T *is continuous.*

Then there exists $x_0\in X$ *such that* $Tx_0=x_0$*. Moreover, if* $Tz=z$*, then* $p(z,z)=0$*.*

Proof. In the case of (a), we define a function q by $q(x,y)=\max\{p(Tx,x),p(x,y)\}$. In the case of (b), we define a function q by $q(x,y)=p(Tx,x)+p(x,y)$. By Proposition 2.3, q is a τ-distance on X. In both cases, we have

$$q(Tx,T^2x)\le rq(x,Tx) \tag{3.1}$$

for all $x \in X$. Conditions (ii) and (iii) are not connected with τ-distance p. In the case of (i), since

$$p(x,y) \le q(x,y), \qquad p(Tx,x) \le q(x,Tx), \tag{3.2}$$

for all $x, y \in X$, T has a fixed point in X by Theorem 3.1. If $Tz = z$, then $q(z,z) = 0$, and hence $p(z,z) = 0$. This completes the proof. □

We now prove a generalization of Kannan's fixed point theorem [12]. Let X be a metric space, let p be a τ-distance on X, and let T be a mapping on X. Then T is called a *Kannan mapping* with respect to p if there exists $\alpha \in [0,1/2)$ such that either (a) or (b) holds:

(a) $p(Tx,Ty) \le \alpha p(Tx,x) + \alpha p(Ty,y)$ for all $x, y \in X$;
(b) $p(Tx,Ty) \le \alpha p(Tx,x) + \alpha p(y,Ty)$ for all $x, y \in X$.

Theorem 3.3. *Let (X,d) be a complete metric space, let p be a τ-distance on X, and let T be a Kannan mapping on X with respect to p. Then T has a unique fixed point $x_0 \in X$. Further, such x_0 satisfies $p(x_0,x_0) = 0$.*

Proof. In the case of (a), there exists $\alpha \in [0,1/2)$ such that $p(Tx,Ty) \le \alpha p(Tx,x) + \alpha p(Ty,y)$ for $x, y \in X$. Since

$$p(T^2x,Tx) \le \alpha p(T^2x,Tx) + \alpha p(Tx,x), \tag{3.3}$$

we have

$$p(T^2x,Tx) \le \frac{\alpha}{1-\alpha} p(Tx,x) \le p(Tx,x) \tag{3.4}$$

for $x \in X$. Putting $r = 2\alpha \in [0,1)$, we have

$$\begin{aligned} \max\{p(T^2x,Tx), p(Tx,T^2x)\} &\le \alpha p(T^2x,Tx) + \alpha p(Tx,x) \\ &\le r p(Tx,x) \\ &\le r \max\{p(Tx,x), p(x,Tx)\} \end{aligned} \tag{3.5}$$

for all $x \in X$. We assume $\lim_n \sup\{p(x_n,x_m) : m > n\} = 0$, $\lim_n p(Tx_n,x_n) = 0$, $\lim_n p(x_n,Tx_n) = 0$, and $\lim_n p(x_n,y) = 0$. Then, by Lemma 2.6, $\{x_n\}$ and $\{Tx_n\}$ are p-Cauchy and

$$\lim_{n\to\infty} d(x_n,y) = \lim_{n\to\infty} d(Tx_n,y) = 0. \tag{3.6}$$

Now we have

$$\begin{aligned} p(Ty,y) &\le \liminf_{n\to\infty} p(Ty,Tx_n) \\ &\le \liminf_{n\to\infty} \{\alpha p(Ty,y) + \alpha p(Tx_n,x_n)\} \\ &= \alpha p(Ty,y), \end{aligned} \tag{3.7}$$

and hence $p(Ty,y) = 0$. Since $p(T^2y,Ty) \le p(Ty,y) = 0$ and $p(T^2y,y) \le p(T^2y,Ty) + p(Ty,y) = 0$, we have $Ty = y$ by Lemma 2.5. Therefore, by Theorem 3.2, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0,x_0) = 0$. Further, a fixed point of T is unique. In fact, if $Tz = z$, then $p(z,z) = 0$ by Theorem 3.2. So we have

$$\begin{aligned} p(x_0,z) &= p(Tx_0,Tz) \le \alpha p(Tx_0,x_0) + \alpha p(Tz,z) \\ &= \alpha p(x_0,x_0) + \alpha p(z,z) = 0. \end{aligned} \tag{3.8}$$

