EXISTENCE OF FIXED POINTS ON COMPACT EPILIPSCHITZ SETS WITHOUT INVARIANCE CONDITIONS

MIKHAIL KAMENSKII AND MARC QUINCAMPOIX

Received 4 April 2005

We provide a new result of existence of equilibria of a single-valued Lipschitz function f on a compact set K of $\mathbb{R}^n$ which is locally the epigraph of a Lipschitz functions (such a set is called epilipschitz set). Equivalently this provides existence of fixed points of the map $x \mapsto x - f(x)$. The main point of our result lies in the fact that we do not impose that $f(x)$ is an "inward vector" for all point x of the boundary of K. Some extensions of the existence of equilibria result are also discussed for continuous functions and set-valued maps.

1. Introduction

This paper is devoted to the following result.

Theorem 1.1. *Let K be an epilipschitz compact subset of $\mathbb{R}^n$; $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a (locally) Lipschitz function. Assume that K_s is closed and that the Euler characteristic $\chi(K_s)$ is well defined.*

If $\chi(K) \neq \chi(K_s)$ then there exists an equilibria in K that is a point $x \in K$ such that $f(x) = 0$.

In the above Theorem 1.1, the set K_s (or $K_s(f)$) is the set of elements x of the boundary of K such that the solution to the Cauchy problem

$$x'(t) = f(x(t)), \quad t \geq 0,\ x(0) = x, \tag{1.1}$$

leaves K immediately (that is there exists $\sigma > 0$ such that $(x((0,\sigma)) \cap K = \emptyset)$). Epilipschitz sets are sets which are locally the epigraph of a Lipschitz function (an equivalent formulation is given in [25]).

It is worth pointing out that when $f(x)$ is "inward" for any $x \in \partial K$, we have that K is invariant by the differential equation

$$x'(t) = f(x(t)), \quad t \geq 0, \tag{1.2}$$

Copyright © 2005 Hindawi Publishing Corporation
Fixed Point Theory and Applications 2005:3 (2005) 267–279
DOI: 10.1155/FPTA.2005.267

and consequently $K_s = \varnothing$. So our theorem, contains for example the famous fixed point Brouwer theorem, viewed as an existence result for equilibria of the map $x \mapsto x - g(x)$ for convex compact closed sets. It contains also several results of existence of equilibria which impose inwardness conditions of the type

$$\forall x \in \partial K, \quad f(x) \in C_K(x) \tag{1.3}$$

where $C_K(x)$ denotes Clarke's tangent cone.

Since pioneering results of Fan and Browder [5, 15], several theorems have been obtained in this direction [10, 12, 13, 19, 18, 23, 22], among them we wish to quote one of the most recent result (in a version adapted for single valued map).

Proposition 1.2 [11, Corollary 4.1]. *If f continuous, K is a compact epilipschitz subset of $\mathbb{R}^n$ with $\chi(K) \neq 0$ and if (1.3) holds true, then there is an equilibria of f in K.*

We also wish to underline that more general results with condition (1.3) have been obtained for set-valued maps and for normed spaces more general than $\mathbb{R}^n$ (cf. for instance for L retracts in normed spaces).

We are mainly interested to weaken the condition (1.3) for a class of epilischitz sets of $\mathbb{R}^n$ which is large enough because it contains for instance convex sets with nonempty interiors, C^1 submanifolds with boundary.

Our approach is mainly based on properties of trajectories of the differential equation associated with f. Indeed the set K_s appears in the so called topological Ważewski principle, which gives sufficient conditions for existence of trajectories of (1.2) remaining in K (cf. [16]). We also would like to mention the approach of [21] for regular sets by using Conley index theory.

We explain how the paper is organized. In the preliminary section we present some relevant tools (differential equations and degree theories) for proving our main theorem. The next section is devoted to proof of our main result. In the last section, we discuss some extensions for a quite large class of f (but still for compact epilipschitz sets of $\mathbb{R}^n$).

