

CONVERGENCE THEOREMS FOR FIXED POINTS OF DEMICONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

C. E. CHIDUME AND H. ZEGEYE

Received 26 August 2004

Let D be an open subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a demicontinuous pseudocontractive mapping satisfying an appropriate condition, where \bar{D} denotes the closure of D . Then, it is proved that (i) $\bar{D} \subseteq \mathcal{R}(I + r(I - T))$ for every $r > 0$; (ii) for a given $y_0 \in D$, there exists a unique path $t \rightarrow y_t \in \bar{D}$, $t \in [0, 1]$, satisfying $y_t := tTy_t + (1 - t)y_0$. Moreover, if $F(T) \neq \emptyset$ or there exists $y_0 \in D$ such that the set $K := \{y \in D : Ty = \lambda y + (1 - \lambda)y_0 \text{ for } \lambda > 1\}$ is bounded, then it is proved that, as $t \rightarrow 1^-$, the path $\{y_t\}$ converges strongly to a fixed point of T . Furthermore, explicit iteration procedures with bounded error terms are proved to converge strongly to a fixed point of T .

1. Introduction

Let D be a nonempty subset of a real linear space E . A mapping $T : D \rightarrow E$ is called a *contraction mapping* if there exists $L \in [0, 1)$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in D$. If $L = 1$ then T is called *nonexpansive*. T is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \quad \forall x, y \in K, \quad (1.1)$$

where J is the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}. \quad (1.2)$$

T is called *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \quad \forall x, y \in K. \quad (1.3)$$

Clearly the class of nonexpansive mappings is a subset of class of pseudocontractive mappings. T is said to be *demicontinuous* if $\{x_n\} \subseteq D$ and $x_n \rightarrow x \in D$ together imply that $Tx_n \rightarrow Tx$, where \rightarrow and \rightharpoonup denote the strong and weak convergences, respectively. We denote by $F(T)$ the set of fixed points of T .

Closely related to the class of pseudocontractive mappings is the class of accretive mappings. A mapping $A : D(A) \subseteq E \rightarrow E$ is called *accretive* if $T := (I - A)$ is pseudocontractive. If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $\mathcal{R}(I + rA)$, the range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $\text{cl}(D(A)) \subseteq \mathcal{R}(I + rA)$, for all $r > 0$, where $\text{cl}(D(A))$ denotes the closure of the domain of A .

Let $z \in D$, then for each $t \in (0, 1)$, and for a nonexpansive map T , there exists a unique point $x_t \in D$ satisfying the condition,

$$x_t = tTx_t + (1 - t)z \quad (1.4)$$

since the mapping $x \rightarrow tTx + (1 - t)z$ is a contraction. When E is a Hilbert space and T is a self-map, Browder [1] showed that $\{x_t\}$ converges strongly to an element of $F(T)$ which is nearest to u as $t \rightarrow 1^-$. This result was extended to various more general Banach spaces by Reich [10], Takahashi and Ueda [11], and a host of other authors. Recently, Morales and Jung [7] proved the existence and convergence of a continuous path to a fixed point of a continuous pseudocontractive mapping in reflexive Banach spaces. More precisely, they proved the following theorem.

THEOREM 1.1 [7, Proposition 2(iv), Theorem 1]. *Suppose D is a nonempty closed convex subset of a reflexive Banach space E and $T : D \rightarrow E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition. Then for $z \in D$, there exists a unique path $t \rightarrow y_t \in D$, $t \in [0, 1)$, satisfying the following condition,*

$$y_t = tTy_t + (1 - t)z. \quad (1.5)$$

Furthermore, suppose E is assumed to have a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of D has the fixed point property for nonexpansive self-mappings. If $F(T) \neq \emptyset$ or there exists $x_0 \in D$ such that the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded, then as $t \rightarrow 1^-$, the path converges strongly to a fixed point of T .

From Theorem 1.1, one question arises quite naturally.

Question. Can the continuity of T be weakened to demicontinuity of T ?

In connection with this, Lan and Wu [3] proved the following theorem in the Hilbert space setting.