By Lemma 2.5 again, we have $x_0 = z$. In the case of (b), there exists $\alpha \in [0,1/2)$ such that $p(Tx,Ty) \le \alpha p(Tx,x) + \alpha p(y,Ty)$ for $x,y \in X$. Then, putting $r = \alpha/(1-\alpha) \in [0,1)$, we have $p(Tx,T^2x) \le rp(Tx,x)$ and $p(T^2x,Tx) \le rp(x,Tx)$ for all $x \in X$. So,

$$p(T^2x,Tx) + p(Tx,T^2x) \le rp(Tx,x) + rp(x,Tx) \tag{3.9}$$

for all $x \in X$. We assume $\lim_n \sup\{p(x_n,x_m) : m > n\} = 0$, $\lim_n p(Tx_n,x_n) = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n,y) = 0$. Then $\{x_n\}$ and $\{Tx_n\}$ are p-Cauchy and $\lim_n d(x_n,y) = \lim_n d(Tx_n,y) = 0$. So we have

$$\begin{aligned} p(Ty,y) &\le \liminf_{n\to\infty} p(Ty,Tx_n) \\ &\le \liminf_{n\to\infty} \{\alpha p(Ty,y) + \alpha p(x_n,Tx_n)\} \\ &= \alpha p(Ty,y), \end{aligned} \tag{3.10}$$

and hence $p(Ty,y) = 0$. Since $p(Ty,T^2y) \le rp(Ty,y) = 0$, we have $y = T^2y$ by Lemma 2.5. So, $p(y,Ty) = p(T^2y,Ty) \le rp(y,Ty)$, and hence $p(y,Ty) = 0$. We also have $p(y,y) \le p(y,Ty) + p(Ty,y) = 0$. So we have $Ty = y$ by Lemma 2.5. Therefore, by Theorem 3.2, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0,x_0) = 0$. As in the case of (a), we obtain that a fixed point of T is unique. □

In general, τ-distance p does not satisfy $p(x,y) = p(y,x)$. So conditions (a) and (b) in the definition of Kannan mappings differ from conditions (c) and (d) in the following theorem. Indeed, there exists a mapping T on a complete metric space X such that (c) and (d) hold, and T has no fixed points; see [19]. However, under the assumption that T is continuous, T has a fixed point.

Theorem 3.4. *Let X be a complete metric space and let T be a continuous mapping on X. Suppose that there exist a τ-distance p on X and $\alpha \in [0,1/2)$ such that either (c) or (d) holds:*

(c) $p(Tx,Ty) \le \alpha p(x,Tx) + \alpha p(Ty,y)$ *for all* $x,y \in X$;
(d) $p(Tx,Ty) \le \alpha p(x,Tx) + \alpha p(y,Ty)$ *for all* $x,y \in X$.

Then there exists a unique fixed point $x_0 \in X$ of T. Moreover, such x_0 satisfies $p(x_0,x_0) = 0$.

Proof. In the case of (c), putting $r = \alpha/(1-\alpha) \in [0,1)$, from $p(Tx,T^2x) \le \alpha p(x,Tx) + \alpha p(T^2x,Tx)$ and $p(T^2x,Tx) \le \alpha p(Tx,T^2x) + \alpha p(Tx,x)$, we have

$$p(T^2x,Tx) + p(Tx,T^2x) \le rp(Tx,x) + rp(x,Tx) \tag{3.11}$$

for all $x \in X$. So, by Theorem 3.2, we prove the desired result. In the case of (d), we have $p(Tx, T^2x) \leq rp(x, Tx)$ for all $x \in X$. Therefore, by Theorem 3.1, we prove the desired result. This completes the proof. □

We next prove a generalization of Meir and Keeler's fixed point theorem [14].

