THE AFTERMATH OF THE INTERMEDIATE VALUE THEOREM

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The solvability of nonlinear equations has awakened great interest among mathematicians for a number of centuries, perhaps as early as the Babylonian culture (3000–300 B.C.E.). However, we intend to bring to our attention that some of the problems studied nowadays appear to be amazingly related to the time of Bolzano's era (1781–1848). Indeed, this Czech mathematician or perhaps philosopher has rigorously proven what is known today as the intermediate value theorem, a result that is intimately related to various classical theorems that will be discussed throughout this work.

1. Introduction

The main motivation of this paper is to establish a close connection between a classical theorem from real analysis (discovered over two centuries ago) and recent works in monotone operator theory for reflexive Banach spaces. Throughout this presentation, we give a brief description of how the original problem has evolved in time, passing through various generalizations obtained in the last thirty years. However, the main purpose of this paper is to generalize Theorem 1 of Minty [17], where the convexity condition on the domain of the operator is no longer required. We also obtain a new result on monotone operators perturbed by compact mappings.

The study of existence of solutions for nonlinear functional equations involving monotone operators has been extensively discussed for forty years or so. Concerning the study of existence of zeros under the boundary condition (2.3), we find, among many contributions, the work of Vaĭnberg and Kačurovskiĭ [27], Minty [16, 17], Browder [4, 5, 6], and Shinbrot [25]. For related work, we also mention Brézis et al. [3], Kačurovskiĭ [11], Leray and Lions [15], and Rockafellar [24]. However, the connection between *Bolzano's boundary condition* and this most recent *condition (2.3)* (known by the early 1950s) has not been explicitly observed. Therefore, our main interest is to identify some of the work done in the *contour* of this condition (2.3) that was, perhaps, first observed by this mathematician of the 19th century.

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We begin with a result for the Euclidean finite-dimensional space $\mathbb{R}^n$, where the symbol $\langle\cdot,\cdot\rangle$ represents the corresponding Euclidean inner product. We continue using the same symbol, although the space of definition will change, passing through Hilbert spaces to end with reflexive Banach spaces. Since, indeed, most of the results will be given for this latter class of spaces, we may assume that both the reflexive Banach space X and its dual X^* are locally uniformly convex after renorming [26]. This very fact implies that the duality mapping J is single valued and strictly monotone. In addition, we will present the main results for demicontinuous operators (i.e., continuous mappings from the strong topology into the weak topology). We now state this classical well-known result which, in fact, has been the inspiration of this paper.

Bolzano's theorem (1817, [8]). *Let $f:[-r,r]\to\mathbb{R}$ be a continuous function, satisfying the following boundary condition:*

$$x\cdot f(x)\geq 0,\quad \text{for } |x|=r. \tag{1.1}$$

Then there exists at least one solution $x_0\in[-r,r]$ of the equation

$$f(x)=0. \tag{1.2}$$

2. Historical background

In 1884, Poincaré [22] has observed that the aforementioned theorem can indeed be extended to a higher finite dimension, where Bolzano's boundary condition is formulated as $f(-r)\leq 0$ and $f(r)\geq 0$. Today, such a result is known as Poincaré-Miranda theorem [23]. Nevertheless, it is our purpose to explore generalizations of Bolzano's theorem as well, but under a different boundary condition (see (2.3)) which appears to be unrelated to the one used by Miranda [18] and Poincaré [23], except for the one-dimensional case. Indeed, the Poincaré-Miranda theorem can be stated as follows.

Proposition 2.1. *Let C be an n-dimensional cube and let $f:C\to\mathbb{R}^n$ be a continuous mapping satisfying the following condition:*

$$f_i(x)\leq 0,\quad f_i(y)\geq 0,\quad \text{for } i=1,\dots,n, \tag{2.1}$$

whenever x and y are in opposite faces of the cube C and $f=(f_1,\dots,f_n)$. Then (1.2) has at least one solution in C.

As can be seen, the boundary condition (2.1) used by Poincaré and Miranda appears to be unrelated to condition (2.3). In fact, condition (2.1) is restricted to n-dimensional rectangles, while condition (2.3) may be imposed on more general domains. Consequently, we initiate our *journey* with an extension to finite dimension, a result that can be derived from Brouwer-Bohl theorem [2, 9]. In what follows, we will use $B(a;r)$ to denote the open ball centered at a with radius r, while ∂A will denote the boundary of the set A.