THEOREM 1.2 [3, Theorems 2.3 and 2.5]. *Let E be a Hilbert space. Suppose D is a nonempty closed convex subset of E and $T : D \rightarrow E$ is a demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Then for $z \in D$, there exists a unique path $t \rightarrow y_t \in D$, $t \in (0, 1)$, satisfying the following condition:*

$$y_t = tTy_t + (1 - t)z. \quad (1.6)$$

Moreover, if (i) D is bounded then $F(T) \neq \emptyset$ and $\{y_t\}$ converges strongly to a fixed point of T as $t \rightarrow 1^-$; (ii) D is unbounded and $F(T) \neq \emptyset$ then $\{y_t\}$ converges strongly to a fixed point of T as $t \rightarrow 1^-$.

Let D be a nonempty open and convex subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a demicontinuous pseudocontractive mapping which satisfies

$$\text{for some } z \in D, \quad Tx - z \neq \lambda(x - z) \quad \text{for } x \in \partial D, \lambda > 1, \tag{1.7}$$

where \bar{D} is the closure of D .

It is our purpose in this paper to give sufficient conditions to ensure that $\bar{D} \subseteq (I + r(I - T))(\bar{D})$ for every $r > 0$ and to prove the existence and convergence of a path to a fixed point of a demicontinuous pseudocontractive mapping in spaces more general than Hilbert spaces. More precisely, we prove that for a given $y_0 \in D$, there exists a unique path $t \rightarrow y_t \in \bar{D}, t \in (0, 1)$, satisfying $y_t := tTy_t + (1 - t)y_0$. Moreover, if $F(T) \neq \emptyset$ or there exists $y_0 \in D$ such that the set $K := \{y \in D : Ty = \lambda y + (1 - \lambda)y_0 \text{ for } \lambda > 1\}$ is bounded, then the path $\{y_t\}$ converges strongly to a fixed point of T . Furthermore, the sequence $\{x_n\}$ generated from $x_1 \in K$ by $x_{n+1} := (1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_1)$, for all integers $n \geq 1$, where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences satisfying appropriate conditions, converges strongly to a fixed point of T . Our theorems provide an affirmative answer to the above question in uniformly smooth Banach spaces and extend Theorem 1.2 to uniformly smooth spaces provided that the interior of D , $\text{int}(D)$, is nonempty.

2. Preliminaries

Let E be a real normed linear space of dimension ≥ 2 . The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}, \quad \tau > 0. \tag{2.1}$$

If there exist a constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are L_p and the Sobolev spaces W_p^m for $1 < p < \infty$. A Banach space E is called *uniformly smooth* if $\lim_{\tau \rightarrow 0} (\rho_E(\tau)/\tau) = 0$. If E is a real uniformly smooth Banach space, then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + \max \{ \|x\|, 1 \} \|y\| b(\|y\|) \tag{2.2}$$

holds for every $x, y \in E$ where $b : [0, \infty) \rightarrow [0, \infty)$ is a continuous strictly increasing function satisfying the following conditions:

- (i) $b(ct) \leq cb(t), \forall c \geq 1,$
- (ii) $\lim_{t \rightarrow 0} b(t) = 0.$ (See, e.g., [8].)

Let D be a nonempty subset of a Banach space E . For $x \in D$, the *inward set* of $x, I_D(x)$, is defined by $I_D(x) := \{x + \lambda(u - x) : u \in D, \lambda \geq 1\}$. A mapping $T : D \rightarrow E$ is called *weakly inward* if $Tx \in \text{cl}[I_D(x)]$ for all $x \in D$, where $\text{cl}[I_D(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $D \subseteq E$ be closed convex and let Q be a mapping of E onto D . A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, *range of Q*. A subset D of E is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto D . If $E = H$, the *metric projection* P_D is a *nonexpansive retraction from H to any closed convex subset D of H*.

In what follows, we will make use of the following lemma and theorems.

LEMMA 2.1 [2]. Let $\{\lambda_n\}$, $\{\gamma_n\}$, and $\{\alpha_n\}$ be sequences of nonnegative numbers satisfying $\sum_1^\infty \alpha_n = \infty$ and $\gamma_n/\alpha_n \rightarrow 0$, as $n \rightarrow \infty$. Let the recursive inequality

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n\psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \dots, \quad (2.3)$$

be given where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that it is positive on $(0, \infty)$ and $\psi(0) = 0$. Then $\lambda_n \rightarrow 0$, as $n \rightarrow \infty$.

THEOREM 2.2 [6]. Let E be a uniformly smooth Banach space and let D be an open subset of E . Suppose $T : \bar{D} \rightarrow E$ is a demicontinuous strongly pseudocontractive mapping which satisfies

$$\text{for some } z \in D : Tx - z \neq \lambda(x - z) \quad \text{for } x \in \partial D, \lambda > 1. \quad (2.4)$$

Then T has a unique fixed point in \bar{D} .