Theorem 3.5. *Let X be a complete metric space, let p be a τ-distance on X, and let T be a mapping on X. Suppose that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in X$, $p(x, y) < \varepsilon + \delta$ implies $p(Tx, Ty) < \varepsilon$. Then T has a unique fixed point x_0 in X. Further, such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. We first show $p(Tx, Ty) \leq p(x, y)$ for all $x, y \in X$. For an arbitrary $\lambda > 0$, there exists $\delta > 0$ such that for every $z, w \in X$, $p(z, w) < p(x, y) + \lambda + \delta$ implies $p(Tz, Tw) < p(x, y) + \lambda$. Since $p(x, y) < p(x, y) + \lambda + \delta$, we have $p(Tx, Ty) < p(x, y) + \lambda$. Since $\lambda > 0$ is arbitrary, we obtain $p(Tx, Ty) \leq p(x, y)$. We next show

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0 \quad \forall x, y \in X. \tag{3.12}$$

In fact, $\{p(T^n x, T^n y)\}$ is nonincreasing and hence converges to some real number r. We assume $r > 0$. Then there exists $\delta > 0$ such that for every $z, w \in X$, $p(z, w) < r + \delta$ implies $p(Tz, Tw) < r$. For such δ, we can choose $m \in \mathbb{N}$ such that $p(T^m x, T^m y) < r + \delta$. So we have $p(T^{m+1}x, T^{m+1}y) < r$. This is a contradiction, and hence (3.12) holds. Let $u \in X$ and put $u_n = T^n u$ for every $n \in \mathbb{N}$. From (3.12), we have $\lim_n p(u_n, u_{n+1}) = 0$. We will show that

$$\lim_{n\to\infty} \sup_{m>n} p(u_n, u_m) = 0. \tag{3.13}$$

Let $\varepsilon > 0$ be arbitrary. Then, without loss of generality, there exists $\delta \in (0, \varepsilon)$ such that for every $z, w \in X$, $p(z, w) < \varepsilon + \delta$ implies $p(Tz, Tw) < \varepsilon$. For such δ, there exists $n_0 \in \mathbb{N}$ such that $p(u_n, u_{n+1}) < \delta$ for every $n \geq n_0$. Assume that there exists $m > \ell \geq n_0$ such that $p(u_\ell, u_m) > 2\varepsilon$. Since

$$p(u_\ell, u_{\ell+1}) < \varepsilon + \delta < p(u_\ell, u_m), \tag{3.14}$$

there exists $k \in \mathbb{N}$ with $\ell < k < m$ such that

$$p(u_\ell, u_k) < \varepsilon + \delta \leq p(u_\ell, u_{k+1}). \tag{3.15}$$

Then, since $p(u_\ell, u_k) < \varepsilon + \delta$, we have $p(u_{\ell+1}, u_{k+1}) < \varepsilon$. On the other hand, we have

$$p(u_\ell, u_{k+1}) \leq p(u_\ell, u_{\ell+1}) + p(u_{\ell+1}, u_{k+1}) < \delta + \varepsilon. \tag{3.16}$$

This is a contradiction. Therefore, $m > n \geq n_0$ implies $p(u_n, u_m) \leq 2\varepsilon$, and hence (3.13) holds. By Lemma 2.6, $\{u_n\}$ is p-Cauchy. So, $\{u_n\}$ is also a Cauchy sequence by Lemma 2.4.

Hence there exists $x_0 \in X$ such that $\{u_n\}$ converges to x_0. From (τ3), we have

$$\begin{aligned}\limsup_{n\to\infty} p(u_n, Tx_0) &\le \limsup_{n\to\infty} p(u_{n-1}, x_0) \\ &= \limsup_{n\to\infty} p(u_n, x_0) \\ &\le \limsup_{n\to\infty}\liminf_{m\to\infty} p(u_n, u_m) \\ &\le \lim_{n\to\infty}\sup_{m>n} p(u_n, u_m) = 0.\end{aligned} \tag{3.17}$$

By Lemma 2.6 again, $\{u_n\}$ converges to Tx_0, and hence $Tx_0 = x_0$. From (3.12), we obtain

$$p(x_0, x_0) = \lim_{n\to\infty} p(T^n x_0, T^n x_0) = 0. \tag{3.18}$$

If $z = Tz$, then

$$p(x_0, z) = \lim_{n\to\infty} p(T^n x_0, T^n z) = 0. \tag{3.19}$$

So, from Lemma 2.5, $x_0 = z$. Therefore, a fixed point of T is unique. This completes the proof. □