Proposition 2.2. *Let $f:\overline{B}(0;r)\subset\mathbb{R}^n\to\mathbb{R}^n$ be a continuous mapping satisfying the following condition:*

$$\langle f(x),x\rangle>0,\quad \text{for } x\in\partial B(0;r). \tag{2.2}$$

Then $\overline{B}(0;\eta) \subset f(\overline{B}(0;r))$ *for some* $\eta > 0$. *In particular, (1.2) has at least one solution in* $B(0;r)$.

The proof of Proposition 2.2 can be found in Morales [20]. Now, as a consequence of this proposition, we obtain the first extension of Bolzano's theorem.

THEOREM 2.3. *Let* $f : \overline{B}(0;r) \to \mathbb{R}^n$ *be a continuous mapping satisfying the following condition:*

$$\langle f(x), x\rangle \geq 0, \quad \text{for } x \in \partial B(0;r). \tag{2.3}$$

Then the equation $f(x) = 0$ *has at least one solution in* $\overline{B}(0;r)$.

A second extension of Bolzano's theorem will involve infinite-dimensional spaces. We begin with Hilbert spaces where the operator is defined for a more general class of domains. Perhaps a result of Altman [1], stated for weakly continuous mappings on separable Hilbert spaces, appears to be one of the earliest results of this type under the above-mentioned condition (2.3). According to the author, the proof of the latter result is based on a combinatorial topology argument concerning the notion of degree theory. To the contrary, a rather elementary proof of our next result can be found in [20].

THEOREM 2.4. *Let* H *be a real Hilbert space and let* D *be a bounded open and convex subset of* H *with* $0 \in D$. *Suppose* $A : \overline{D} \to H$ *is a mapping satisfying the following conditions:*

(i) $I - A$ *is a compact operator;*
(ii) $\langle A(x), x\rangle \geq 0$ *for* $x \in \partial D$.

Then the equation $A(x) = 0$ *has at least one solution in* $\overline{D}$.

However, Theorem 2.4 can be extended to compact operators defined on nonconvex domain for general Banach spaces, under the Leray-Schauder condition (see (2.4)) which is weaker than the corresponding condition (ii) of Theorem 2.4.

THEOREM 2.5. *Let* X *be a Banach space and let* D *be a bounded open subset of* X *with* $0 \in D$. *Suppose* $T : \overline{D} \to X$ *is a compact mapping satisfying*

$$T(x) \neq \lambda x, \quad \text{for } x \in \partial D,\ \lambda > 1. \tag{2.4}$$

Then T *has a fixed point in* $\overline{D}$.

An interesting question is whether we can remove the compactness on the operator $I - A$ of Theorem 2.4 and perhaps replace it with a different type of condition. Indeed, for the past forty years, monotonicity conditions have captured a great deal of interest to solve problems of this nature. In fact, by 1960, Kačurovskiĭ [10] observed that the gradient of a convex function was a monotone operator. Later, Minty [16] formulated the notion of monotone operators in Hilbert spaces. For an extensive recollection on monotone operators, see Kačurovskiĭ [12] and Zeidler [28]. A mapping $A : D \subset H \to H$ is said to be *monotone* if

$$\langle A(x) - A(y), x - y\rangle \geq 0. \tag{2.5}$$

A is said to be *strongly monotone* if there exists a constant $c > 0$ such that

$$\langle A(x) - A(y), x - y \rangle \geq c\|x - y\|^2. \tag{2.6}$$

Notice that if we apply the *Cauchy-Schwarz* inequality to (2.6), we get

$$c\|x - y\| \leq \big|\big|A(x) - A(y)\big|\big|, \tag{2.7}$$

which means that A is expansive, and therefore has an inverse that happens to be Lipschitz. We state our first result for monotone operators in Hilbert spaces, which is a consequence of a theorem of Minty [16]. A rather elementary proof of this fact may be found in [20].

Theorem 2.6. *Let H be a (real) Hilbert space and let $A : \overline{B}(0;r) \to H$ be a continuous monotone operator. Suppose*

$$\langle Ax, x \rangle \geq 0, \quad \text{for } x \in \partial B(0;r). \tag{2.8}$$

Then the equation $Ax = 0$ has a solution in $\overline{B}(0;r)$.

