TRANSFER POSITIVE HEMICONTINUITY AND ZEROS, COINCIDENCES, AND FIXED POINTS OF MAPS IN TOPOLOGICAL VECTOR SPACES

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Let E be a real Hausdorff topological vector space. In the present paper, the concepts of the transfer positive hemicontinuity and strictly transfer positive hemicontinuity of set-valued maps in E are introduced (condition of strictly transfer positive hemicontinuity is stronger than that of transfer positive hemicontinuity) and for maps $F : C \to 2^E$ and $G : C \to 2^E$ defined on a nonempty compact convex subset C of E, we describe how some ideas of K. Fan have been used to prove several new, and rather general, conditions (in which transfer positive hemicontinuity plays an important role) that a single-valued map $\Phi : \bigcup_{c\in C}(F(c) \times G(c)) \to E$ has a zero, and, at the same time, we give various characterizations of the class of those pairs (F,G) and maps F that possess coincidences and fixed points, respectively. Transfer positive hemicontinuity and strictly transfer positive hemicontinuity generalize the famous Fan upper demicontinuity which generalizes upper semicontinuity. Furthermore, a new type of continuity defined here essentially generalizes upper hemicontinuity (the condition of upper demicontinuity is stronger than the upper hemicontinuity). Comparison of transfer positive hemicontinuity and strictly transfer positive hemicontinuity with upper demicontinuity and upper hemicontinuity and relevant connections of the results presented in this paper with those given in earlier works are also considered. Examples and remarks show a fundamental difference between our results and the well-known ones.

1. Introduction

One of the most important tools of investigations in nonlinear and convex analysis is the minimax inequality of Fan [11, Theorem 1]. There are many variations, generalizations, and applications of this result (see, e.g., Hu and Papageorgiou [16, 17], Ricceri and Simons [19], Yuan [21, 22], Zeidler [24] and the references therein). Using the partition of unity, his minimax inequality, introducing in [10, page 236] the concept of upper demicontinuity and giving in [11, page 108] the inwardness and outwardness conditions, Fan initiated a new line of research in coincidence and fixed point theory of set-valued maps in topological vector spaces, proving in [11] the general results ([11, Theorems 3–6]) which extend and unify several well-known theorems (e.g., Browder [7], [5, Theorems 1 and 2]

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and [6, Theorems 3 and 5], Fan [6, 9], [10, Theorem 5] and [8, Theorem 1], Glicksberg [14], Kakutani [18], Bohnenblust and Karlin [3], Halpern and Bergman [15], and others) concerning upper semicontinuous maps and, in particular, inward and outward maps (the condition of upper semicontinuity is stronger than that of upper demicontinuity).

Let C be a nonempty compact convex subset of a real Hausdorff topological vector space E, let $F : C \to 2^E$ and $G : C \to 2^E$ be set-valued maps and let $\Phi : \bigcup_{c\in C}(F(c) \times G(c)) \to E$ be a single-valued map. The purpose of our paper is to introduce the concepts of the transfer positive hemicontinuity and strictly transfer positive hemicontinuity of set-valued maps in E and prove various new results concerning the existence of zeros of Φ, coincidences of F and G and fixed points of F in which transfer positive hemicontinuity and strictly transfer positive hemicontinuity plays an important role (see Section 2). In particular, our results generalize theorems of Fan type (e.g., [11, Theorems 3–6]) and contain fixed point theorems for set-valued transfer positive hemicontinuous maps with the inwardness and outwardness conditions given by Fan [11, page 108]. Transfer positive hemicontinuity and strictly transfer positive hemicontinuity generalize the Fan upper demicontinuity. Furthermore, a new type of continuity defined here essentially generalizes upper hemicontinuity (every upper demicontinuous map is upper hemicontinuous). Comparisons of transfer hemicontinuity and strictly transfer positive hemicontinuity with upper demicontinuity and upper hemicontinuity are given in Sections 3 and 4. The remarks, examples and comparisons of our results with Fan's results and other results concerning coincidences and fixed points of upper hemicontinuous maps given by Yuan et al. [22, 23] (see also the references therein) show that our theorems are new and differ from those given by the above-mentioned authors (see Sections 2–4).

2. Transfer positive hemicontinuity, strictly transfer positive hemicontinuity, zeros, coincidences, and fixed points

Let E be a real Hausdorff topological vector space and let E' denote the vector space of all continuous linear forms on E.

Let C be a nonempty subset of E. A set-valued map $F : C \to 2^E$ is a map which assigns a unique nonempty subset $F(c) \in 2^E$ to each $c \in C$ (here 2^E denotes the family of all nonempty subsets of E).

Definition 2.1. Let C be a nonempty subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$. Let $\Phi : \bigcup_{c\in C}(F(c) \times G(c)) \to E$ be a single-valued map.

(a) We say that a pair (F,G) is Φ-*transfer positive hemicontinuous* (Φ-*t.p.h.c.*) on C if, whenever $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c > 0$ are such that

$$\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0 \quad \text{for any } (u,v) \in F(c) \times G(c), \tag{2.1}$$

there exists a neighbourhood $N(c)$ of c in C such that

$$\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0 \quad \text{for any } x \in N(c) \text{ and any } (u,v) \in F(x) \times G(x). \tag{2.2}$$

(b) We say that a pair (F,G) is Φ-*transfer hemicontinuous* (Φ-*t.h.c.*) on C if, whenever $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ is such that

$$\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0 \quad \text{for any } (u,v) \in F(c) \times G(c), \tag{2.3}$$

there exists a neighbourhood $N(c)$ of c in C such that

$$\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0 \quad \text{for any } x \in N(c) \text{ and any } (u,v) \in F(x) \times G(x). \tag{2.4}$$

(c) We say that a map F is Φ-*t.p.h.c.* or Φ-*t.h.c.* on C if a pair (F,I_E) is Φ-t.p.h.c. or Φ-t.h.c. on C, respectively.

(d) We say that a pair (F,G) is *transfer positive hemicontinuous (t.p.h.c.)* or *transfer hemicontinuous (t.h.c.)* on C if (F,G) is Φ-t.p.h.c. or Φ-t.h.c. on C, respectively, for Φ of the form $\Phi(u,v) = u - v$ where $(u,v) \in F(c) \times G(c)$ and $c \in C$.

(e) We say that a map F is *t.p.h.c.* or *t.h.c.* on C if a pair (F,I_E) is t.p.h.c. or t.h.c. on C, respectively.

Recall that an *open half-space* H in E is a set of the form $H = \{x \in E : \varphi(x) < t\}$ where $\varphi \in E' \setminus \{0\}$ and $t \in \mathbb{R}$.

Remark 2.2. The geometric meaning of the Φ-transfer positive hemicontinuity and Φ-transfer hemicontinuity is clear.

Really define

$$\begin{aligned} H_{c,\varphi_c,\lambda_c,\varepsilon_c} &= \{w \in E : \varphi_c(w) < (1+\varepsilon_c)\lambda_c\}, \quad \varepsilon_c \geq 0, \\ W_{c,\varphi_c,\lambda_c,\Phi} &= \{x \in C : (\varphi_c \circ \Phi)(u,v) < \lambda_c \text{ for any } (u,v) \in F(x) \times G(x)\}, \\ U_{c,\varphi_c,\lambda_c,\Phi} &= \left\{x \in C : \sup_{(u,v)\in F(x)\times G(x)} (\varphi_c \circ \Phi)(u,v) \leq \lambda_c\right\} \end{aligned} \tag{2.5}$$

when $\lambda_c < 0$,

$$\begin{aligned} H_{c,\varphi_c,\lambda_c,\varepsilon_c} &= \{w \in E : \varphi_c(w) > (1+\varepsilon_c)\lambda_c\}, \quad \varepsilon_c \geq 0, \\ W_{c,\varphi_c,\lambda_c,\Phi} &= \{x \in C : (\varphi_c \circ \Phi)(u,v) > \lambda_c \text{ for any } (u,v) \in F(x) \times G(x)\}, \\ U_{c,\varphi_c,\lambda_c,\Phi} &= \left\{x \in C : \inf_{(u,v)\in F(x)\times G(x)} (\varphi_c \circ \Phi)(u,v) \geq \lambda_c\right\} \end{aligned} \tag{2.6}$$

when $\lambda_c > 0$.

By Definition 2.1, we see that the pair (F,G) is Φ-t.p.h.c. or Φ-t.h.c. on C if, whenever $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c \geq 0$ are such that the set $\Phi(F(c) \times G(c))$ is contained in open half-space $H(c,\varphi_c,\lambda_c,\varepsilon_c)$ (here $\varepsilon_c > 0$ in the case of Φ-transfer positive hemicontinuity and $\varepsilon_c = 0$ in the case of Φ-transfer hemicontinuity), then the following hold: (i) there exists a neighbourhood $N(c)$ of c in C such that, for any $x \in N(c)$, the set $\Phi(F(x)\times G(x))$ is contained in open half-space $H_{c,\varphi_c,\lambda_c,0}$; (ii) c is an interior point of the sets $W_{c,\varphi_c,\lambda_c,\Phi}$ and $U_{c,\varphi_c,\lambda_c,\Phi}$. Indeed, then $\lambda_c[(\varphi_c\circ\Phi)(u,v)-\lambda_c] > 0$ for any $x\in N(c)$ and any $(u,v)\in F(x)\times G(x)$.