Let X be a metric space and let p be a τ-distance on X. For $\varepsilon \in (0,\infty]$, X is called *ε-chainable* with respect to p if, for each $(x,y) \in X \times X$, there exists a finite sequence $\{u_0, u_1, u_2, \dots, u_\ell\}$ in X such that $u_0 = x$, $u_\ell = y$, and $p(u_{i-1}, u_i) < \varepsilon$ for $i = 1,2,\dots,\ell$. We will prove a generalization of Edelstein's fixed point theorem [5].

Theorem 3.6. *Let X be a complete metric space. Suppose that X is ε-chainable with respect to p for some $\varepsilon \in (0,\infty]$ and for some τ-distance p on X. Let T be a mapping on X. Suppose that there exists $r \in [0,1)$ such that $p(Tx, Ty) \le rp(x,y)$ for all $x, y \in X$ with $p(x,y) < \varepsilon$. Then there exists a unique fixed point $x_0 \in X$ of T. Further, such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. We first show

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0 \tag{3.20}$$

for all $x, y \in X$. Let $x, y \in X$ be fixed. Then there exist $u_0, u_1, u_2, \dots, u_\ell \in X$ such that $u_0 = x$, $u_\ell = y$, and $p(u_{i-1}, u_i) < \varepsilon$ for $i = 1,2,\dots,\ell$. Since $p(u_{i-1}, u_i) < \varepsilon$, we have $p(Tu_{i-1}, Tu_i) \le rp(u_{i-1}, u_i) < \varepsilon$. Thus

$$p(T^n u_{i-1}, T^n u_i) \le rp(T^{n-1} u_{i-1}, T^{n-1} u_i) \le \cdots \le r^n p(u_{i-1}, u_i). \tag{3.21}$$

Therefore

$$\begin{aligned}\limsup_{n\to\infty} p(T^n x, T^n y) &\le \limsup_{n\to\infty} \sum_{i=1}^{\ell} p(T^n u_{i-1}, T^n u_i) \\ &\le \lim_{n\to\infty} \sum_{i=1}^{\ell} r^n p(u_{i-1}, u_i) = 0.\end{aligned} \tag{3.22}$$

We have shown (3.20). Let $x \in X$ be fixed. From (3.20), there exists $n_0 \in \mathbb{N}$ such that

$$p(T^n x, T^{n+1}x) < \varepsilon \tag{3.23}$$

for $n \geq n_0$. Then, for $m > n \geq n_0$, we have

$$\begin{aligned} p(T^n x, T^m x) &\leq \sum_{k=n}^{m-1} p(T^k x, T^{k+1}x) \\ &\leq \sum_{k=n}^{m-1} r^{k-n_0} p(T^{n_0}x, T^{n_0+1}x) \\ &\leq \frac{r^{n-n_0}}{1-r} p(T^{n_0}x, T^{n_0+1}x). \end{aligned} \tag{3.24}$$

Hence, $\lim_n \sup\{p(T^n x, T^m x) : m > n\} = 0$. By Lemma 2.6, $\{T^n x\}$ is p-Cauchy. By Lemma 2.4, $\{T^n x\}$ is a Cauchy sequence. So, $\{T^n x\}$ converges to some $x_0 \in X$. Since

$$\begin{aligned} \limsup_{n\to\infty} p(T^n x, x_0) &\leq \limsup_{n\to\infty} \liminf_{m\to\infty} p(T^n x, T^m x) \\ &\leq \lim_{n\to\infty} \sup_{m>n} p(T^n x, T^m x) = 0, \end{aligned} \tag{3.25}$$

we have

$$\limsup_{n\to\infty} p(T^n x, Tx_0) \leq \lim_{n\to\infty} r p(T^{n-1}x, x_0) = 0. \tag{3.26}$$