2. Preliminaries

We denote by $\mathrm{cl}(A)$ the closure of a set A, $\mathrm{int}(A)$ its interior, $\mathrm{co}(A)$ its closed convex hull, ∂A its boundary and by $x \mapsto d_A(x)$ the distance function to A. The unit closed ball of $\mathbb{R}^n$ is denoted by B, S is the unit sphere. The number $\chi(A)$ denotes the Euler characteristics of A. The set of elements of A which are not in B is denoted by $A \backslash B$. For a closed set $K \subset \mathbb{R}^n$, and $x \in K$ we denote

$$C_K(x) := \left\{ v \in \mathbb{R}^n \mid \lim_{h \to 0^+, y \in K \to x} \frac{d_K(y + hv)}{h} = 0 \right\} \tag{2.1}$$

Clarke's tangent cone, $N_K(x) := (C_K(x))^-$ the corresponding Clarke's normal cone and the following contingent (Bouligand's) cone:

$$T_K(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{h \to 0^+} \frac{d_K(x + hv)}{h} = 0 \right\}. \tag{2.2}$$

Definition 2.1. A nonempty closed subset $K \subset \mathbb{R}^n$ is epilipschitz if and only if the interior $\operatorname{int}(C_K(x))$ of the Clarke tangent cone is nonempty for any $x \in K$ (or equivalently iff the normal cone does not contain straight lines).

We recall some well-known facts about epilipschitz sets in the following.

Lemma 2.2 (cf. for instance [11]). *Let $K \subset \mathbb{R}^n$ be closed epilipschitz. Then $K = \operatorname{cl}(\operatorname{int}(K))$, the set valued maps $x \mapsto C_K(x)$ $x \mapsto \operatorname{int}(C_K(x))$ are lower semicontinuous with nonempty closed convex values, the map $x \mapsto N_K(x) \cap S$ is upper semicontinuous with nonempty compact values and $T_K(x) \supset C_K(x)$ for any $x \in K$.*

Recall also that for any $x \in \partial K$, $C_K(x) \neq \mathbb{R}^n$ and $N_K(x) \neq \{0\}$.

We shall need a suitable definition of the degree of a mapping on closed sets which are the closure of their interior and for set-valued maps. For such a definition we refer the reader to [9]. Also there are many algebraic topology books with definition of the Euler characteristics (cf. [14] for instance), but we want to stress that—for regular sets—the Euler characteristic is also the degree of the field of normals [20]. One recent statement of this fact can be find in [11, Theorem 4.1].

Lemma 2.3. *Let K be compact epilipschitz, F be an upper semicontinuous set-valued map with nonempty convex compact values such that*

$$0 \notin F(x), \quad F(x) \cap C_K(x) \neq \emptyset, \quad \forall x \in \partial K. \tag{2.3}$$

Then $\chi(K) = \deg(-F, K, 0)$.

Also we recall in an adapted version the following well-known fact for differential inclusions (cf. for instance [1] or [24]).

Lemma 2.4. *Let K be a closed set, O be an open set, F be an upper semicontinuous set-valued map with nonempty convex compact values. The two following properties are equivalent:*

$$\forall x \in \partial K \cap O, \quad F(x) \cap T_K(x) \neq \emptyset. \tag{2.4}$$

For any initial condition $x_0 \in K \cap O$, there exists at least one trajectory of $x'(t) \in F(x(t))$ starting from x_0 remaining in K for all $t \geq 0$ until it possibly leaves O.

3. Proof of the main result

Throughout this section K_s is assumed to be closed and k is the lipschitz constant of f in $K + B$.

3.1. About properties of epilipschitz sets and of the set K_s.

First we state a lemma which easily follows from the lower semicontinuity of the Clarke tangent cone for epilipschitz set.

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be epilipschitz compact and g be a continuous function. If for some set A*

$$g(y) \in \operatorname{int}(C_K(y)), \quad \forall y \in K \cap A, \tag{3.1}$$

then there exists an open neighborhood V of A and an $\alpha > 0$ such that

$$g(y) + \alpha d_{K\setminus V}(y)B \subset \operatorname{int}\big(C_K(y)\big), \quad \forall y \in K \cap V. \tag{3.2}$$

Lemma 3.2. *Let $K \subset \mathbb{R}^n$ be epilipschitz compact. There exists a continuous map $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that*

$$0 \neq g(x) \in \operatorname{int}\big(C_K(x)\big), \quad \forall x \in \partial K. \tag{3.3}$$

Proof. With any $x \in K$ we can associate a vector $0 \neq l_x \in \operatorname{int}(C_K(x))$. By Michael's selection theorem [2, Theorem 9.1.2] there exists a continuous map $y \mapsto g_x(y)$ with

$$g_x(x) = l_x,\ g_x(y) \in C_K(y), \quad \forall y \in K. \tag{3.4}$$

By virtue of Lemma 3.1, there exists $\alpha_x > 0$ and V_x an open neighborhood of x such that

$$g_x(y) + \alpha_x B \subset C_K(y), \quad \forall y \in V_x \cap K. \tag{3.5}$$