Remark 2.3. We observe that, in Theorem 2.2, if, in addition, D is convex, then any weakly inward map satisfies condition (2.4).

THEOREM 2.4 (Reich [10]). Let E be uniformly smooth. Let $A \subset E \times E$ be accretive with $\text{cl}(D(A))$ convex. Suppose A satisfies the range condition. Let $J_t := (I + tA)^{-1}$, $t > 0$ be the resolvent of A and assume that $A^{-1}(0)$ is nonempty. Then, for each $x \in \mathcal{R}(I + rA)(\bar{D})$, $\lim_{t \rightarrow \infty} J_t x = Px \in A^{-1}(0)$, where P is the sunny nonexpansive retraction of $\text{cl}(D(A))$ onto $A^{-1}(0)$.

Remark 2.5. From the proof of Theorem 2.4, we observe that we may replace the assumption that $A^{-1}(0) \neq \emptyset$ with the assumption that $x_t = J_t x$ is bounded, for each $x \in \mathcal{R}(I + tA)$ and $t > 0$.

3. Main results

We first prove the following results which will be used in the sequel.

PROPOSITION 3.1. Let D be an open subset of a real uniformly smooth Banach space E and let $T : \bar{D} \rightarrow E$ be a demicontinuous pseudocontractive mapping which satisfies condition (2.4). Let $A_T : \bar{D} \rightarrow E$ be defined by $A_T := I + r(I - T)$ for any $r > 0$. Then $\bar{D} \subseteq A_T[\bar{D}]$.

Proof. Let $z \in \bar{D}$. Then it suffices to show that there exists $x \in \bar{D}$ such that $z = A_T(x)$. Define $g : \bar{D} \rightarrow E$ by $g(x) := (1/(1+r))(rT(x) + z)$ for some $r > 0$. Then clearly g is demicontinuous and for $x, y \in \bar{D}$ we have that $\langle g(x) - g(y), j(x - y) \rangle \leq (r/(1+r))\|x - y\|^2$. Thus, g is a strongly pseudocontractive mapping which satisfies condition (2.4). Therefore, by Theorem 2.2, there exists $x \in \bar{D}$ such that $g(x) = x$, that is, $z = A_T(x)$. The proof is complete. \square

COROLLARY 3.2. Let E be a real uniformly smooth Banach space and let $A : E \rightarrow E$ be demicontinuous accretive mapping. Then A is m -accretive.

Proof. Set $T := (I - A)$. Then, we obtain that T is a demicontinuous pseudocontractive self-map of E . Clearly, condition (2.4) is satisfied. The conclusion follows from Proposition 3.1. \square

Corollary 3.2 was proved by Minty [5] in a Hilbert space setting for continuous accretive mappings and this was extended to general Banach spaces by Martin [4].

We now prove the following theorems.

THEOREM 3.3. *Let D be an open and convex subset of a real uniformly smooth Banach space E . Let $T : \bar{D} \rightarrow E$ be a demicontinuous pseudocontractive mapping satisfying condition (2.4). Then for a given $y_0 \in D$, there exists a unique path $t \rightarrow y_t \in \bar{D}$, $t \in (0, 1)$, satisfying*

$$y_t = tTy_t + (1 - t)y_0. \tag{3.1}$$

Furthermore, if $F(T) \neq \emptyset$ or there exists $z \in D$ such that the set $K := \{y \in D : Ty = \lambda y + (1 - \lambda)z \text{ for } \lambda > 1\}$ is bounded, then the path $\{y_t\}$ described by (3.1) converges strongly to a fixed point of T as $t \rightarrow 1^-$.