By Lemma 2.6, we obtain $Tx_0 = x_0$. If z is a fixed point of T, then we have

$$p(x_0, z) = \lim_{n\to\infty} p(T^n x_0, T^n z) = 0 \tag{3.27}$$

from (3.20). We also have $p(x_0, x_0) = 0$. Therefore, $z = x_0$ by Lemma 2.5. This completes the proof. □

Let X be a metric space and let p be a τ-distance on X. Then, a set-valued mapping T from X into itself is called *p-contractive* if Tx is nonempty for each $x \in X$ and there exists $r \in [0,1)$ such that

$$Q(Tx, Ty) \leq r p(x, y) \tag{3.28}$$

for all $x, y \in X$, where

$$Q(A, B) = \sup_{a\in A} \inf_{b\in B} p(a, b). \tag{3.29}$$

The following theorem is a generalization of Nadler's fixed point theorem [15].

Theorem 3.7. *Let (X,d) be a complete metric space and let p be a τ-distance on X. Let T be a set-valued p-contractive mapping from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$ and $p(x_0, x_0) = 0$.*

Remark 3.8. $z \in Tz$ does not necessarily imply $p(z,z) = 0$; see Example 3.9.

Proof. By the assumption, there exists $r' \in [0,1)$ such that $Q(Tx,Ty) \le r'p(x,y)$ for all $x,y \in X$. Put $r = (1+r')/2 \in [0,1)$ and fix $x,y \in X$ and $u \in Tx$. Then, in the case of $p(x,y) > 0$, there is $v \in Ty$ satisfying $p(u,v) \le rp(x,y)$. In the case of $p(x,y) = 0$, we have $Q(Tx,Ty) = 0$. Then there exists a sequence $\{v_n\}$ in Ty satisfying $\lim_n p(u,v_n) = 0$. By Lemma 2.5, $\{v_n\}$ is p-Cauchy, and hence $\{v_n\}$ is Cauchy. Since X is complete and Ty is closed, $\{v_n\}$ converges to some point $v \in Ty$. Then we have

$$p(u,v) \le \lim_{n\to\infty} p(u,v_n) = 0 = rp(x,y). \tag{3.30}$$

Hence, we have shown that for any $x,y \in X$ and $u \in Tx$, there is $v \in Ty$ with $p(u,v) \le rp(x,y)$. Fix $u_0 \in X$ and $u_1 \in Tu_0$. Then there exists $u_2 \in Tu_1$ such that $p(u_1,u_2) \le rp(u_0,u_1)$. Thus, we have a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and $p(u_n,u_{n+1}) \le rp(u_{n-1},u_n)$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$p(u_n,u_{n+1}) \le rp(u_{n-1},u_n) \le r^2p(u_{n-2},u_{n-1}) \le \cdots \le r^np(u_0,u_1), \tag{3.31}$$

and hence, for any $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} p(u_n,u_m) &\le p(u_n,u_{n+1}) + p(u_{n+1},u_{n+2}) + \cdots + p(u_{m-1},u_m) \\ &\le r^np(u_0,u_1) + r^{n+1}p(u_0,u_1) + \cdots + r^{m-1}p(u_0,u_1) \\ &\le \frac{r^n}{1-r}p(u_0,u_1). \end{aligned} \tag{3.32}$$

By Lemma 2.6, $\{u_n\}$ is a p-Cauchy sequence. Hence, by Lemma 2.4, $\{u_n\}$ is a Cauchy sequence. So, $\{u_n\}$ converges to some point $v_0 \in X$. For $n \in \mathbb{N}$, from $(\tau 3)$, we have

$$p(u_n,v_0) \le \liminf_{m\to\infty} p(u_n,u_m) \le \frac{r^n}{1-r}p(u_0,u_1). \tag{3.33}$$

By hypothesis, we also have $w_n \in Tv_0$ such that $p(u_n,w_n) \le rp(u_{n-1},v_0)$ for $n \in \mathbb{N}$. So we have

$$\begin{aligned} \limsup_{n\to\infty} p(u_n,w_n) &\le \limsup_{n\to\infty} rp(u_{n-1},v_0) \\ &\le \lim_{n\to\infty} \frac{r^n}{1-r}p(u_0,u_1) = 0. \end{aligned} \tag{3.34}$$