By compactness of K we can extract finite covering $(V_{x_i})_{i=1}^N$ of K. Let consider λ_i an associated partition of unity. Define the continuous function

$$y \in K \longmapsto g(y) := \sum_{i=1}^{N} \lambda_i(y) g_{x_i}(y). \tag{3.6}$$

Let $y \in K$ and $\lambda_j,\ j \in J \subset [1,N]$ the non zero terms of the partition evaluated in y. For any $j \in J$, we have

$$g_{x_j}(y) + \alpha B \subset C_K(y), \tag{3.7}$$

where $\alpha = \min\{\alpha_j \mid j \in J\}$. So by convexity of the Clarke cone

$$g(y) + \alpha B = \sum_{j\in J} \lambda_j\big(g_{x_j}(y) + \alpha B\big) \subset C_K(y). \tag{3.8}$$

This complete the proof if one notices that g cannot take value 0 on ∂K. Indeed, suppose, contrary to our claim, that $g(x) = 0$ for some $x \in \partial K$. Then $0 \in \operatorname{int}(C_K(x))$. Because $C_K(x)$ is a closed convex cone, we infer $C_K(x) = \mathbb{R}^n$. Consequently $x \in \operatorname{int}(K)$ a contradiction. □

Now we need a more precise property of f on the relative boundary of $\partial_K K_s$ of K_s in K.

Lemma 3.3. *Let $K \subset \mathbb{R}^n$ be epilipschitz compact and let $x_0 \in \partial_K K_s$. Then*

$$\mathbb{R} f(x_0) \cap \operatorname{int}\big(C_K(x_0)\big) = \varnothing. \tag{3.9}$$

Proof. Note that the solution to (1.2) starting from x_0 must leave K immediately so $f(x_0) \neq 0$. We prove the lemma by contradiction, if the wished claim is false then either $f(x_0) \in \operatorname{int} C_K(x_0)$ or $-f(x_0) \in \operatorname{int} C_K(x_0)$.

Case a ($f(x_0) \in \operatorname{int} C_K(x_0)$). From Lemma 3.1 there exist α, η positive numbers such that

$$f(x) + \alpha B \subset C_K(x), \quad \forall x \in x_0 + \eta B. \tag{3.10}$$

This implies that $f(x) \in T_K(x)$ for all $x \in x_0 + \eta B$. So by the local viability theorems (cf. Lemma 2.4), the trajectory of (1.2) starting from x_0 remains in K for a small time. This is a contradiction with $x_0 \in K_s$.

Case b ($-f(x_0) \in \operatorname{int} C_K(x_0)$). From Lemma 3.1 there exist α, η positive numbers such that

$$-f(x) + \alpha B \subset C_K(x), \quad \forall x \in x_0 + \eta B. \tag{3.11}$$

Fix $x \in ((x_0 + \eta B) \cap \partial K)) \backslash K_s$. From (3.11) one can easily deduce that there exist $\tau > 0$ small enough such that

$$\begin{gathered} x + [0,\tau]\left(- f(x) + \left(\frac{\alpha}{2}\right) B\right) \subset K, \\ \left(x + [0,\tau]\left(- f(x) + \left(\frac{\alpha}{2}\right) B\right)\right) \cap K_s = \varnothing. \end{gathered} \tag{3.12}$$

An easy estimation for the solutions to the differential equation

$$y'(t) = -f(y(t)), \quad t \geq 0 \tag{3.13}$$

will provide the existence of some $\tau' > 0$ small enough such that any solution $y(\cdot)$ of (3.13) starting from x satisfies the following estimation

$$y(t) \in x + [0,\tau]\left(- f(x) + \left(\frac{\alpha}{4}\right) B\right) \subset K, \quad \forall t \in [0,\tau']. \tag{3.14}$$

Fix $z \in (x_0 + \kappa B) \backslash K$ and

$$0 < \kappa < \min\left\{\tau'\left(\frac{\alpha}{4}\right) e^{-k\tau'}, \operatorname{dist}\left(K_s, x + [0,\tau]\left(- f(x) + \left(\frac{\alpha}{2}\right) B\right)\right)\right\}. \tag{3.15}$$