Definition 2.3. Let C be a nonempty subset of E, let $F: C \to 2^E$ and let $G: C\to 2^E$. Let $\Phi: \bigcup_{c\in C}(F(c)\times G(c)) \to E$ be a single-valued map.

(a) We say that a pair (F,G) is Φ-*strictly transfer positive hemicontinuous* (Φ-*s.t.p.h.c.*) on C if, whenever $(c,\varphi_c,\lambda_c)\in C\times E'\times\mathbb{R}$ and $\varepsilon_c > 0$ are such that

$$\lambda_c\big[(\varphi_c\circ\Phi)(u,v)-(1+\varepsilon_c)\lambda_c\big] > 0 \quad \text{for any } (u,v)\in F(c)\times G(c), \tag{2.7}$$

then c is an interior point of the set $V_{c,\varphi_c,\lambda_c,\Phi}$, where

$$\begin{aligned} V_{c,\varphi_c,\lambda_c,\Phi} &= \Big\{x\in C: \sup_{(u,v)\in F(x)\times G(x)}(\varphi_c\circ\Phi)(u,v) < \lambda_c\Big\} \quad \text{if } \lambda_c < 0,\\ V_{c,\varphi_c,\lambda_c,\Phi} &= \Big\{x\in C: \inf_{(u,v)\in F(x)\times G(x)}(\varphi_c\circ\Phi)(u,v) > \lambda_c\Big\} \quad \text{if } \lambda_c > 0. \end{aligned} \tag{2.8}$$

(b) We say that a pair (F,G) is Φ-*strictly transfer hemicontinuous* (Φ-*s.t.h.c.*) on C if, whenever $(c,\varphi_c,\lambda_c)\in C\times E'\times\mathbb{R}$ is such that

$$\lambda_c\big[(\varphi_c\circ\Phi)(u,v)-\lambda_c\big] > 0 \quad \text{for any } (u,v)\in F(c)\times G(c), \tag{2.9}$$

then c is an interior point of the set $V_{c,\varphi_c,\lambda_c,\Phi}$.

(c) We say that a map F is Φ-*s.t.p.h.c.* or Φ-*s.t.h.c.* on C if a pair (F,I_E) is Φ-s.t.p.h.c. or Φ-s.t.h.c. on C, respectively.

(d) We say that a pair (F,G) is *strictly transfer positive hemicontinuous* (*s.t.p.h.c.*) or *strictly transfer hemicontinuous* (*s.t.h.c.*) on C if (F,G) is Φ-s.t.p.h.c. or Φ-s.t.h.c. on C, respectively, for Φ of the form $\Phi(u,v)=u-v$ where $(u,v)\in F(c)\times G(c)$ and $c\in C$.

(e) We say that a map F is *s.t.p.h.c.* or *s.t.h.c.* on C if a pair (F,I_E) is s.t.p.h.c. or s.t.h.c. on C, respectively.

Proposition 2.4. *Let C be a nonempty subset of E, let $F: C\to 2^E$ and let $G: C\to 2^E$. Let $\Phi:\bigcup_{c\in C}(F(c)\times G(c))\to E$ be a single-valued map.*

(i) *If (F,G) is Φ-t.h.c. on C, then (F,G) is Φ-t.p.h.c. on C.*

(ii) *If (F,G) is Φ-t.p.h.c. on C and, for each $x\in C$, $\Phi(F(x)\times G(x))$ is compact, then (F,G) is Φ-t.h.c. on C.*

(iii) *If (F,G) is Φ-s.t.h.c. on C, then (F,G) is Φ-s.t.p.h.c. on C.*

(iv) *If (F,G) is Φ-s.t.p.h.c. on C and, for each $x\in C$, $\Phi(F(x)\times G(x))$ is compact, then (F,G) is Φ-s.t.h.c. on C.*

(v) *If (F,G) is Φ-s.t.p.h.c. (Φ-s.t.h.c., resp.) on C, then (F,G) is Φ-t.p.h.c. (Φ-t.h.c., resp.) on C.*

(vi) *If* (F,G) *is* Φ*-t.p.h.c. (*Φ*-t.h.c., resp.) on* C *and, for each* $x \in C$, $\Phi(F(x) \times G(x))$ *is compact, then* (F,G) *is* Φ*-s.t.p.h.c. (*Φ*-s.t.h.c., resp.) on* C.

Proof. (i) Let (F,G) be Φ-t.h.c. on C and assume that there exist $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c > 0$ such that $\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ or, equivalently, $(1+\varepsilon_c)\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $(u,v) \in F(c) \times G(c)$. Then, by Φ-transfer hemicontinuity, there exists a neighbourhood $N(c)$ of c in C such that $(1+\varepsilon_c)\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$. This implies, in particular, that $\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$, that is, (F,G) is Φ-t.p.h.c. on C.

(ii) Let (F,G) be Φ-t.p.h.c. on C and let there exists $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ such that, for any $(u,v) \in F(c) \times G(c)$, $\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0$ or, equivalently, for any $(u,v) \in F(c) \times G(c)$, $(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$ and $(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$. Since, for each $x \in C$, $\Phi(F(x) \times G(x))$ is compact, thus $\sup_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$ and $\inf_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$, so there is some $\varepsilon_c > 0$ such that $\sup_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) < (1+\varepsilon_c)\lambda_c$ if $\lambda_c < 0$ and $\inf_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) > (1+\varepsilon_c)\lambda_c$ if $\lambda_c > 0$. Therefore, for any $(u,v) \in F(c) \times G(c)$, $(\varphi_c \circ \Phi)(u,v) < (1+\varepsilon_c)\lambda_c$ if $\lambda_c < 0$ and $(\varphi_c \circ \Phi)(u,v) > (1+\varepsilon_c)\lambda_c$ if $\lambda_c > 0$ or, equivalently, $\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $(u,v) \in F(c) \times G(c)$. Then, by Φ-transfer positive hemicontinuity, there exists a neighbourhood $N(c)$ of c in C such that $\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$, that is, (F,G) is Φ-t.h.c. on C.

(iii) Let (F,G) be Φ-s.t.h.c. on C and assume that there exist $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c > 0$ such that $\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ or, equivalently, $(1+\varepsilon_c)\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $(u,v) \in F(c) \times G(c)$. Then, by Φ-strictly transfer hemicontinuity, c is an interior point of the set $V_{c,\varphi_c,(1+\varepsilon_c)\lambda_c,\Phi}$. But $V_{c,\varphi_c,(1+\varepsilon_c)\lambda_c,\Phi} \subset V_{c,\varphi_c,\lambda_c,\Phi}$. This implies, in particular, that c is an interior point of the set $V_{c,\varphi_c,\lambda_c,\Phi}$, that is, (F,G) is Φ-s.t.p.h.c. on C.

(iv) Let (F,G) be Φ-s.t.p.h.c. on C and let there exists $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ such that, for any $(u,v) \in F(c) \times G(c)$, $\lambda_c[(\varphi_c \circ \Phi)(u,v) - \lambda_c] > 0$ or, equivalently, for any $(u,v) \in F(c) \times G(c)$, $(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$ and $(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$. Since, for each $x \in C$, $\Phi(F(x) \times G(x))$ is compact, thus $\sup_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$ and $\inf_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$, so there is some $\varepsilon_c > 0$ such that $\sup_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) < (1+\varepsilon_c)\lambda_c$ if $\lambda_c < 0$ and $\inf_{(u,v)\in F(c)\times G(c)}(\varphi_c \circ \Phi)(u,v) > (1+\varepsilon_c)\lambda_c$ if $\lambda_c > 0$. Therefore, for any $(u,v) \in F(c) \times G(c)$, $(\varphi_c \circ \Phi)(u,v) < (1+\varepsilon_c)\lambda_c$ if $\lambda_c < 0$ and $(\varphi_c \circ \Phi)(u,v) > (1+\varepsilon_c)\lambda_c$ if $\lambda_c > 0$ or, equivalently, $\lambda_c[(\varphi_c \circ \Phi)(u,v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $(u,v) \in F(c) \times G(c)$. Then, by Φ-strictly transfer positive hemicontinuity, c is an interior point of the set $V_{c,\varphi_c,\lambda_c,\Phi}$, that is, (F,G) is Φ-s.t.p.h.c. on C.

(v) By Definitions 2.1 and 2.3 and Remark 2.2, we see that $V_{c,\varphi_c,\lambda_c,\Phi} \subset W_{c,\varphi_c,\lambda_c,\Phi}$.