By Lemma 2.6, $\{w_n\}$ converges to v_0. Since Tv_0 is closed, we have $v_0 \in Tv_0$. For such v_0, there exists $v_1 \in Tv_0$ such that $p(v_0,v_1) \le rp(v_0,v_0)$. Thus, we also have a sequence $\{v_n\}$ in X such that $v_{n+1} \in Tv_n$ and $p(v_0,v_{n+1}) \le rp(v_0,v_n)$ for all $n \in \mathbb{N}$. So we have

$$p(v_0,v_n) \le rp(v_0,v_{n-1}) \le \cdots \le r^np(v_0,v_0). \tag{3.35}$$

Hence

$$\limsup_{n\to\infty} p(u_n,v_n) \le \lim_{n\to\infty} \left(p(u_n,v_0) + p(v_0,v_n)\right) = 0. \tag{3.36}$$

By Lemma 2.6 again, $\{v_n\}$ is a p-Cauchy sequence and converges to v_0. So we have

$$p(v_0, v_0) \le \lim_{n\to\infty} p(v_0, v_n) = 0. \tag{3.37}$$

This completes the proof. □

Example 3.9. Put $X = \{0,1\}$ and define a τ-distance p on X by $p(x,y) = y$ for all $x, y \in X$, and a set-valued p-contractive mapping T from X into itself by $T(x) = X$ for all $x \in X$. Then $1 \in X$ is a fixed point of T and $p(1,1) \ne 0$.

4. Other examples of τ-distances

In this section, we give other examples of τ-distances generated by either some τ-distance p or a family of τ-distances.

Proposition 4.1. *Let p be a τ-distance on a metric space X. Fix $c > 0$. Define a function q from $X \times X$ into $[0,\infty)$ by*

$$q(x,y) = \min\{p(x,y), c\} \tag{4.1}$$

for $x, y \in X$. Then q is also a τ-distance on X.

Proof. Fix $x, y, z \in X$. In the case of $p(x,y) < c$ and $p(y,z) < c$, we have

$$q(x,z) \le p(x,z) \le p(x,y) + p(y,z) = q(x,y) + q(y,z). \tag{4.2}$$

In the case of $p(x,y) \ge c$ or $p(y,z) \ge c$, we have

$$q(x,z) \le c \le q(x,y) + q(y,z). \tag{4.3}$$

Therefore, we have shown $(\tau 1)_q$. So, by Proposition 2.2, we obtain the desired result. □

Proposition 4.2. *Let (X,d) be a metric space. Let $\{p_n\}$ be a sequence of τ-distances on X. Then the following hold.*

(i) *A function q_1, defined by*

$$q_1(x,y) = \max\{p_1(x,y), p_2(x,y)\} \tag{4.4}$$

for $x, y \in X$, is a τ-distance on X.

(ii) *A function q_2, defined by*

$$q_2(x,y) = p_1(x,y) + p_2(x,y) \tag{4.5}$$

for $x, y \in X$, is a τ-distance on X.

(iii) *For each $c > 0$, a function q_3, defined by*

$$q_3(x,y) = \min\left\{\sup_{n\in\mathbb{N}} p_n(x,y), c\right\} \tag{4.6}$$

for $x, y \in X$, is a τ-distance on X.

(iv) *For each $c > 0$, a function q_4, defined by*

$$q_4(x,y) = \min\left\{\sum_{n=1}^{\infty} p_n(x,y), c\right\} \tag{4.7}$$

for $x,y \in X$, is a τ-distance on X.

(v) *If a function q_5, defined by*

$$q_5(x,y) = \sup_{n\in\mathbb{N}} p_n(x,y) \tag{4.8}$$

for $x,y \in X$, is a real-valued function, then q_5 is a τ-distance on X.

(vi) *If a function q_6, defined by*

$$q_6(x,y) = \sum_{n=1}^{\infty} p_n(x,y) \tag{4.9}$$

for $x,y \in X$, is a real-valued function, then q_6 is a τ-distance on X.