By Lipschitz continuous dependence result of the solution of a differential eqution with respect to the initial data, one obtains that the solution $z(\cdot)$ of (3.13) with $z(0) = z$ satisfies

$$\forall t \in [0,\tau'], \quad \|y(t) - z(t)\| \leq \|y - z\| e^{kt}. \tag{3.16}$$

In view of (3.12)–(3.14), we obtain $z(\tau') \in K$ and $z([0,\tau']) \cap K_s = \varnothing$. Hence the function $t \mapsto z(\tau' - t)$ is a trajectory to (1.2) starting from a point of K and leaving K before the time τ' without crossing K_s. This is a contradiction with the very definition of K_s. $\square$

Proposition 3.4. *Assume that $K \subset \mathbb{R}^n$ is epilipschitz compact, K_s is closed and that there is no equilibrium point of f on the boundary of K. Then there exists an upper semicontinuous (multi-valued) map $\Psi : \partial K \mapsto \mathbb{R}^n$ with nonempty convex valued compact values such that*

(i) $\Psi(x) = f(x)$, *for all* $x \in \partial K \backslash K_s$

(ii) $\Psi(x) \cap C_K(x) \neq \emptyset$, *for all* $x \in \partial K$

(iii) $0 \notin \Psi(x)$, *for all* $x \in \partial K$.

Proof. Consider g obtained in Lemma 3.2. Define Ψ as follows:

$$\Psi(x) = \begin{cases} f(x), & \forall x \in \partial K \backslash K_s, \\ g(x), & \forall x \in K_s \backslash \partial_K K_s, \\ [f(x), g(x)], & \forall x \in \partial_K K_s. \end{cases} \tag{3.17}$$

Clearly Ψ is upper semicontinuous with nonempty convex compact values. By [2, Theorem 4.1.9], and by the very construction of g, statements (i) and (ii) are obtained.

For obtaining (iii), we have to prove that $0 \notin [f(x), g(x)]$ if $x \in \partial_K K_s$ which is a direct consequence of Lemma 3.3. □

3.2. Construction of the set W_m. We shall construct an epilipschitz subset W_m of K which has the same Euler characteristic that K_s. This construction will be made under the following supposition:

$$\forall x \in K \backslash K_s, \quad f(x) \in \operatorname{int}(C_K(x)). \tag{3.18}$$

Before doing this we recall that if K_s is closed then the function

$$\tau_K(x_0) := \inf\{t > 0, x(t, x_0) \notin K\} \tag{3.19}$$

is continuous (where $x(\cdot, x_0)$ denotes the unique solution to (1.1) (see [1, Lemma 4.2.2] and [16, Lemma 1.8]).

Fix a positive integer m sufficiently large. Observe that K_s is contained in the interior (with respect to K) of the set

$$U_{m+1} := \left\{x \in K,\ \tau_K(x) \leq \frac{1}{m+1}\right\}. \tag{3.20}$$

Choose Z_{K_s} an open neighborhood of K_s with

$$K_s \subset Z_{K_s} \subset \operatorname{cl}(Z_{K_s}) \subset U_{m+1}. \tag{3.21}$$

By compactness of $\operatorname{cl}(Z_{K_s})$, and continuous dependance of the solution to a differential inclusion with respect to the right-hand side and the initial condition, there exists some $\eta > 0$, some open neighborhood U of $\operatorname{cl}(Z_{K_s})$ such that all trajectories of the differential inclusion

$$x'(t) \in f(x(t)) + \eta B \tag{3.22}$$

starting from points in U, leave K in a time smaller than $1/m$. From condition (3.18) and Lemma 3.1 (applied to $A = K\backslash Z_{K_s}$ and $g = f$), there exists $\alpha > 0$ such that for all x with $d_{Z_{K_s}}(x) < \eta$ we have

$$f(x) + \alpha d_{Z_{K_s}}(x)B \subset C_K(x), \quad \forall x \in \partial K\backslash Z_{K_s}. \tag{3.23}$$

Define then the Lipschitz set-valued map

$$F_m(x) := f(x) + \frac{\alpha}{2} d_{Z_{K_s}}(x)B, \tag{3.24}$$

and $S_{F_m}(x_0)$ the set of—absolutely continuous—solutions to

$$x'(t) \in F_m(x(t)), \quad t \geq 0,\ x(0) = x_0. \tag{3.25}$$

We claim that K_s is both

(a) the set of all points $x_0 \in \partial K$ such that all solutions to (3.25) leave K immediately;

(b) the set of all points $x_0 \in \partial K$ such *that there exists at least one solution* to (3.25) leaving K immediately.