Now, we ask ourselves whether we can extend Theorem 2.6 beyond Hilbert spaces. To answer this question, we need to find a proper interpretation of the boundary condition (2.3). However, continuous linear functionals appear to be the right tool for such an interpretation, since every vector in an arbitrary Hilbert space can be uniquely identified with a continuous functional. Therefore, if an operator A with domain $D(A)$ takes values in the corresponding dual space X^* of X (with $D(A) \subset X$), we may state the following: a mapping $A : D(A) \subset X \to X^*$ is said to be *monotone* if

$$\langle A(x) - A(y), x - y \rangle \geq 0, \quad \text{for } x, y \in D(A), \tag{2.9}$$

where the pairing $\langle \cdot, \cdot \rangle$ denotes the action of a functional on an element of X. If (2.9) holds locally, that is, if each $z \in D(A)$ has a neighborhood U such that the restriction of A to U is a monotone mapping, then A is said to be a *locally monotone* mapping. On the other hand, if we still wish to have the operator A mapping its domain $D(A)$ into the space X itself, then condition (2.9) requires a different interpretation, which leads to an entire new class of operators. These are known as *accretive* operators. In fact, by 1967, Browder [7] and Kato [14] introduced, independently, this new family of mappings which has been extensively studied in recent years.

3. Recent results

We begin with an extension of Theorem 2.6 to reflexive Banach spaces, which was originally stated by Minty [17]. In this case, Theorem 3.3 gives a sharper conclusion in the sense of assuring that the solution belongs to a nonconvex domain. We first prove a restrictive case of the theorem, which is vital for the proof of this result. In addition, we improve Theorem 4 of Morales [20].

PROPOSITION 3.1. *Let X be a reflexive (real) Banach space and let D be a bounded open subset of X such that $0 \in D$. Suppose $A : \overline{D} \to X^*$ is a demicontinuous monotone operator satisfying*

$$\langle Ax, x\rangle > 0, \quad \text{for } x \in \partial D. \tag{3.1}$$

Then the equation $Ax = 0$ has a solution in D.

Proof. We first show that there exists $z \in \overline{\text{co}}(D)$ such that

$$\langle Ax, x - z\rangle \geq 0, \quad \text{for } x \in \overline{D}. \tag{3.2}$$

To this end, define $C(x) = \{y \in \overline{\text{co}}(D) : \langle Ax, x - y\rangle \geq 0\}$. This means inequality (3.2) has a solution if $\cap\{C(x) : x \in \overline{D}\} \neq \phi$. Since $\overline{\text{co}}(D)$ is weakly compact, it suffices to show that the collection $\{C(x) : x \in \overline{D}\}$ enjoys the finite intersection property.

Let $C(x_1), \ldots, C(x_m)$ be an arbitrary finite collection and let $z_1, \ldots, z_n$ be a basis of the finite vector space $Y = \text{span}\{x_1, \ldots, x_m\}$. Let $G = Y \cap D$ and define the mapping

$$g : \overline{G}_Y \longrightarrow Y \text{ by } g(x) = \Sigma_{j=1}^n \langle Ax, z_j\rangle z_j. \tag{3.3}$$

Then g is continuous. To see this, let $x_n \to x$ for some $x \in G$. Since A is monotone, it is locally bounded on G, and then $\{Ax_n\}$ is bounded. This means that there exists a subsequence $\{x_{n_k}\}$ such that $Ax_{n_k} \xrightarrow{w} y$. Therefore, $g(x_{n_k}) \to g(x)$ and, consequently, g is continuous on G. In addition, $I - g$ satisfies the *Leray-Schauder condition* on $\partial_Y G$. To see this, let $x \in \partial_Y G$ so that $g(x) = tx$ for some $t < 0$. Then

$$\Sigma_{j=1}^n \langle Ax, z_j\rangle z_j = tx, \tag{3.4}$$

which implies that

$$\Sigma_{j=1}^n \left[\langle Ax, z_j\rangle\right]^2 = t\langle Ax, x\rangle < 0. \tag{3.5}$$

This is a contradiction! Therefore, by Theorem 2.5, there exists $x_0 \in G$ such that $g(x_0) = 0$. Since $\langle Ax_0, z_j\rangle = 0$ for each $j = 1, \ldots, n$, we have

$$\langle Ax_i, x_i - x_0\rangle = \langle Ax_i - Ax_0, x_i - x_0\rangle \geq 0, \tag{3.6}$$

for $i = 1, \ldots, m$. Hence, $\cap_{i=1}^m C(x_i) \neq \phi$. This means that inequality (3.2) has a solution $z \in \overline{\text{co}}(D)$. If $z \notin D$, then there exists $z_0 \in \text{seg}[0, z] \cap D$ such that $z = \lambda z_0$ for some $\lambda > 1$. Consequently, $\langle Az_0, z_0\rangle \leq 0$, which contradicts the assumption on z_0. Therefore, $z \in D$.