(vi) By Definition 2.1, the pair (F,G) is Φ-t.p.h.c. or Φ-t.h.c. on C if, whenever $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c \geq 0$ are such that the set $\Phi(F(c) \times G(c))$ is contained in open half-space $H(c,\varphi_c,\lambda_c,\varepsilon_c)$ (here $\varepsilon_c > 0$ in the case of Φ-transfer positive hemicontinuity and $\varepsilon_c = 0$ in the case of Φ-transfer hemicontinuity), then there exists a neighbourhood $N(c)$ of c in C such that, for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$, $(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$ and $(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$. Since, for each $x \in C$, $\Phi(F(x) \times G(x))$ is compact, thus, for each $x \in N(c)$, $\sup_{(u,v)\in F(x)\times G(x)}(\varphi_c \circ \Phi)(u,v) < \lambda_c$ if $\lambda_c < 0$

and $\inf_{(u,v)\in F(x)\times G(x)}(\varphi_c \circ \Phi)(u,v) > \lambda_c$ if $\lambda_c > 0$. Consequently, $N(c) \subset V_{c,\varphi_c,\lambda_c,\Phi}$, that is, c is an interior point of the set $V_{c,\varphi_c,\lambda_c,\Phi}$. □

Remark 2.5. This proves, in particular, that the condition of strictly transfer positive hemicontinuity is stronger than that of transfer positive hemicontinuity.

Definition 2.6. Let C be a nonempty compact convex subset of E. We say that $(c,\varphi) \in C \times (E' \setminus \{0\})$ is *admissible* if $\varphi(c) = \min_{x\in C}\varphi(x)$; thus if (c,φ) is admissible, then this means that the *closed hyperplane* determined by φ of the form $\{x \in E : \varphi(x) = \varphi(c)\}$ is a *supporting hyperplane* of C at c.

Definition 2.7. Let C be a nonempty subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$. Let $\Phi : \bigcup_{c\in C}(F(c)\times G(c)) \to E$ be a single-valued map.

(a) A pair (F,G) is called Φ-*inward* (Φ-*outward*, resp.) if, for any admissible $(c,\varphi) \in C \times (E' \setminus \{0\})$ there is a point $(u,v) \in F(c)\times G(c)$ such that $(\varphi \circ \Phi)(u,v) \geq 0$ ($(\varphi \circ \Phi)(u,v) \leq 0$, resp.).

(b) A map F is called Φ-*inward* (Φ-*outward*, resp.) if the pair (F,I_E) is Φ-inward (Φ-outward, resp.).

(c) A pair (F,G) is called *inward* (*outward*, resp.) if the pair (F,G) is Φ-inward (Φ-outward, resp.) for Φ of the form $\Phi(u,v) = u - v$ where $(u,v) \in F(c)\times G(c)$ and $c \in C$.

(d) A map F is called *inward* (*outward*, resp.) (see Fan [11, page 108]) if a pair (F,I_E) is inward (outward, resp.).

Definition 2.8. Let C be a nonempty subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$. Let $\Phi : \bigcup_{c\in C}(F(c)\times G(c)) \to E$ be a single-valued map.

(a) We say that a pair (F,G) has a Φ-*coincidence* if there exist $c \in C$ and $(u,v) \in F(c) \times G(c)$, such that $\Phi(u,v) = 0$, that is, $(u,v) \in F(c)\times G(c)$ is a zero of Φ; this point c is called a Φ-*coincidence point* for (F,G).

(b) We say that a map F has a Φ-*fixed point* (a pair (F,I_E) has a Φ-coincidence) if there exist $c \in C$ and $u \in F(c)$, such that $\Phi(u,c) = 0$; this point c is called a Φ-*fixed point* for F.

(c) We say that a pair (F,G) has a *coincidence* if there exist $c \in C$ and $(u,v) \in F(c) \times G(c)$, such that $u = v$; this point c is called a *coincidence point* for (F,G).

(d) We say that F has a *fixed point* if there exists $c \in C$ such that $c \in F(c)$; this point c is called a *fixed point* for F.

With the background given, the first result of our paper can now be presented.

THEOREM 2.9. *Let E be a real Hausdorff topological vector space. Let C be a nonempty compact convex subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$. Let $\Phi : \bigcup_{c\in C}(F(c)\times G(c)) \to E$ be a single-valued map.*

(i) *Let the pair (F,G) be Φ-t.p.h.c. on C. If (F,G) is Φ-inward or Φ-outward, then there exists $c_0 \in C$ such that, for any $\varphi \in E'$, there is no $\lambda \in \mathbb{R}$ such that $\lambda[(\varphi \circ \Phi)(u,v) - \lambda] > 0$ for all $(u,v) \in F(c_0)\times G(c_0)$.*

(ii) *Let F be Φ-t.p.h.c. on C. If F is Φ-inward or Φ-outward, then there exists $c_0 \in C$ such that, for any $\varphi \in E'$, there is no $\lambda \in \mathbb{R}$ such that $\lambda[(\varphi \circ \Phi)(u,c_0) - \lambda] > 0$ for all $u \in F(c_0)$.*

(iii) *Let the pair (F,G) be t.p.h.c. on C. If (F,G) is inward or outward, then there exists $c_0 \in C$ such that, for any $\varphi \in E'$, there is no $\lambda \in \mathbb{R}$ such that $\lambda[\varphi(u - v) - \lambda] > 0$ for all $(u,v) \in F(c_0)\times G(c_0)$.*

(iv) *Let F be t.p.h.c. on C. If F is inward or outward, then there exists $c_0 \in C$ such that, for any $\varphi \in E'$, there is no $\lambda \in \mathbb{R}$ such that $\lambda[\varphi(u - c_0) - \lambda] > 0$ for all $u \in F(c_0)$.*

Proof. (i) Assume that, for any admissible $(c,\varphi) \in C \times (E' \setminus \{0\})$, there exists $(u,v) \in F(c) \times G(c)$ such that

$$(\varphi \circ \Phi)(u,v) \geq 0 \tag{2.10}$$

and assume that the assertion does not hold, that is, without loss of generality, for any $c \in C$, there exist $\varphi_c \in E' \setminus \{0\}$, $\lambda_c < 0$ and $\varepsilon_c \geq 0$, such that

$$(\varphi_c \circ \Phi)(u,v) < (1+\varepsilon_c)\lambda_c \quad \forall (u,v) \in F(c) \times G(c). \tag{2.11}$$

By Definition 2.1(a), there exists a neighbourhood $N(c)$ of c in C such that

$$(\varphi_c \circ \Phi)(u,v) < \lambda_c \quad \text{for any } x \in N(c) \text{ and any } (u,v) \in F(x) \times G(x). \tag{2.12}$$

Since the family $\{N(c) : c \in C\}$ is an open cover of a compact set C, there exists a finite subset $\{c_1,\dots,c_n\}$ of C such that the family $\{N(c_j) : j = 1,2,\dots,n\}$ covers C. Let $\{\beta_1,\dots,\beta_n\}$ be a partition of unity with respect to this cover, that is, a finite family of real-valued nonnegative continuous maps β_j on C such that β_j vanish outside $N(c_j)$ and are less than or equal to one everywhere, $1 \leq j \leq n$, and $\sum_{j=1}^{n} \beta_j(c) = 1$ for all $c \in C$.

Define $\eta(c) = \sum_{j=1}^{n} \beta_j(c)\varphi_{c_j}$ for $c \in C$. Then $\eta(c) \in E'$ for each $c \in C$. Therefore

$$([\eta(c)] \circ \Phi)(u,v) < \lambda \tag{2.13}$$

for any $c \in C$ and $(u,v) \in F(c) \times G(c)$, where $\lambda = \max_{1 \leq j \leq n} \lambda_{c_j} < 0$ since

$$([\eta(c)] \circ \Phi)(u,v) = \sum_{j=1}^{n} \beta_j(c)(\varphi_{c_j} \circ \Phi)(u,v) < \sum_{j=1}^{n} \beta_j(c)\lambda_{c_j}. \tag{2.14}$$

Let now $k : C \times C \to \mathbb{R}$ be a continuous map of the form $k(c,x) = [\eta(c)](c - x)$ for $(c,x) \in C \times C$. Since, for each $c \in C$, the map $k(c,\cdot)$ is quasi-concave on C, therefore, by [11, page 103], the following minimax inequality

$$\min_{c \in C} \max_{x \in C} k(c,x) \leq \max_{c \in C} k(c,c) \tag{2.15}$$

holds. But $k(c,c) = 0$ for each $c \in C$, so there is some $c_0 \in C$ such that $k(c_0,x) \leq 0$ for all $x \in C$. Since

$$[\eta(c_0)](c_0) = \min_{x \in C} [\eta(c_0)](x), \tag{2.16}$$

we have that $(c_0, \eta(c_0)) \in C \times (E' \setminus \{0\})$ is admissible and, by (2.13),

$$([\eta(c_0)] \circ \Phi)(u,v) < \lambda \quad \text{for any } (u,v) \in F(c_0) \times G(c_0), \tag{2.17}$$

which is impossible by (2.10).