Proof. Let $\{\eta_n\}$ be a sequence of functions satisfying $(\tau 2)_{p_n,\eta_n}$, $(\tau 3)_{p_n,\eta_n}$, $(\tau 4)_{p_n,\eta_n}$, and $(\tau 5)_{p_n,\eta_n}$ for $n \in \mathbb{N}$. We first prove that q_5 is a τ-distance on X. Since

$$\sup_{n\in\mathbb{N}} p_n(x,z) \le \sup_{n\in\mathbb{N}} \left(p_n(x,y) + p_n(y,z)\right) \le \sup_{n\in\mathbb{N}} p_n(x,y) + \sup_{n\in\mathbb{N}} p_n(y,z), \tag{4.10}$$

we have $q_5(x,z) \le q_5(x,y) + q_5(y,z)$ for $x,y,z \in X$. Define a function θ from $X \times [0,\infty)$ into $[0,\infty)$ by

$$\theta(x,t) = t + \sum_{n=1}^{\infty} 2^{1-n} \min\{\eta_n(x,t), 1\} \tag{4.11}$$

for $x \in X$ and $t \in [0,\infty)$. Fix $x \in X$. For any $\varepsilon > 0$, we choose $k_1 \in \mathbb{N}$ with $1/k_1 + 2^{1-k_1} < \varepsilon/2$. Then there exists $t_1 \in (0,\varepsilon/2)$ satisfying

$$\sum_{n=1}^{k_1} 2^{1-n}\eta_n(x,t_1) < \frac{1}{k_1}. \tag{4.12}$$

Hence

$$\theta(x,t_1) < t_1 + \frac{1}{k_1} + \sum_{n=k_1+1}^{\infty} 2^{1-n} \min\{\eta_n(x,t_1), 1\} \le \frac{\varepsilon}{2} + \frac{1}{k_1} + 2^{1-k_1} < \varepsilon. \tag{4.13}$$

Therefore, $\theta(x,\cdot)$ is continuous at 0. Hence, $(\tau 2)_\theta$ is shown. We suppose $\lim_n x_n = x$ and $\lim_n \sup\{\theta(z_n, q_5(z_n,x_m)) : m \ge n\} = 0$. Then, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{m\ge n} \min\{\eta_k(z_n, p_k(z_n,x_m)), 1\} \\ &\quad\le \limsup_{n\to\infty} \sup_{m\ge n} \min\{\eta_k(z_n, q_5(z_n,x_m)), 1\} \\ &\quad\le \lim_{n\to\infty} \sup_{m\ge n} 2^{k-1}\theta(z_n, q_5(z_n,x_m)) = 0, \end{aligned} \tag{4.14}$$

and hence

$$\lim_{n\to\infty}\sup_{m\geq n}\eta_k(z_n, p_k(z_n,x_m)) = 0. \tag{4.15}$$

From $(\tau 3)_{p_k,\eta_k}$,

$$p_k(w,x) \leq \liminf_{n\to\infty} p_k(w,x_n) \tag{4.16}$$

for all $w \in X$. Therefore, we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} p_k(w,x) &\leq \sup_{k\in\mathbb{N}}\liminf_{n\to\infty} p_k(w,x_n)\\ &\leq \liminf_{n\to\infty}\sup_{k\in\mathbb{N}} p_k(w,x_n),\end{aligned} \tag{4.17}$$

and hence $q_5(w,x) \leq \liminf_n q_5(w,x_n)$ for all $w \in X$. We have shown $(\tau 3)_{q_5,\theta}$. We prove $(\tau 4)_{q_5,\theta}$. We assume that $\lim_n \sup\{q_5(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \theta(x_n,t_n) = 0$. Then we have $\lim_n \sup\{p_k(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \eta_k(x_n,t_n) = 0$ for all $k \in \mathbb{N}$. From $(\tau 4)_{p_k,\eta_k}$, we have $\lim_n \eta_k(y_n,t_n) = 0$ for all $k \in \mathbb{N}$. For any $\varepsilon > 0$, we choose $k_2 \in \mathbb{N}$ with $1/k_2 + 2^{1-k_2} < \varepsilon/2$. Then there exists $n_2 \in \mathbb{N}$ satisfying