Proof. For each $t \in (0, 1)$ the mapping T_t defined by $T_t x := tT(t_n)x + (1 - t)y_0$ is demicontinuous and strongly pseudocontractive. By Theorem 2.2, it has a unique fixed point y_t in \bar{D} , that is, for each $t \in (0, 1)$ there exists $y_t \in \bar{D}$ satisfying (3.1). Continuity of y_t follows as in [7]. Now we show the convergence of $\{y_t\}$ to a fixed point of T . Let $A := I - T$. Then A is accretive and by Proposition 3.1, $\bar{D} \subseteq (I + rA)(\bar{D})$ for all $r > 0$ and hence A satisfies the range condition. Moreover, from (3.1), $y_t + (t/(1 - t))Ay_t = y_0$. But this implies that $y_t = (I + (t/(1 - t))A)^{-1}y_0 = J_{(t/(1-t))}y_0$. Furthermore, since $A^{-1}(0) \neq \emptyset$ or the fact that K is bounded implies that $\{y_t\}$ is bounded (see, e.g., [7]), we have by Theorem 2.4 that $y_t \rightarrow y^* \in A^{-1}(0)$ and hence $y_t \rightarrow y^* \in F(T)$ as $t \rightarrow 1^-$. This completes the proof of the theorem. \square

Remark 3.4. We note that, in Theorem 3.3, the requirement that T satisfies condition (2.4) may be replaced with the weakly inward condition. Furthermore, Theorem 3.3 extends [3, Theorems 2.3 and 2.5] to the more general Banach spaces which include $l_p, L_p, W_p^m, 1 < p < \infty$, spaces, provided that $\text{int}(D)$ is nonempty.

For our next theorem and corollary, $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ are real sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \lim_{n \rightarrow \infty} (b(\lambda_n)/\theta_n) = 0$;
- (iii) $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n - 1)/\lambda_n \theta_n) = 0, c_n = o(\lambda_n \theta_n)$.

THEOREM 3.5. *Let D be an open and convex subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a bounded demicontinuous pseudocontractive mapping satisfying condition (2.4). Suppose \bar{D} is a nonexpansive retract of E with Q as the nonexpansive retraction. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_0) - c_n (x_n - u_n)), \tag{3.2}$$

for all positive integers n , where $\{u_n\}$ is a sequence of bounded error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By Theorem 3.3, $F(T) \neq \emptyset$. Let $x^* \in F(T)$. Let $r > 1$ be sufficiently large such that $x_0 \in B_{r/2}(x^*)$.

Claim 3.6. $\{x_n\}$ is bounded.

It suffices to show by induction that $\{x_n\}$ belongs to $B = \overline{B}_r(x^*)$ for all positive integers. Now, $x_0 \in B$ by assumption. Hence we may assume that $x_n \in B$ and set $M := 2r + \sup\{\|(I - T)x_i\| + \|x_i - u_i\|, \text{ for } i \leq n\}$. We prove that $x_{n+1} \in B$. Suppose x_{n+1} is not in B . Then $\|x_{n+1} - x^*\| > r$ and thus from (3.2) we have that $\|x_{n+1} - x^*\| \leq \|x_n - x^* - \lambda_n((I - T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)\| \leq \|x_n - x^*\| + \lambda_n\|(I - T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\| \leq r + M$. Moreover, from (3.2) and inequality (2.2), and using the fact that $\theta_n \leq 1$, we get that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|Q((1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)) - x^*\|^2 \\ &\leq \|x_n - x^* - \lambda_n((I - T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\lambda_n\langle(I - T)x_n, j(x_n - x^*)\rangle \\ &\quad - 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle - 2c_n\langle x_n - u_n, j(x_n - x^*)\rangle \\ &\quad + \max\{\|x_n - x^*\|, 1\}\lambda_n\left\|\left|(I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right\| \quad (3.3) \\ &\quad \times b\left(\lambda_n\left\|\left|(I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right\|\right) \\ &\leq \|x_n - x^*\|^2 - 2\lambda_n\langle(I - T)x_n, j(x_n - x^*)\rangle \\ &\quad - 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle - 2c_n\langle x_n - u_n, j(x_n - x^*)\rangle \\ &\quad + (r + 1)\lambda_nMb(\lambda_nM). \end{aligned}$$

Since T is pseudocontractive and $x^* \in F(T)$, we have $\langle(I - T)x_n, j(x_n - x^*)\rangle \geq 0$. Hence (3.3) gives

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle \\ &\quad + 2c_n\|x_n - u_n\| \cdot \|x_n - x^*\| + (r + 1)\lambda_nM^2b(\lambda_n). \end{aligned} \quad (3.4)$$

Choose $L > 0$ sufficiently small such that $L \leq r^2/(2\sqrt{D^*} + 2M)^2$, where $D^* = (r + 1)M^2$. Set $d := \sqrt{L}$. Then since $\|x_{n+1} - x^*\| > \|x_n - x^*\|$ by our assumption, from (3.4) we get that $2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle \leq (r + 1)M^2\lambda_nb(\lambda_n) + 2c_nMr$ which gives $\langle x_n - x_0, j(x_n - x^*)\rangle \leq D^*L$, since $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq L = d^2$, for all $n \geq 1$ by our assumption.