(ii)–(iv) The argumentation is analogous and will be omitted. □

Two sets X and Y in E can be *strictly separated by a closed hyperplane* if there exist $\varphi \in E'$ and $\lambda \in \mathbb{R}$, such that $\varphi(x) < \lambda < \varphi(y)$ for each $(x,y) \in X \times Y$.

Theorem 2.9 has the following consequence.

Theorem 2.10. *Let E be a real Hausdorff topological vector space. Let C be a nonempty compact convex subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$. Let $\Phi : \bigcup_{c \in C}(F(c) \times G(c)) \to E$ be a single-valued map.*

(i) *Let the pair (F,G) be Φ-t.p.h.c. on C and inward or outward. Then there exists $c_0 \in C$ such that $\Phi(F(c_0) \times G(c_0))$ and $\{0\}$ cannot be strictly separated by any closed hyperplane in E. If, additionally, E is locally convex and, for each $c \in C$, the set $\Phi(F(c) \times G(c))$ is closed and convex, then a pair (F,G) has a Φ-coincidence.*

(ii) *Let F be Φ-t.p.h.c. on C and inward or outward. Then there exists $c_0 \in C$ such that $\Phi(F(c_0) \times \{c_0\})$ and $\{0\}$ cannot be strictly separated by any closed hyperplane in E. If, additionally, E is locally convex and, for each $c \in C$, the set $\Phi(F(c) \times \{c\})$ is closed and convex, then a map F has a Φ-fixed point.*

(iii) *Let the pair (F,G) be t.p.h.c. on C and inward or outward. Then, the following hold:*

(iii_1) *if, for each $c \in C$, at least one of the sets $F(c)$ or $G(c)$ is compact, then there exists $c_0 \in C$ such that $F(c_0)$ and $G(c_0)$ cannot be strictly separated by any closed hyperplane in E;*

(iii_2) *if E is locally convex and, for each $c \in C$, the sets $F(c)$ and $G(c)$ are convex and closed and at least one of them is compact, then there exists $c_0 \in C$ such that $F(c_0)$ and $G(c_0)$ have a nonempty intersection.*

(iv) *Let $F : C \to 2^E$ be t.p.h.c. on C and inward or outward. Then, the following hold:*

(iv_1) *there exists $c_0 \in C$ such that $F(c_0)$ and $\{c_0\}$ cannot be strictly separated by any closed hyperplane in E;*

(iv_2) *if E is locally convex and, for each $c \in C$, the set $F(c)$ is closed and convex, then there exists $c_0 \in C$ such that $c_0 \in F(c_0)$.*

Proof. (i) Let us observe that if we assume that the following condition holds:

$$(1+\varepsilon)\lambda\big[(\varphi \circ \Phi)(u,v) - (1+\varepsilon)\lambda\big] > 0 \tag{2.18}$$

for some $\lambda \in \mathbb{R}$, $\varphi \in E'$ and $\varepsilon \geq 0$, and for all $(u,v) \in F(c_0) \times G(c_0)$, then we obtain that, for all $(u,v) \in F(c_0) \times G(c_0)$, $(\varphi \circ \Phi)(u,v) < (1+\varepsilon)\lambda \leq \lambda < \varphi(0)$ if $\lambda < 0$ and $(\varphi \circ \Phi)(u,v) > (1+\varepsilon)\lambda \geq \lambda > \varphi(0)$ if $\lambda > 0$, that is, the sets $\Phi(F(c_0) \times G(c_0))$ and $\{0\}$ are strictly separated by a closed hyperplane in E.

Otherwise, assume that, for all $(u,v) \in F(c_0) \times G(c_0)$, $(\varphi \circ \Phi)(u,v) < t_1 < \varphi(0)$ for some $t_1 \in \mathbb{R}$ or $(\varphi \circ \Phi)(u,v) > t_2 > \varphi(0)$ for some $t_2 \in \mathbb{R}$. Then we obtain that, for all $(u,v) \in F(c_0) \times G(c_0)$, $(\varphi \circ \Phi)(u,v) < (1+\varepsilon)\lambda_1 < 0$ where $(1+\varepsilon)\lambda_1 = t_1$ or $(\varphi \circ \Phi)(u,v) > (1+\varepsilon)\lambda_2 > 0$ where $(1+\varepsilon)\lambda_2 = t_2$. Therefore condition (2.18) is then satisfied.

The above considerations, Theorem 2.9(i) and the separation theorem yield the assertion.

(ii) This is a consequence of (i).

(iii) Assume, without loss of generality, that $G(c_0)$ is compact.

Let us observe that if we assume that the following condition holds:

$$(1+\varepsilon)\lambda[\varphi(u-v)-(1+\varepsilon)\lambda]>0 \tag{2.19}$$

for some $\lambda \in \mathbb{R}$ and $\varepsilon \geq 0$ and for all $(u,v)\in F(c_0)\times G(c_0)$, then we obtain that, for all $(u,v)\in F(c_0)\times G(c_0)$, $\varphi(u)<t_2<\varphi(v)$ where $t_2=(1+\varepsilon)\lambda+\min_{w\in G(c_0)}\varphi(w)$ if $\lambda<0$ and $\varphi(u)>t_1>\varphi(v)$ where $t_1=(1+\varepsilon)\lambda+\max_{w\in G(c_0)}\varphi(w)$ if $\lambda>0$, that is, the sets $F(c_0)$ and $G(c_0)$ are strictly separated by a closed hyperplane in E.

Otherwise, assume that, for all $(u,v)\in F(c_0)\times G(c_0)$, $\varphi(u)>t_1>\varphi(v)$ for some $t_1\in\mathbb{R}$ or $\varphi(u)<t_2<\varphi(v)$ for some $t_2\in\mathbb{R}$. Then we obtain that, for all $(u,v)\in F(c_0)\times G(c_0)$, $\varphi(u-v)>(1+\varepsilon)\lambda_1>0$ where $(1+\varepsilon)\lambda_1=t_1-\max_{w\in G(c_0)}\varphi(w)$ or $\varphi(u-v)<(1+\varepsilon)\lambda_2<0$ where $(1+\varepsilon)\lambda_2=t_2-\min_{w\in G(c_0)}\varphi(w)$, respectively. Therefore condition (2.19) is then satisfied.

The above considerations, Theorem 2.9(iii) and the separation theorem yield the assertion.

(iv) This is a consequence of (iii). □

We now prove the result under stronger condition.

Theorem 2.11. *Let E be a real Hausdorff topological vector space, let C be a nonempty compact convex subset of E and suppose that $F: C\to 2^E$ and $G: C\to 2^E$.*

(i) *Denote by Φ a single-valued map of $\bigcup_{c\in C}(F(c)\times G(c))$ into E such that, for each $c\in C$, $\Phi(F(c)\times G(c))$ is convex and compact and let the pair (F,G) be Φ-t.h.c. on C. Then the following hold:* (i_1) *either (F,G) has a Φ-coincidence or there exists $\lambda\in\mathbb{R}$ and, for any $c\in C$, there exists $\varphi_c\in E'$ such that $\lambda[(\varphi_c\circ\Phi)(u,v)-\lambda]>0$ for all $(u,v)\in F(c)\times G(c)$;* (i_2) *if the pair (F,G) is Φ-inward or Φ-outward, then (F,G) has a Φ-coincidence.*

(ii) *Denote by Φ a single-valued map of $\bigcup_{c\in C}(F(c)\times\{c\})$ into E such that, for each $c\in C$, $\Phi(F(c)\times\{c\})$ is convex and compact and assume that F is Φ-t.h.c. on C. Then the following hold:* (ii_1) *either F has a Φ-fixed point or there exists $\lambda\in\mathbb{R}$ and, for any $c\in C$, there exists $\varphi_c\in E'$ such that $\lambda[(\varphi_c\circ\Phi)(u,c)-\lambda]>0$ for all $u\in F(c)$;* (ii_2) *if F is Φ-inward or Φ-outward, then F has a Φ-fixed point.*

(iii) *Suppose that $F(c)$ and $G(c)$ are compact subsets of E and $F(c)-G(c)$ is convex for each $c\in C$ and assume that the pair (F,G) is t.h.c. on C. Then the following hold:* (iii_1) *either (F,G) has a coincidence or there exists $\lambda\in\mathbb{R}$ and, for any $c\in C$, there exists $\varphi_c\in E'$ such that $\lambda[\varphi_c(u-v)-\lambda]>0$ for all $(u,v)\in F(c)\times G(c)$;* (iii_2) *if the pair (F,G) is inward or outward, then (F,G) has a coincidence;* (iii_3) *either (F,G) has a coincidence or, for any $c\in C$, the sets $F(c)$ and $G(c)$ are strictly separated by a closed hyperplane in E.*

(iv) *Suppose that F is a t.h.c. map on C such that, for each $c\in C$, $F(c)$ is convex and compact. Then the following hold:* (iv_1) *either F has a fixed point or there exists $\lambda\in\mathbb{R}$ and, for any $c\in C$, there exists $\varphi_c\in E'$ such that $\lambda[\varphi_c(u-c)-\lambda]>0$ for all $u\in F(c)$;* (iv_2) *if F is inward or outward, then F has a fixed point;* (iv_3) *either F has a fixed point or, for any $c\in C$, the sets $F(c)$ and $\{c\}$ are strictly separated by a closed hyperplane in E.*

Proof. (i_1) Assume that (F,G) has no Φ-coincidence in C. Then, for all $c \in C$, the set D_c, $D_c = \Phi(F(c) \times G(c))$, is convex, compact and $0 \notin D_c$.