We prove our claim. Fix $x_0 \in \partial K$. We consider the two following cases.

Case I ($x_0 \in \partial K\backslash Z_{K_s}$). From (3.23), we know that any trajectory to (3.25) starting from x_0 enters in K.

Case II ($x_0 \in \partial K \cap Z_{K_s}$). In this—set relatively open in ∂K—we have $F_m = f$. By the very definition of K_s we know that a solution to (3.25) starting from x_0 (which is locally also a solution to (1.2) because $F_m = f$ on the open set Z_{K_s}) leaves K immediately if and only if x_0 belongs to K_s.

This ends the proof of our claim.

Once again by [1, Lemma 4.2.2] and [16, Lemma 1.8], the function

$$\tau_m(x_0) := \sup_{x(\cdot)\in S_{F_m}(x_0)} \inf\{t > 0,\ x(t) \notin K\} \tag{3.26}$$

is continuous on K

$$W_m := \left\{x_0 \in K,\ \tau_m(x_0) < \frac{1}{m}\right\}. \tag{3.27}$$

Lemma 3.5. *Assume that (3.18) holds true. Then*

(i) *the set* $\text{cl}(W_m)$ *is epilipschitz and it contains* $\text{cl}(Z_{K_s})$,

(ii) $\chi(\text{cl}(W_m)) = \chi(K_s)$.

Proof. Remark that the choice of α and η implies $\text{cl}(Z_{K_s}) \subset W_m$.

Proof of (i). We claim that $\text{cl}(W_m)\backslash K_s$ is—locally in time—invariant by trajectories of the differential inclusion (3.25).

Indeed, let $x_0 \in \text{cl}(W_m)\backslash K_s$ and $x(\cdot) \in S_{F_m}(x_0)$. One can easily remark that for every $t \in [0, \tau_m(x_0)]$ and $y(\cdot) \in S_{F_m}(x(t))$,

$$\inf\{s > 0, y(s) \notin K\} \leq \tau_m(x(t)) + t \leq \tau_m(x_0) \leq \frac{1}{m}. \tag{3.28}$$

So $x(t) \in \text{cl}(W_m)$ for any $t \in [0, \tau_m(x_0)]$. Because $\tau_m(x_0) \neq 0$ our claim is proved.

Now we know that for any element v of the continuous convex map F_m, there exists a C^1 solution $x(\cdot) \in S_{F_m}(x_0)$ with $x'(0) = v$ (this can be viewed as a consequence of Michael selection theorem, see also [1, Corollary 5.3.2]). Because such a solution remains in $\mathrm{cl}(W_m)$ for small time, we have

$$\forall x \in \mathrm{cl}(W_m) \backslash K_s, \quad f(x) \in F_m(x) \subset T_{\mathrm{cl}(W_m)}(x). \tag{3.29}$$

Because (3.29) is valid for points in $\partial W_m \backslash \mathrm{cl}(Z_{K_s})$, by [2, Theorem 4.1.9]

$$F_m(x) \subset \liminf_{y \to x, x \in \partial W_m} T_{\mathrm{cl}(W_m)}(y) \subset C_{\mathrm{cl}(W_m)}(x). \tag{3.30}$$

So $\mathrm{int}(C_K(x_0)) \neq \emptyset$ for any $x \in \partial W_m \backslash \mathrm{cl}(Z_{K_s})$ because for such an x, the set $F_m(x)$ has a nonempty interior.

Consider $x \in \partial W_m \cap \mathrm{cl}(Z_{K_s}) \cap \partial K$. By the very definition of W_m, we have

$$((Z_{K_s} \cap \partial K) + rB) \cap K \subset W_m \tag{3.31}$$

for $r > 0$ small enough. Hence $C_K(x) \subset C_{\mathrm{cl}(W_m)}(x)$, these sets have nonempty interiors because K is epilipschitz.

Thus $\mathrm{int}(C_{\mathrm{cl}(W_m)}(x)) \neq \emptyset$ for any $x \in \partial W_m$. Hence $\mathrm{cl}(W_m)$ is epilipschitz, this completes the proof of (i).