Now adding $\langle x_0 - x^*, j(x_n - x^*) \rangle$ to both sides of this inequality, we get that

$$\begin{aligned} \|x_n - x^*\|^2 &\leq LD^* + \langle x_0 - x^*, j(x_n - x^*) \rangle \\ &\leq LD^* + \|x_0 - x^*\| \|x_n - x^*\| \leq LD^* + \frac{r}{2} \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

Solving this quadratic inequality for $\|x_n - x^*\|$ and using the estimate $\sqrt{r^2/16 + LD^*} \leq r/4 + \sqrt{LD^*}$, we obtain that $\|x_n - x^*\| \leq r/2 + \sqrt{LD^*}$. But in any case, $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \lambda_n \|(I - T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\|$ so that $\|x_{n+1} - x^*\| \leq r/2 + \sqrt{LD^*} + \lambda_n M \leq r$, by the original choices of L and λ_n , and this contradicts the assumption that x_{n+1} is not in B . Therefore, $x_n \in B$ for all positive integers n . Thus $\{x_n\}$ is bounded. Now we show that $x_n \rightarrow x^*$. Let $\{y_n\}$ be a subsequence of $\{y_t : t \in [0, 1]\}$, such that $y_n := y_{t_n}$, $t_n = 1/(1 + \theta_n)$. Then from (3.2) and inequality (2.2) and using the fact that $y_n \in \bar{D}$ for all $n \geq 0$, we get

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &= \|Q((1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n(x_n - x_0) - c_n(x_n - u_n)) - y_n\|^2 \\ &\leq \|x_n - y_n - \lambda_n((I - T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)\|^2 \\ &\leq \|x_n - y_n\|^2 - 2\lambda_n \langle (I - T)x_n + \theta_n(x_n - x_0), j(x_n - y_n) \rangle \\ &\quad - 2c_n \langle x_n - u_n, j(x_n - y_n) \rangle \\ &\quad + \max\{\|x_n - y_n\|, 1\} \lambda_n \left\| (I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n) \right\| \\ &\quad \times b \left(\lambda_n \left\| (I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n) \right\| \right) \\ &\leq (1 - 2\lambda_n \theta_n) \|x_n - y_n\|^2 - 2\lambda_n \langle (I - T)x_n + \theta_n(y_n - x_0), j(x_n - y_n) \rangle \\ &\quad + 2c_n \|x_n - u_n\| \cdot \|x_n - y_n\| \\ &\quad + \max\{\|x_n - y_n\|, 1\} \lambda_n \left\| (I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n) \right\| \\ &\quad \times b \left(\lambda_n \left\| (I - T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n) \right\| \right). \end{aligned} \tag{3.6}$$

Since $T y_n = y_n + \theta_n(y_n - x_0)$ and T is pseudocontractive, we get that $\langle (I - T)x_n + \theta_n(y_n - x_0), j(x_n - y_n) \rangle \geq 0$. Moreover, since $\{x_n\}$, $\{y_n\}$, and hence $\{T x_n\}$, are bounded, there exists $M_0 > 0$ such that $\max\{\|x_n - y_n\|, 1, \|x_n - y_n\| \cdot \|x_n - u_n\|, \|(I - T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\|\} \leq M_0$. Therefore, (3.6) with property of b gives

$$\|x_{n+1} - y_n\|^2 \leq (1 - 2\lambda_n \theta_n) \|x_n - y_n\|^2 + M_0 \lambda_n b(\lambda_n) + c_n M_0. \tag{3.7}$$

On the other hand, by the pseudocontractivity of T and the fact that $\theta_n(y_n - x_0) + (y_n - T y_n) = 0$, we have that

$$\begin{aligned} \|y_{n-1} - y_n\| &\leq \left\| y_{n-1} - y_n + \frac{1}{\theta_n} ((I - T)y_{n-1} - (I - T)y_n) \right\| \\ &\leq \frac{\theta_{n-1} - \theta_n}{\theta_n} (\|y_{n-1}\| + \|z\|) = \left(\frac{\theta_{n-1}}{\theta_n} - 1 \right) (\|y_{n-1}\| + \|z\|). \end{aligned} \tag{3.8}$$