For $(c,w) \in C \times D_c$, there exists $\varphi_{c,w} \in E'$ such that $\varphi_{c,w}(w) \neq 0$ and we assume, without loss of generality, that, $\varphi_{c,w}(w) > 0$ for each $(c,w) \in C \times D_c$.

First, let us observe that:

(a) *for each $c \in C$, there exist $\varphi_c \in E'$ and $\lambda_c > 0$, such that*

$$(\varphi_c \circ \Phi)(u,v) > \lambda_c \quad \text{for any } (u,v) \in F(c) \times G(c). \tag{2.20}$$

Indeed, by the continuity of $\varphi_{c,w}$, we define a neighbourhood $M_c(w)$ of w in D_c such that

$$M_c(w) \subset \{x \in D_c : \varphi_{c,w}(x) > \varphi_{c,w}(w)/2\}. \tag{2.21}$$

Clearly, there exists a finite subset $\{w_1,\dots,w_m\}$ of D_c such that $M_c(w_i)$ are nonempty, $1 \le i \le n$, and $D_c = \bigcup_{i=1}^m M_c(w_i)$. Let $\{\alpha_1,\dots,\alpha_m\}$ be a partition of unity with respect to this cover, that is, a finite family of real-valued nonnegative continuous maps α_i on D_c such that α_i vanish outside $M_c(w_i)$ and are less than or equal to one everywhere, $1 \le i \le m$, and $\sum_{i=1}^m \alpha_i(w) = 1$ for all $w \in D_c$. Define

$$\psi_c(w) = \sum_{i=1}^m \alpha_i(w)\varphi_{c,w_i} \quad \text{for } w \in D_c. \tag{2.22}$$

Then $\psi_c(w) \in E'$ for each $w \in D_c$.

Now, let $h_c : D_c \times D_c \to \mathbb{R}$ be of the form

$$h_c(w,y) = [\psi_c(w)](w-y) \quad \text{for } (w,y) \in D_c \times D_c. \tag{2.23}$$

Thus h_c is continuous on $D_c \times D_c$ and, for each $w \in D_c$, the map $h_c(w,\cdot)$ is quasi-concave on D_c. By [11, page 103], the following minimax inequality

$$\min_{w\in D_c} \max_{y\in D_c} h_c(w,y) \le \max_{w\in D_c} h_c(w,w) \tag{2.24}$$

holds. But $h_c(w,w) = 0$ for each $w \in D_c$, so there is some $w_c \in D_c$ such that $h_c(w_c,y) \le 0$ for all $y \in D_c$. Then

$$[\psi(w_c)](w_c) = \min_{y\in D_c} [\psi(w_c)](y). \tag{2.25}$$

Since $w_c \in M_c(w_i)$ for some $1 \le i \le m$, therefore $\alpha_i(w_c) > 0$ and

$$[\psi_c(w_c)](w_c) = \alpha_i(w_c)\varphi_{c,w_i}(w_c) \ge \alpha_i(w_c)\varphi_{c,w_i}(w_i)/2 > 0. \tag{2.26}$$

Consequently, we may assume that

$$\varphi_c = \psi_c(w_c), \qquad \lambda_c = \alpha_i(w_c)\varphi_{c,w_i}(w_i)/4, \tag{2.27}$$

where $\lambda_c > 0$. Thus (a) is proved.

Using (a), since (F,G) is Φ-t.h.c. on C, we get:

(b) *for each* $c \in C$, *there exist* $\varphi_c \in E'$, $\lambda_c > 0$ *and a neighbourhood* $N(c)$ *of* c *in* C, *such that*

$$(\varphi_c \circ \Phi)(u,v) > \lambda_c \quad \text{for any } x \in N(c) \text{ and any } (u,v) \in F(x) \times G(x). \tag{2.28}$$

Now, we prove:

(c) *there exists* $\lambda > 0$ *and, for any* $c \in C$, *there exists* $\varphi_c \in E'$ *such that*

$$(\varphi_c \circ \Phi)(u,v) > \lambda \quad \forall (u,v) \in F(c) \times G(c). \tag{2.29}$$

Indeed, for each $c \in C$, let φ_c, λ_c and $N(c)$ be as in (b). Since the family $\{N(c) : c \in C\}$ is an open cover of a compact set C, there exists a finite subset $\{c_1,\dots,c_n\}$ of C such that the family $\{N(c_j) : j = 1,2,\dots,n\}$ covers C. Let $\{\beta_1,\dots,\beta_n\}$ be a partition of unity with respect to this cover, that is, a finite family of real-valued nonnegative continuous maps β_j on C such that β_j vanish outside $N(c_j)$ and are less than or equal to one everywhere, $1 \le j \le n$, and $\sum_{j=1}^n \beta_j(c) = 1$ for all $c \in C$.

Define $\eta(c) = \sum_{j=1}^n \beta_j(c)\varphi_{c_j}$ for $c \in C$. Then $\eta(c) \in E'$ for each $c \in C$.

If $c \in C$ and the index j are such that $\beta_j(c) > 0$, then

$$c \in N(c_j) \subset \{x \in C : \varphi_{c_j}(w) > \lambda_{c_j} \ \forall w \in D_c\}. \tag{2.30}$$

Consequently, for any $c \in C$ and $w \in D_c$, we have

$$[\eta(c)](w) = \sum_{j=1}^{n} \beta_j(c)\varphi_{c_j}(w) > \sum_{j=1}^{n} \beta_j(c)\lambda_{c_j} \ge \min_{1\le j\le n} \lambda_{c_j}, \tag{2.31}$$

whence it follows that we may assume that $\lambda = (1/2)\min_{1\le j\le n}\lambda_{c_j} > 0$ and, for any $c \in C$, $\varphi_c = \eta(c)$.

(i_2) First, let us observe that if $k : C \times C \to \mathbb{R}$ is a map of the form $k(c,x) = [\eta(c)](c-x)$ for $(c,x) \in C \times C$, where $\eta(c)$ is constructed in the proof of (i_1), then k is continuous on $C \times C$ and, for each $c \in C$, the map $k(c,\cdot)$ is quasi-concave on C. By [11, page 103], the following minimax inequality

$$\min_{c\in C}\max_{x\in C} k(c,x) \le \max_{c\in C} k(c,c) \tag{2.32}$$

holds. But $k(c,c) = 0$ for each $c \in C$, so there is some $c_0 \in C$ such that $k(c_0,x) \le 0$ for all $x \in C$. Since

$$[\eta(c_0)](c_0) = \min_{x\in C} [\eta(c_0)](x), \tag{2.33}$$

we have that $(c_0,\eta(c_0)) \in C \times (E' \setminus \{0\})$ is admissible.

Assume now that, for any admissible $(c,\varphi) \in C \times (E' \setminus \{0\})$, there exists $(u,v) \in F(c) \times G(c)$ such that $(\varphi \circ \Phi)(u,v) \ge 0$ (or $(\varphi \circ \Phi)(u,v) \le 0$) but (F,G) has no Φ-coincidence. From assertion (i) and its proof we then have that there exists $\lambda < 0$ (or $\lambda > 0$) such that $([\eta(c_0)] \circ \Phi)(u,v) < \lambda$ (or $([\eta(c_0)] \circ \Phi)(u,v) > \lambda$) for all $(u,v) \in F(c_0) \times G(c_0)$. We obtain a contradiction.

(ii_1), (ii_2), (iii_1) and (iii_2) This is a consequence of (i).

(iii_3) By (iii_1), if (F,G) has no coincidence, then, for any $c \in C$ and for any $(u,v) \in F(c) \times G(c)$, we have that $\varphi_c(u) < \lambda + \min_{w\in G(c)} \varphi_c(w) < \varphi_c(v)$ if $\lambda < 0$, and $\varphi_c(u) > \lambda + \max_{w\in G(c)} \varphi_c(w) > \varphi_c(v)$ if $\lambda > 0$.

(iv_1)–(iv_3) This is a consequence of (iii). □

3. Comparison of transfer positive hemicontinuity and strictly transfer positive hemicontinuity with upper demicontinuity and upper hemicontinuity

We say that $F : C \to 2^E$ is *upper semicontinuous* (*u.s.c.*) (see Berge [2, Chapter VI]) if, for each $c \in C$ and an arbitrary neighbourhood V of $F(c)$, there is a neighbourhood $N(c)$ of c in C such that $F(x) \subset V$ for each $x \in N(c)$.

A map $F : C \to 2^E$ is called *upper demicontinuous* (*u.d.c.*) on C (after Fan [10]) if, for each $c \in C$ and any open half-space H in E containing $F(c)$, there is a neighbourhood $N(c)$ of c in C such that $F(x) \subset H$ for each $x \in N(c)$.