$$\sum_{k=1}^{k_2} 2^{1-k}\eta_k(y_n,t_n) < \frac{1}{k_2} \tag{4.18}$$

and $t_n < \varepsilon/2$ for $n \geq n_2$. We now have

$$\theta(y_n,t_n) \leq t_n + \frac{1}{k_2} + 2^{1-k_2} < \varepsilon \tag{4.19}$$

for $n \geq n_2$. This implies $\lim_n \theta(y_n,t_n) = 0$. We prove $(\tau 5)_{q_5,\theta}$. We assume $\lim_n \theta(z_n, q_5(z_n,x_n)) = 0$ and $\lim_n \theta(z_n,q_5(z_n,y_n)) = 0$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\min\{\eta_1(z_n,p_1(z_n,x_n)),1\} &\leq \limsup_{n\to\infty}\theta(z_n,p_1(z_n,x_n))\\ &\leq \lim_{n\to\infty}\theta(z_n,q_5(z_n,x_n)) = 0,\end{aligned} \tag{4.20}$$

and hence $\lim_n \eta_1(z_n,p_1(z_n,x_n)) = 0$. We can similarly prove $\lim_n \eta_1(z_n,p_1(z_n,y_n)) = 0$. Therefore, we obtain $\lim_n d(x_n,y_n) = 0$. We have shown that q_5 is a τ-distance on X. We next prove that q_6 is a τ-distance on X. Since

$$\sum_{n=1}^{\infty} p_n(x,z) \leq \sum_{n=1}^{\infty}(p_n(x,y)+p_n(y,z)) = \sum_{n=1}^{\infty} p_n(x,y) + \sum_{n=1}^{\infty} p_n(y,z), \tag{4.21}$$

we have $q_6(x,z) \leq q_6(x,y) + q_6(y,z)$ for $x,y,z \in X$. We note that $q_5(x,y) \leq q_6(x,y)$ for $x,y \in X$. We suppose $\lim_n x_n = x$ and $\lim_n \sup\{\theta(z_n,q_6(z_n,x_m)) : m \geq n\} = 0$. Then we have $\lim_n \sup\{\theta(z_n, q_5(z_n, x_m)) : m \geq n\} = 0$. In such case, we have already shown that $p_k(w,x) \leq \liminf_n p_k(w,x_n)$ for $w \in X$ and $k \in \mathbb{N}$. Fix λ with

$$\lambda < \sum_{k=1}^{\infty} \liminf_{n\to\infty} p_k(w,x_n). \tag{4.22}$$

Then there exist $k_3, n_3 \in \mathbb{N}$ such that $\lambda < \sum_{k=1}^{k_3} p_k(w,x_n)$ for $n \geq n_3$. Hence

$$\lambda \leq \liminf_{n\to\infty} \sum_{k=1}^{k_3} p_k(w,x_n) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} p_k(w,x_n). \tag{4.23}$$

Therefore, we have

$$\sum_{k=1}^{\infty} p_k(w,x) \leq \sum_{k=1}^{\infty} \liminf_{n\to\infty} p_k(w,x_n) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} p_k(w,x_n), \tag{4.24}$$

and hence $q_6(w,x) \leq \liminf_n q_6(w,x_n)$ for $w \in X$. By Proposition 2.1, q_6 is a τ-distance on X. Since

$$\begin{aligned} q_1(x,y) &= \sup\{p_1(x,y), p_2(x,y), p_2(x,y), p_2(x,y), \ldots\}, \\ q_2(x,y) &= p_1(x,y) + \frac{1}{2}p_2(x,y) + \frac{1}{4}p_2(x,y) + \frac{1}{8}p_2(x,y) + \cdots, \end{aligned} \tag{4.25}$$

q_1 and q_2 are τ-distances on X. Since

$$\begin{aligned} q_3(x,y) &= \sup_{n\in\mathbb{N}} \min\{p_n(x,y), c\}, \\ q_4(x,y) &= \sup_{n\in\mathbb{N}} \min\left\{\sum_{k=1}^{n} p_k(x,y), c\right\}, \end{aligned} \tag{4.26}$$

q_3 and q_4 are τ-distances on X. This completes the proof. □

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