However,

$$\|x_n - y_n\|^2 \leq \|x_n - y_{n-1}\|^2 + \|y_{n-1} - y_n\|(\|y_{n-1} - y_n\| + 2\|y_{n-1} - x_n\|). \quad (3.9)$$

Therefore, these estimates with (3.7) give that

$$\|x_{n+1} - y_n\|^2 \leq (1 - 2\lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + M_1\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + M_1\lambda_nb(\lambda_n) + c_nM_1, \quad (3.10)$$

for some $M_1 > 0$. Thus, by Lemma 2.1, $x_{n+1} - y_n \rightarrow 0$. Hence, since $y_n \rightarrow x^*$ by Theorem 3.3, we have that $x_n \rightarrow x^*$, this completes the proof of the theorem. \square

With the help of Remark 2.3 and Theorem 3.5 we obtain the following corollary.

COROLLARY 3.7. *Let D be an open and convex subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a bounded demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Suppose \bar{D} is a nonexpansive retract of E with Q as the nonexpansive retraction. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \quad (3.11)$$

for all positive integers n , where $\{u_n\}$ is a sequence of error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded then, there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a fixed point of T .

Remark 3.8. For the case where E is q -uniformly smooth, where $q > 1$, and $t \leq M$ for some $M > 0$, the function b in (2.2) is estimated by $b(t) \leq ct^{q-1}$ for some $c > 0$ (see [9]). Thus, we have the following corollary.

COROLLARY 3.9. *Let D be an open and convex subset of a real q -uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a bounded demicontinuous pseudocontractive mapping satisfying condition (2.4). Suppose \bar{D} is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lim_{n \rightarrow \infty} (\lambda_n^{(q-1)}/\theta_n) = 0$;
- (iii) $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n - 1)/\lambda_n\theta_n) = 0$, $c_n = o(\lambda_n\theta_n)$.

Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \quad (3.12)$$

for all positive integers n , where $\{u_n\}$ is a bounded sequence of error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, \lambda_n^{(q-1)}/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a fixed point of T .

Remark 3.10. Examples of sequences $\{\lambda_n\}$ and $\{\theta_n\}$ satisfying conditions of Corollary 3.9 are as follows: $\lambda_n = 2(n + 1)^{-a}$, $\theta_n = 2(n + 1)^{-b}$, and $c_n = 2(n + 1)^{-1}$ with $0 < b < a$ and $a + b < 1$ if $2 \leq q < \infty$, and with $0 < b < a(q - 1)$ and $a + b(q - 1) < 1$ if $1 < q < 2$.

If in Theorem 3.5, T is a self-map of \bar{D} , then the projection operator Q is replaced with I , the identity map on E . Moreover, T satisfies condition (2.4). As a consequence, we have the following corollaries.

COROLLARY 3.11. *Let D be an open and convex subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Suppose $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ are real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5 and $\lambda_n(1 + \theta_n) + c_n \leq 1$, $\forall n \geq 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n), \tag{3.13}$$

for all positive integers n , where $\{u_n\}$ is a sequence of bounded error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a fixed point of T .

Proof. The conditions on λ_n , θ_n , and c_n imply that the sequence $\{x_n\}$ is well defined. Thus, the proof follows from Theorem 3.5. □

If in Theorem 3.5, D is assumed to be bounded, then the conditions $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for some $d > 0$ which guarantee the boundedness of the sequence $\{x_n\}$ are not needed. In fact, we have the following corollary.

COROLLARY 3.12. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow E$ is a bounded demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Suppose \bar{D} is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_nTx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \tag{3.14}$$

for all positive integers n , where $\{u_n\}$ is a sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. Since D , and hence \bar{D} , is bounded we have that $\{x_n\}$ is bounded. Thus the conclusion follows from Theorem 3.5. □

COROLLARY 3.13. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5 and $\lambda_n(1 + \theta_n) + c_n \leq 1$, $\forall n \geq 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$*

by

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n - \lambda_n \theta_n (x_n - x_0) - c_n (x_n - u_n), \tag{3.15}$$

for all positive integers n , where $\{u_n\}$ is a sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T .

Remark 3.14. If in Theorem 3.5, D is bounded, T is a self-map, and $c_n \equiv 1$ for all $n \geq 1$, that is, the error term is ignored, then the following corollary holds.