The upper demicontinuity for set valued maps, defined by Fan, generalizes the upper demicontinuity studied by Browder [4] for single valued maps.

A map $F : C \to 2^E$ is called *upper hemicontinuous* (*u.h.c.*) on C (see Aubin and Ekeland [1]) if for each $\varphi \in E' \setminus \{0\}$ and any $\lambda \in \mathbb{R}$ the set

$$\left\{x \in C : \sup_{u\in F(x)} \varphi(u) < \lambda\right\} \tag{3.1}$$

is open in C.

It is clear that every u.s.c. map is u.d.c. and each u.d.c. is u.h.c.

The following result says that the conditions of upper demicontinuity and upper hemicontinuity are stronger than that of transfer positive hemicontinuity.

Proposition 3.1. *Let C be a nonempty subset of E, let $F : C \to 2^E$ and let $G : C \to 2^E$.*

(i) *If F and G are u.d.c., then the pair (F,G) is t.p.h.c.*

(ii) *If F is u.d.c., then F is t.h.c.*

(iii) *If F and G are u.h.c., then the pair (F,G) is t.p.h.c.*

(iv) *If F is u.h.c., then F is t.p.h.c.*

Proof. (i) Assume that $F : C \to 2^E$ and $G : C \to 2^E$ are u.d.c. on C and assume that there exist $(c,\varphi_c,\lambda_c) \in C \times E' \times \mathbb{R}$ and $\varepsilon_c > 0$, such that

$$\lambda_c\left[\varphi_c(u-v) - (1+\varepsilon_c)\lambda_c\right] > 0 \quad \text{for any } (u,v) \in F(c) \times G(c). \tag{3.2}$$

Assume that

$$\lambda_c > 0, \quad \varphi_c(u-v) > (1+\varepsilon_c)\lambda_c \quad \text{for any } (u,v) \in F(c) \times G(c) \tag{3.3}$$

(if we replace assumption (3.3) by $\lambda_c < 0$ and $\varphi_c(u-v) < (1+\varepsilon_c)\lambda_c$ for any $(u,v) \in F(c) \times G(c)$, then the argumentation is analogous). Let, for some $\eta \geq \varepsilon_c$,

$$\inf_{(u,v)\in F(c)\times G(c)} \varphi_c(u-v) = (1+\eta)\lambda_c \tag{3.4}$$

and let $\tau > 0$ be such that $\tau < \eta\lambda_c$. Then there exists $(u_0, v_0) \in F(c) \times G(c)$ such that

$$\varphi_c(u_0 - v_0) < (1+\eta)\lambda_c + \tau/2 \tag{3.5}$$

and, for any $(u, v) \in F(c) \times G(c)$, we get

$$\varphi_c(u - v) > (1+\eta)\lambda_c - \tau/4. \tag{3.6}$$

Hence, for any $(u, v) \in F(c) \times G(c)$,

$$\varphi_c(u) > (1+\eta)\lambda_c - \tau/4 + \varphi_c(v_0), \qquad -\varphi_c(v) > (1+\eta)\lambda_c - \tau/4 - \varphi_c(u_0). \tag{3.7}$$

By the upper demicontinuity of F and G, there exist neighbourhoods $U(c)$ and $V(c)$ of c in C such that

$$\begin{aligned} \varphi_c(u) &> (1+\eta)\lambda_c - \tau/4 + \varphi_c(v_0) \quad \text{for any } u \in F(x) \text{ and any } x \in U(c), \\ -\varphi_c(v) &> (1+\eta)\lambda_c - \tau/4 - \varphi_c(u_0) \quad \text{for any } v \in G(x) \text{ and any } x \in V(c). \end{aligned} \tag{3.8}$$

Therefore we obtain

$$\varphi_c(u - v) > 2(1+\eta)\lambda_c - \tau/2 - \varphi_c(u_0 - v_0) \tag{3.9}$$

for any $(u, v) \in F(x) \times G(x)$ and $x \in N(c)$ where $N(c) = U(c) \cap V(c)$. Consequently,

$$\varphi_c(u - v) \geq 2(1+\eta)\lambda_c - \tau/2 - (1+\eta)\lambda_c - \tau/2 = \lambda_c + \eta\lambda_c - \tau > \lambda_c \tag{3.10}$$

for any $(u, v) \in F(x) \times G(x)$ and $x \in N(c)$. The assertion has thus been proved.

(ii) Assume that $F : C \to 2^E$ is u.d.c. on C, and that there exists $(c, \varphi_c, \lambda_c) \in C \times (E' \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ such that $\lambda_c[\varphi_c(u - c) - \lambda_c] > 0$ for any $u \in F(c)$.

If $\lambda_c > 0$, then, by the upper demicontinuity of F, there exists a neighbourhood $N(c)$ of c in C such that $\varphi_c(u) > \lambda_c + \varphi_c(c)$ or, equivalently, $\lambda_c[\varphi_c(u - c) - \lambda_c] > 0$ for any $x \in N(c)$ and any $u \in F(x)$.

If $\lambda_c < 0$, then the argumentation is analogous.

(iii) Let F and G be u.h.c. on C and let there exist $(c, \varphi_c, \lambda_c) \in C \times E' \times (\mathbb{R} \setminus \{0\})$ and $\varepsilon_c > 0$ such that $\lambda_c[\varphi_c(u - v) - (1+\varepsilon_c)\lambda_c] > 0$ for any $(u, v) \in F(c) \times G(c)$.

Assume that

$$\lambda_c > 0, \quad \varphi_c(u - v) > (1+\varepsilon_c)\lambda_c \quad \text{for any } (u, v) \in F(c) \times G(c) \tag{3.11}$$

(if we replace condition (3.11) by $\lambda_c < 0$ and $\varphi_c(u - v) < (1+\varepsilon_c)\lambda_c$ for any $(u, v) \in F(c) \times G(c)$, then the argumentation is analogous and will be omitted) and let η be such that $\eta \geq \varepsilon_c$ and $\inf_{(u,v) \in F(c) \times G(c)} \varphi_c(u - v) = (1+\eta)\lambda_c$. Assume also that $\tau > 0$ satisfies $\tau < (1/3)\eta\lambda_c$. Then there exists $(u_0, v_0) \in F(c) \times G(c)$ such that

$$\varphi_c(u_0 - v_0) < (1+\eta)\lambda_c + \tau/2. \tag{3.12}$$

Obviously $\varphi_c(u-v) > (1+\eta)\lambda_c - \tau/4$ for any $(u,v) \in F(c) \times G(c)$. Hence, in particular, we have that $\varphi_c(u) > (1+\eta)\lambda_c - \tau/4 + \varphi_c(v_0)$ and $-\varphi_c(v) > (1+\eta)\lambda_c - \tau/4 - \varphi_c(u_0)$ for any $(u,v) \in F(c) \times G(c)$, that is, $-\varphi_c(u) < -(1+\eta)\lambda_c + \tau/4 - \varphi_c(v_0)$ and $\varphi_c(v) < -(1+\eta)\lambda_c + \tau/4 + \varphi_c(u_0)$ for any $(u,v) \in F(c) \times G(c)$.

Now, let us observe that, by upper hemicontinuity, the sets $U(c) = \{c \in C : \sup_{u\in F(x)} -\varphi_c(u) < -(1+2\eta/3)\lambda_c + \tau/4 - \varphi_c(v_0)\}$ and $V(c) = \{c \in C : \sup_{v\in G(x)} \varphi_c(v) < -(1+2\eta/3)\lambda_c + \tau/4 + \varphi_c(u_0)\}$ are open in C. Of course, $c \in U(c) \cap V(c)$ since $\sup_{u\in F(c)} -\varphi_c(u) \le -(1+\eta)\lambda_c + \tau/4 - \varphi_c(v_0) < -(1+2\eta/3)\lambda_c + \tau/4 - \varphi_c(v_0)$ and $\sup_{v\in G(c)} \varphi_c(v) \le -(1+\eta)\lambda_c + \tau/4 + \varphi_c(u_0) < -(1+2\eta/3)\lambda_c + \tau/4 + \varphi_c(u_0)$. Hence, if we denote $N(c) = U(c) \cap V(c)$, then we conclude that $\varphi_c(u-v) > 2(1+2\eta/3)\lambda_c - \tau/2 - \varphi_c(u_0 - v_0)$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$ and, by (3.12), we obtain $\varphi_c(u-v) > 2(1+2\eta/3)\lambda_c - \tau/2 - (1+\eta)\lambda_c - \tau/2 = \lambda_c + \eta\lambda_c/3 - \tau > \lambda_c$ since $\tau < (1/3)\eta\lambda_c$. Consequently, we have shown that if $\lambda_c > 0$, then $\varphi_c(u-v) - \lambda_c > 0$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$, that is, $\lambda_c[\varphi_c(u-v) - \lambda_c] > 0$ for any $x \in N(c)$ and any $(u,v) \in F(x) \times G(x)$. Thus the pair (F,G) is t.p.h.c. on C.