Proof of (ii). For doing this we follow the same lines that in the proof of [16, Theorem A]. Define the following (multivalued) homotopy H:

$$\begin{aligned} &\mathrm{cl}(W_m) \times [0,1] \longmapsto \mathrm{cl}(W_m), \\ &(x_0, t) \longrightarrow H(x_0, t) := \bigcup_{x(\cdot) \in S_{F_m}(x_0)} x(t\theta(x(\cdot))) \end{aligned} \tag{3.32}$$

where for any absolutely continuous function $x(\cdot)$, we denote

$$\theta(x(\cdot)) := \inf\{s > 0,\ x(s) \notin K\}. \tag{3.33}$$

Clearly for any $x \in K_s$ we have $H(x,1) = x$ and $H(\cdot,0)$ is the identity map. Moreover H is an admissible map, in the sense or Gorniewicz [18] (cf. also [16]). So the Cech homology groups of K_s and $\mathrm{cl}(W_m)$ do coincide, so $\chi(\mathrm{cl}(W_m)) = \chi(K_s)$. This completes the proof. □

Remark 3.6. In the above proof, we have shown that the Euler characteristics of K_s and $\mathrm{cl}(W_m)$ do coincide *when characteristics are defined through Cech homology*. We underline that epilipschitz set (as K and $\mathrm{cl}(W_m)$) are absolute neighborhood retracts [4] and consequently Cech (co)homology and Singular homology are the same for these sets, hence so are Euler characteristics defined by Singular or Cech homologies.

3.3. Degree of f on K. We are now ready to prove the following crucial result.

Proposition 3.7. *Let K be epilipschitz compact. Assume that K_s is closed and that*

$$f(x) \neq 0, \quad \forall x \in \partial K. \tag{3.34}$$

Then

$$\deg(-f, K, 0) = \chi(K) - \chi(K_s). \tag{3.35}$$

Clearly our main result Theorem 1.1 is a direct consequence of the above proposition because if $\deg(-f,K,0) \neq 0$ then f has an equilibrium point in K (cf. for instance [11]).

Proof of Proposition 3.7. We shall argue in two separate case.
Case 1. We assume here that condition (3.18) holds true. Let m be large enough such that

$$0 \neq f(x), \quad \forall x \in \partial W_m. \tag{3.36}$$

Let consider Ψ given by Proposition 3.4. By defining $\Psi(x) = f(x)$ for $x \in K \setminus \operatorname{int}(W_m)$, one obtains an upper semicontinuous map with convex compact nonempty values which can be extended on K (cf. [8]) in a multivalued map denoted $\widehat{\Psi}$ with the same regularity. Thus

$$\deg(-\widehat{\Psi}, K, 0) = \deg\big(-\widehat{\Psi}, K \setminus \operatorname{cl}(W_m), 0\big) + \deg\big(-\widehat{\Psi}, \operatorname{cl}(W_m), 0\big). \tag{3.37}$$

By [11, Theorem 4.1] (or Lemma 2.3), Proposition 3.4 does imply

$$\chi(K) = \deg\big(-\widehat{\Psi}, K, 0\big). \tag{3.38}$$

The construction of W_m and (3.29) enables us to obtain

$$\widehat{\Psi}(x) \cap C_{\operatorname{cl}(W_m)}(x) \neq \varnothing, \quad \forall x \in \partial W_m. \tag{3.39}$$

Thus by the same argument (Lemma 2.3) applied to the epilipschitz set $\operatorname{cl}(W_m)$, we obtain $\chi(\operatorname{cl}(W_m)) = \deg(-\widehat{\Psi}, \operatorname{cl}(W_m), 0)$. Lemma 3.5 yields

$$\chi\big(\operatorname{cl}(W_m)\big) = \chi(K_s) = \deg\big(-\widehat{\Psi}, \operatorname{cl}(W_m), 0\big). \tag{3.40}$$

Moreover, since $f = \widehat{\Psi}$ on $\partial(K \setminus \operatorname{cl}(W_m))$ and because f has no equilibria on $\operatorname{cl}(W_m)$

$$\deg\big(-\widehat{\Psi}, K \setminus \operatorname{cl}(W_m), 0\big) = \deg\big(-f, K \setminus \operatorname{cl}(W_m), 0\big) = \deg(-f, K, 0). \tag{3.41}$$