COROLLARY 3.15. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E . Suppose $T : \bar{D} \rightarrow \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5 with $c_n \equiv 0$ for all $n \geq 1$ and $\lambda_n(1 + \theta_n) \leq 1$, for all $n \geq 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n - \lambda_n \theta_n (x_n - x_0), \tag{3.16}$$

for all positive integers n . Then $\{x_n\}$ converges strongly to a fixed point of T .

The following convergence theorem is for the approximation of solution of demicontinuous accretive mappings.

THEOREM 3.16. *Let D be an open and convex subset of a real uniformly smooth Banach space E . Suppose $A : \bar{D} \rightarrow E$ is a bounded demicontinuous accretive mapping which satisfies, for some $x_0 \in D$, $Ax \neq t(x - x_0)$ for all $x \in \partial D$ and $t < 0$. Suppose \bar{D} is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q(x_n - \lambda_n (Ax_n + \theta_n (x_n - x_0))) - c_n (x_n - u_n), \tag{3.17}$$

for all positive integers n , where $\{u_n\}$ is a sequence of bounded error terms. Suppose either $N(A) \neq \emptyset$ ($N(A)$ is the null space of A) or the set $K := \{x \in D : (I - A)x = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded. Then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a zero of A .

Proof. Set $T := (I - A)$. Then, we have that for some $x_0 \in D$, $(I - T)x \neq t(x - x_0)$ for $x \in \partial D$ and $t < 0$. This implies that $Tx - x_0 = \lambda(x - x_0)$ for all $x \in \partial D$ and $\lambda > 1$. Moreover, $F(T) \neq \emptyset$ or the set $K = \{x \in D : Tx = \lambda x + (1 - \lambda)x_0, \text{ for } \lambda = (1 - t) > 1\}$ is bounded. Therefore, by Theorem 3.5, $\{x_n\}$ converges strongly to $x^* \in F(T)$. But $F(T) = N(A)$. Hence, $\{x_n\}$ converges strongly to $x^* \in N(A)$. The proof of the theorem is complete. \square

The following corollary follows from Theorem 3.16.

COROLLARY 3.17. *Let E be a real uniformly smooth Banach space and suppose $A : E \rightarrow E$ is a bounded demicontinuous accretive mapping. Let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0, 1]$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from*

$x_0 \in E$ by

$$x_{n+1} = x_n - \lambda_n(Ax_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n), \quad (3.18)$$

for all positive integers n , where $\{u_n\}$ is a sequence of bounded error terms. If either $N(A) \neq \emptyset$ or the set $K := \{x \in E : (I - A)x = \lambda x + (1 - \lambda)x_0 \text{ for } \lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a point of $N(A)$.

Acknowledgment

The second author undertook this work when he was visiting The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, as a Junior Associate.

References

- [1] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.
- [2] C. E. Chidume and H. Zegeye, *Approximation of solutions of nonlinear equations of monotone and Hammerstein type*, Appl. Anal. **82** (2003), no. 8, 747–758.
- [3] K. Q. Lan and J. H. Wu, *Convergence of approximants for demicontinuous pseudo-contractive maps in Hilbert spaces*, Nonlinear Anal. Ser. A: Theory Methods **49** (2002), no. 6, 737–746.
- [4] R. H. Martin Jr., *A global existence theorem for autonomous differential equations in a Banach space*, Proc. Amer. Math. Soc. **26** (1970), 307–314.
- [5] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346.
- [6] C. H. Morales, *Zeros for accretive operators satisfying certain boundary conditions*, J. Math. Anal. Appl. **105** (1985), no. 1, 167–175.
- [7] C. H. Morales and J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3411–3419.
- [8] S. Reich, *An iterative procedure for constructing zeros of accretive sets in Banach spaces*, Nonlinear Anal. **2** (1978), no. 1, 85–92.
- [9] ———, *Constructive techniques for accretive and monotone operators*, Applied Nonlinear Analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex., 1978), Academic Press, New York, 1979, pp. 335–345.
- [10] ———, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), no. 1, 287–292.
- [11] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl. **104** (1984), no. 2, 546–553.

C. E. Chidume: The Abdus Salam International Centre for Theoretical Physics, 34014 Trieste, Italy
E-mail address: chidume@ictp.trieste.it

H. Zegeye: The Abdus Salam International Centre for Theoretical Physics, 34014 Trieste, Italy
E-mail address: habtuzh@yahoo.com