(iv) Assume that $F : C \to 2^E$ is u.h.c. on C and assume that there exist $(c,\varphi_c,\lambda_c) \in C \times (E' \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ and $\varepsilon_c > 0$, such that

$$\lambda_c[\varphi_c(u-c) - (1+\varepsilon_c)\lambda_c] > 0 \quad \text{for any } u \in F(c). \tag{3.13}$$

If $\lambda_c < 0$, then the above condition implies that $\varphi_c(u-c) < (1+\varepsilon_c)\lambda_c$ for any $u \in F(c)$, that is, $\varphi_c(u) < \varphi_c(c) + (1+\varepsilon_c)\lambda_c < \varphi_c(c) + \lambda_c < \varphi_c(c)$ for any $u \in F(c)$. Now, by upper hemicontinuity, the set $N(c) = \{x \in C : \sup_{u\in F(x)} \varphi_c(u) < \varphi_c(c) + \lambda_c\}$ is open in C. Moreover, $c \in N(c)$ since $\sup_{u\in F(c)} \varphi_c(u) \le \varphi_c(c) + (1+\varepsilon_c)\lambda_c < \varphi_c(c) + \lambda_c$. Obviously $N(c) \subset \{x \in C : \varphi_c(u) < \varphi_c(c) + \lambda_c, u \in F(x)\}$. This gives $\varphi_c(u-c) < \lambda_c < 0$ for any $x \in N(c)$ and any $u \in F(x)$, that is, $\lambda_c[\varphi_c(u-c) - \lambda_c] > 0$ for any $x \in N(c)$ and any $u \in F(x)$.

If we assume that $\lambda_c > 0$ and $\varphi_c(u-c) > (1+\varepsilon_c)\lambda_c$ for any $u \in F(c)$, then, by analogous argumentation, we show that there exists a neighbourhood $N(c)$ of c in C such that $\lambda_c[\varphi_c(u-c) - \lambda_c] > 0$ for any $x \in N(c)$ and any $u \in F(x)$.

The assertion has thus been proved. □

Remark 3.2. (a) Let the map $F : C \to 2^E$ be t.p.h.c. or t.h.c. on C. Denote

$$\begin{aligned} H_{c,\varphi_c,\lambda_c,\varepsilon_c} &= \{w \in E : \varphi_c(w) < \varphi_c(c) + (1+\varepsilon_c)\lambda_c\}, \quad \varepsilon_c \ge 0, \\ W_{c,\varphi_c,\lambda_c} &= \{x \in C : \varphi_c(u) < \varphi_c(c) + \lambda_c \text{ for any } u \in F(x)\}, \\ U_{c,\varphi_c,\lambda_c} &= \Big\{x \in C : \sup_{u\in F(x)} \varphi_c(u) \le \varphi_c(c) + \lambda_c\Big\} \end{aligned} \tag{3.14}$$

when $\lambda_c < 0$;

$$\begin{aligned} H_{c,\varphi_c,\lambda_c,\varepsilon_c} &= \{w \in E : \varphi_c(w) > \varphi_c(c) + (1+\varepsilon_c)\lambda_c\}, \quad \varepsilon_c \ge 0, \\ W_{c,\varphi_c,\lambda_c} &= \{x \in C : \varphi_c(u) > \varphi_c(c) + \lambda_c \text{ for any } u \in F(x)\}, \\ U_{c,\varphi_c,\lambda_c} &= \Big\{x \in C : \inf_{u\in F(x)} \varphi_c(u) \ge \varphi_c(c) + \lambda_c\Big\} \end{aligned} \tag{3.15}$$

when $\lambda_c > 0$. By Definition 2.1 and Remark 2.2, we see that if the set $F(c)$ is contained in open half-space $H_{c,\varphi_c,\lambda_c,\varepsilon_c}$ (here $\varepsilon_c > 0$ in the case of transfer positive hemicontinuity and $\varepsilon_c = 0$ in the case of transfer hemicontinuity), then there exists a neighbourhood $N(c)$ of c in C such that, for any $x \in N(c)$, the set $F(x)$ is contained in open half-space $H_{c,\varphi_c,\lambda_c,0}$ and c is an interior point of the sets of $W_{c,\varphi_c,\lambda_c}$ and $U_{c,\varphi_c,\lambda_c}$.

This fact means that transfer positive hemicontinuity essentially generalizes upper semicontinuity and upper demicontinuity.

(b) Let the map $F : C \to 2^E$ be s.t.p.h.c. or s.t.h.c. on C. Denote

$$V_{c,\varphi_c,\lambda_c} = \left\{x \in C : \sup_{u\in F(x)} \varphi_c(u) < \varphi_c(c) + \lambda_c\right\} \quad \text{when } \lambda_c < 0,$$
$$V_{c,\varphi_c,\lambda_c} = \left\{x \in C : \inf_{u\in F(x)} \varphi_c(u) > \varphi_c(c) + \lambda_c\right\} \quad \text{when } \lambda_c > 0. \tag{3.16}$$

By Definition 2.3 and Remark 2.5, we see that if the set $F(c)$ is contained in open half-space $H_{c,\varphi_c,\lambda_c,\varepsilon_c}$ (here $\varepsilon_c > 0$ in the case of stictly transfer positive hemicontinuity and $\varepsilon_c = 0$ in the case of strictly transfer hemicontinuity), then c is an interior point of the set $V_{c,\varphi_c,\lambda_c}$.

This fact means that strictly transfer positive hemicontinuity essentially generalizes upper hemicontinuity.

(c) If set-valued map is compact-valued, then upper hemicontinuity implies upper demicontinuity. If the space of set-valued map with compact-valued is compact, then the definition of upper semicontinuity, upper demicontinuity and upper hemicontinuity are equivalent. For more details concerning comparisons of these three concepts of continuity, see, for example, Yuan et al. [20, 22, 23].

Analogous properties do not hold between upper hemicontinuity and strictly transfer positive hemicontinuity (transfer positive hemicontinuity). Indeed, in Example 4.1 we show that the sets C, $F_3(C)$, $G_3(C)$, $F_3(c)$ and $G_3(c)$, $c \in C$, are compact and convex and F_3 and G_3 are s.t.p.h.c. and t.p.h.c. on C (thus also s.t.h.c. and t.h.c. by Proposition 2.4, Remark 2.2 and Definitions 2.1 and 2.3) but not u.h.c. on C.

4. Examples and remarks

Let $E = \{x = (x_1, x_2) : x \in \mathbb{R}^2\}$ be a normed space with the Euclidean norm $\|\cdot\|$ and let $C = \{c = (c_1, c_2) \in E : \|c\| \le 1\}$. Note that if $(w_0, \varphi_0) \in C \times (E' \setminus \{0\})$ is admissible, then $w_0 = (-\alpha/\theta, -\beta/\theta)$, $\theta = (\alpha^2 + \beta^2)^{1/2}$, $\varphi_0(c) = \alpha c_1 + \beta c_2$, $c \in E$, $|\alpha| + |\beta| > 0$ and $\varphi_0(w_0) = \min_{c\in C} \varphi_0(c) = -\theta$.

Example 4.1. For $c = (c_1, c_2) \in C$, define:

$$F_1(c) = G_1(c) = \{x = (x_1, x_2) \in C : -1/2 < x_1 < 1/2\} \quad \text{if } c_1 = 0,$$
$$F_1(c) = \{x \in C : x_2 > c_2\}, \quad G_1(c) = \{x \in C : x_2 < -c_2\} \quad \text{if } c_1 \ne 0,\ c_2 \ge 0,$$
$$F_1(c) = \{x \in C : x_2 < c_2\}, \quad G_1(c) = \{x \in C : x_2 > -c_2\} \quad \text{if } c_1 \ne 0,\ c_2 < 0; \tag{4.1}$$
$$F_2(c) = \operatorname{Int}(F_1(c)), \quad G_2(c) = \operatorname{Int}(G_1(c)); \qquad F_3(c) = \overline{F_1(c)}, \quad G_3(c) = \overline{G_1(c)};$$
$$F_4(c) = \operatorname{Int}(F_1(c)), \quad G_4(c) = \overline{G_1(c)}; \qquad F_5(c) = F_1(c), \quad G_5(c) = \overline{G_1(c)}.$$

The pair (F_i,G_i) is t.p.h.c. on C, $i=1-5$. Indeed, if $(c,\varphi_c,\lambda_c)\in(C\setminus\{0\})\times E'\times\mathbb{R}$ and $\varepsilon_c>0$ are such that $\lambda_c[\varphi_c(u-v)-(1+\varepsilon_c)\lambda_c]>0$ for any $(u,v)\in F_i(c)\times G_i(c)$, there exists a neighbourhood $N_i(c)$ of c in C such that $\lambda_c[\varphi_c(u-v)-\lambda_c]>0$ for any $x\in N_i(c)$ and any $(u,v)\in F_i(x)\times G_i(x)$, $i=1-5$.