To complete the proof of Proposition 2.2, let $h \in X$ and $t > 0$ such that $z + th \in D$. Then $\langle A(z + th), h\rangle \geq 0$ for all $t > 0$ sufficiently small. Therefore, $\langle Az, h\rangle \geq 0$. On the other hand, since h is arbitrary, we easily obtain that $Az = 0$. □

LEMMA 3.2. *Let X be a reflexive (real) Banach space, let D be a subset of X, and let $A : D \to X^*$ be a monotone operator. Suppose there exists a bounded sequence $\{x_n\}$ such that $A(x_n) + \epsilon_n J(x_n) = 0$ for each $n \in \mathbb{N}$ with $\epsilon_n \to 0^+$ as $n \to \infty$. Then $x_n \to x$ for some $x \in \overline{D}$.*

We are now ready to state and prove Bolzano's theorem for monotone operators defined on reflexive Banach spaces.

THEOREM 3.3. *Let X be a reflexive (real) Banach space and let D be a bounded open subset of X such that $0 \in D$. Suppose $A : \overline{D} \to X^*$ is a demicontinuous monotone operator satisfying*

$$\langle Ax, x\rangle \geq 0, \quad \text{for } x \in \partial D. \tag{3.7}$$

Then the equation $Ax = 0$ has a solution in $\overline{D}$.

Proof. Let $A_\epsilon = A + \epsilon J$, where J is the duality mapping and $\epsilon > 0$. Then A_ϵ satisfies condition (3.1) on ∂D. Therefore, by Proposition 3.1, there exists $x \in D$ such that $A(x) + \epsilon J(x) = 0$. By selecting a sequence $\{\epsilon_n\}$ that converges to zero, we find $x_n \in D$ such that

$$A(x_n) + \epsilon_n J(x_n) = 0, \quad \text{for each } n \in \mathbb{N}. \tag{3.8}$$

Therefore, by Lemma 3.2, $x_n \to x$ for some $x \in \overline{D}$. Hence, $A(x) = 0$. □

As a consequence of Theorem 3.3, we may derive an invariance of the domain theorem, whose proof [20] uses a rather elementary argument and extends, among others, [21, Theorem 2.3], where a degree theory argument is used.

THEOREM 3.4. *Let A be an open subset of a reflexive (real) Banach space X. Suppose $A : D \to X^*$ is a demicontinuous, locally closed, locally one-to-one, and locally monotone operator. Then $A(D)$ is open in X^*.*

We should mention that the operator A, in Theorem 3.4, is *closed* if it maps closed sets onto closed sets, and the property holds *locally* if, for each $x \in D$, there exists a closed ball B such that A restricted to B is closed.

COROLLARY 3.5. *Let X be a reflexive (real) Banach space and let $A : X \to X^*$ be a demicontinuous and α-strongly monotone operator; that is,*

$$\langle A(x) - A(y), x - y\rangle \geq \alpha(\|x - y\|)\|x - y\|, \tag{3.9}$$

where $\alpha : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function with $\alpha(0) = 0$, while $\alpha(r) > 0$ for $r > 0$. Then A is surjective.

Finally, we will study a compact perturbation of a strongly monotone operator under the same boundary condition discussed throughout this paper. For additional related results, see Kartsatos [13] and Morales [19].

THEOREM 3.6. *Let X be a reflexive Banach space and let $A : X \to X^*$ be a demicontinuous and α-strongly monotone operator. Suppose D is a bounded open subset of X (with $0 \in D$) such that the mapping $g : \overline{D} \to X^*$ is compact and satisfies*

$$\langle A(x) + g(x), x\rangle \geq 0, \tag{3.10}$$

for all $x \in \partial D$. Then $0 \in \mathcal{R}(A + g)$.

Proof. Since A is a bijection from X to X^* and $\alpha(t)$ is a continuous increasing function, then A^{-1} exists and is also continuous. Now, let $h : \overline{D} \to X$ be defined by $h(x) = A^{-1}(-g(x))$. Indeed, to prove our conclusion is equivalent to showing that h has a fixed point in $\overline{D}$. To this end, we will prove that h satisfies the Leray-Schauder condition on $\overline{D}$.

Let $h(x) = \lambda x$ for $x \in \partial D$ and $\lambda > 1$. Then $A(\lambda x) + g(x) = 0$ and hence

$$\langle A(x) + g(x), x\rangle + \langle A(\lambda x) - A(x), x\rangle = 0. \tag{3.11}$$

However, due to the assumption on $A + g$, we obtain that

$$\langle A(\lambda x) - A(x), x\rangle \le 0, \tag{3.12}$$

but since A is α-strongly monotone, we may derive that $\lambda = 1$, which is a contradiction. Therefore, by Theorem 2.5, h has a fixed point, implying that $A + g$ has a zero in $\overline{D}$. □

We should remark that if g is a constant function, then the conclusion of Theorem 3.6 follows directly from Corollary 3.5.

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