The maps F_i and G_i are t.p.h.c. on C, $i=1-5$. Indeed, if $(c,\varphi_c,\lambda_c)\in(C\setminus(0,0)),\times E'\times\mathbb{R}$ and $\varepsilon_c>0$ are such that $\lambda_c[\varphi_c(c-v)-(1+\varepsilon_c)\lambda_c]>0$ for any $v\in G_i(c)$, there exists a neighbourhood $N_i(c)$ of c in C such that $\lambda_c[\varphi_c(c-v)-\lambda_c]>0$ for any $x\in N_i(c)$ and any $v\in G_i(x)$, $i=1-5$.

The pair (F_i,G_i) and the maps F_i and G_i are not t.h.c. on C, $i=1,2,4,5$. The pair (F_3,G_3) and the maps F_3 and G_3 are t.h.c. on C.

The pair (F_3,G_3) is s.t.h.c. on C. Indeed, assume that $(c,\varphi_c,\lambda_c)\in(C\setminus\{0\})\times E'\times\mathbb{R}$ is such that $\lambda_c[\varphi_c(u-v)-\lambda_c]>0$ for any $(u,v)\in F_3(c)\times G_3(c)$. Since $W_{c,\varphi_c,\lambda_c}\subset V_{c,\varphi_c,\lambda_c}$, Remark 2.2 yields that c is an interior point of the set $V_{c,\varphi_c,\lambda_c}$. Here $V_{c,\varphi_c,\lambda_c}=\{x\in C:\sup_{(u,v)\in F_3(x)\times G_3(x)}\varphi_c(u-v)<\lambda_c\}$ if $\lambda_c<0$, $V_{c,\varphi_c,\lambda_c}=\{x\in C:\inf_{(u,v)\in F_3(x)\times G_3(x)}\varphi_c(u-v)>\lambda_c\}$ if $\lambda_c>0$, $W_{c,\varphi_c,\lambda_c}=\{x\in C:\varphi_c(u-v)<\lambda_c$ for any $(u,v)\in F_3(x)\times G_3(x)\}$ if $\lambda_c<0$ and $W_{c,\varphi_c,\lambda_c}=\{x\in C:\varphi_c(u-v)>\lambda_c$ for any $(u,v)\in F_3(x)\times G_3(x)\}$ if $\lambda_c>0$. This proves that (F_3,G_3) is s.t.h.c. on C. The maps F_3 and G_3 are s.t.h.c. on C; the argumentation is analogous and will be omitted.

Obviously, F_i and G_i are not u.h.c. on C, $i=1-5$. Indeed, for $\varphi\in E'$ of the form $\varphi(x)=x_1$, $x=(x_1,x_2)\in E$, and $\lambda=2/3$ we have that $0\in U_i=\{x\in C:\sup_{u\in F_i(x)}\varphi(u)<\lambda\}$ and $0\in V_i=\{x\in C:\sup_{v\in G_i(x)}\varphi(v)<\lambda\}$. But U_i and V_i are not open in C since if $N(0)$ is an arbitrary and fixed neighbourhood of 0 in C, then $N(0)$ is contained neither in U_i nor V_i, $i=1-5$.

The pair (F_i,G_i) and the maps F_i and G_i satisfy the assumptions of Theorems 2.9(iii) and 2.9(iv), respectively, $i=1-5$. The pair (F_i,G_i) and the maps F_i and G_i satisfy the assumptions of Theorems 2.10(iii) and 2.10(iv), respectively, $i=3-5$. The pair (F_3,G_3) and the maps F_3 and G_3 satisfy the assumptions of Theorems 2.11(iii) and 2.11(iv), respectively.

Remark 4.2. (a) Theorems 2.10(iii$_2$) and 2.10(iv$_2$) includes Theorems 5 and 6 of Fan [11], respectively; this follows from Proposition 3.1.

(b) Example 4.1 shows that Theorems 5 and 6 of Fan [11] for pairs (F_3,G_3) and maps F_3 and G_3, respectively, hold if we replace upper demicontinuity by strictly transfer positive hemicontinuity or transfer positive hemicontinuity.

Example 4.3. Define the maps F and G as follows:

$$F(0)=\{x=(x_1,x_2)\in\operatorname{Int}(C):x_2>0\},$$

$$F(c)=\{x\in\operatorname{Int}(C):|\operatorname{Arg}(x_1+ix_2)-\operatorname{Arg}(c_1+ic_2)|<\pi/2\}\quad\text{if } c\neq 0; \tag{4.2}$$

$$G(c)=-F(c)\quad\text{if } c\in C.$$

The pair (F,G) and the maps F and G are t.p.h.c. on C, neither F nor G is not u.h.c. on C and $\varphi_0(u_0-v_0)<0$ for each $(u_0,v_0)\in F(w_0)\times G(w_0)$. Thus the pair (F,G) and the maps F and G satisfy the assumptions of Theorems 2.9(iii) and 2.9(iv), respectively.

Remark 4.4. Example 4.3 shows that Theorems 3 and 4 of Fan [11] not hold if we replace upper demicontinuity by transfer positive hemicontinuity.

Example 4.5. For $c \in C$, let $F_1(c) = F(c)$, $F_2(c) = \overline{F(c)}$ and $G_1(c) = G_2(c) = \overline{G(c)}$ where F and G are defined in Example 4.3. Then the t.p.h.c. pairs (F_1, G_1) and (F_2, G_2) satisfy the assumptions of Theorems 2.10(iii$_1$) and 2.10(iii$_2$), respectively, and all the maps F_1, F_2, G_1 and G_2 are not u.h.c. on C.

Example 4.6. For $c = (c_1, c_2) \in C$, define:

$$\begin{aligned} &F(c) = G(c) = \{x \in C : |x_1| \leq 1/4\} \quad \text{if } c_1 = 0; \\ &F(c) = \{x \in E : \|(x_1, x_2) - (c_1, 1/2)\| \leq 1/2\} \quad \text{if } c_1 \neq 0; \\ &G(c) = \{x \in E : \|(x_1, x_2) - (c_1, -1/2)\| \leq 1/2\} \quad \text{if } c_1 \neq 0. \end{aligned} \tag{4.3}$$

Obviously, C, $F(c)$ and $G(c)$, $c \in C$, are compact and convex, the pair (F, G) and the maps F and G are t.h.c. on C, and $u_0 = v_0 = (-\alpha/\theta, 0) \in F(w_0) \cap G(w_0)$, $\varphi_0(u_0 - v_0) = 0$ and $\varphi_0(u_0 - w_0) = \varphi_0(v_0 - w_0) = \beta^2/\theta^2 \geq 0$.

All the assumptions of Theorems 2.11(iii$_3$) and 2.11(iv$_3$) for the pair (F, G) and for the maps F and G, respectively, are satisfied, each $c \in C$ is a coincidence for (F, G) and each $c \in C$ is a fixed point of F or G.

Obviously, neither F nor G is not u.h.c. on C. Indeed, for $\varphi \in E'$ of the form $\varphi(x) = x_1$, $x = (x_1, x_2) \in E$, and $\lambda = 1/2$ we have that $0 \in U = \{x \in C : \sup_{u \in F(x)} \varphi(u) < \lambda\}$ and $0 \in V = \{x \in C : \sup_{v \in G(x)} \varphi(v) < \lambda\}$. But U and V are not open in C since if $N(0)$ is an arbitrary and fixed neighbourhood of 0 in C, then $N(0)$ is contained neither in U nor V.

Example 4.7. For $c = (c_1, c_2) \in C$, define $G(c) = -F(c)$ where

$$\begin{gathered} F(0) = \{x \in C : |x_1| < 1/2\}, \\ F(c) = \{x = (x_1, x_2) \in C : |\operatorname{Arg}(x_1 + ix_2) - \operatorname{Arg}(c_1 + ic_2)| < \pi/4 \text{ if } c \neq 0\}. \end{gathered} \tag{4.4}$$

The pair (F, G) and the maps F and G are t.p.h.c. on C, $w_0 \in F(w_0)$ and $\varphi_0(w) \geq \varphi_0(w_0) = -\theta$ for all $w \in F(w_0) \cup G(w_0)$. Thus (F, G) satisfies the assumptions of Theorem 2.9(iii) and F satisfies the assumptions of Theorem 2.10(iv$_1$).

Obviously, neither F nor G is not u.h.c. on C. Indeed, for $\varphi \in E'$ of the form $\varphi(x) = x_1$, $x = (x_1, x_2) \in E$, and $\lambda = 2^{1/2}/2$ we have that $0 \in U = \{x \in C : \sup_{u \in F(x)} \varphi(u) < \lambda\}$ and $0 \in V = \{x \in C : \sup_{v \in G(x)} \varphi(v) < \lambda\}$. But U and V are not open in C since if $N(0)$ is an arbitrary and fixed neighbourhood of 0 in C, then $N(0)$ is contained neither in U nor V.

Remark 4.8. Our theorems concern maps which satisfy a more general condition of continuity than those existing in a large literature; cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] (see also references therein).

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