In view of (3.37)–(3.41) we obtain (3.35).
Case 2. General case: $f(x) \in C_K(x)$ for any $x \in K \setminus K_s$. Consider g as given in Lemma 3.2. There exists $\bar{\varepsilon} > 0$ small enough such that

$$0 \notin f(x) + [0, \bar{\varepsilon}]\big(g(x) - f(x)\big), \quad \forall x \in \partial K. \tag{3.42}$$

Define the following continuous function:

$$\bar{f}(x) := f(x) + \min\{\bar{\varepsilon}, d_{K_s}(x)\}(g(x) - f(x)). \tag{3.43}$$

Thus $\deg(-f,K,0) = \deg(-\bar{f},K,0)$.

Because for $x \in \partial K \backslash K_s, f(x) \in C_K(x), g(x) \in \operatorname{int}(C_K(x))$ and $C_K(x)$ is convex, then $\bar{f}(x) \in \operatorname{int}(C_K(x))$. Since $f = \bar{f}$ on K_s then $K_s(f) = K_s(\bar{f})$, so we can apply the Case 1 for completing the proof. □

Remark 3.8. It is worth pointing out that in order to apply our main theorem, one has to check that K_s is closed. Note that K_s cannot, in general be described through geometric conditions but it is only defined through the behavior of trajectories of (1.2). But for instance, K_s can be approached by formulas like the following one:

$$K_{\Rightarrow} \subset K_s \subset \overline{K_{\Rightarrow}}, \tag{3.44}$$

where $K_{\Rightarrow} := \{x \in \partial K \mid f(x) \notin T_K(x)\}$. This approximation together with other more precise formulas were used and studied in [6, 16] (cf. [7] for the proofs).

Nevertheless, when K is more smooth, one can expect an analytic description of K_s in several cases. Suppose that $K = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\}$ where $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is of class C^2 with nonvanishing gradient on points x where $\varphi(x) = 0$. If the following condition holds true

$$[\varphi(x) = 0 \text{ and } \langle \nabla\varphi(x), f(x)\rangle = 0] \implies [\langle \nabla^2\varphi(x)f(x), f(x)\rangle > 0], \tag{3.45}$$

then one can easily check that K_s is closed and

$$K_s = \{x \in \mathbb{R}^n \mid \varphi(x) = 0 \text{ and } \langle \nabla\varphi(x), f(x)\rangle \geq 0\}. \tag{3.46}$$

4. Further extensions

Throughout this section $K \subset \mathbb{R}^n$ is epilipschitz compact.

One can expect that previous results could be extended to continuous functions and set-valued maps. In fact these two cases are related because Cauchy problem (1.1) can have more that one solution. We indicate several way of extensions of our results. For a set valued map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ upper semicontinuous with convex compact values, define

$$\begin{aligned} K_s(F) &:= \{x_0 \in \partial K,\ \forall x(\cdot) \in S_F(x_0),\ \exists \sigma > 0,\ x((0,\sigma]) \cap K = \varnothing\}, \\ K_e(F) &:= \{x_0 \in \partial K,\ \exists x(\cdot) \in S_F(x_0),\ \exists \sigma > 0,\ x((0,\sigma]) \cap K = \varnothing\}. \end{aligned} \tag{4.1}$$

The first set is the set of initial position such that *all* solutions of the differential inclusion leave K immediately while the second set is the set of intial conditions such that *at least one* solution leaves K immediately. Clearly these two sets reduces to K_s when F is single-valued Lipschitz.

Proposition 4.1. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous function. Suppose that $K_s(f)$ is closed. If $\chi(K_s(f)) \neq \chi(K)$ then an equilibrium of f exists in K.*

Proof. Suppose that f has no equilibrium in ∂K (else the proof is finished). For any $x_0 \in \partial K \backslash K_s(f)$ athere exists a solution remaining locally in K (i.e., a solution $x(\cdot)$ to (1.2) and $x(0) = x_0$ such that there exists $s > 0$ with $x[0,s] \subset K$). So $f(x_0) \in C_K(x_0)$ (same argument that (3.30)). Consider g given in Lemma 3.2. Pose

$$\tilde{f}(x) = f(x) + \delta d_{K_s(f)}(x)\big(f(x) - g(x)\big) \tag{4.2}$$

for $\delta > 0$ sufficiently small such that $\tilde{f}$ has no equilibrium in a neighborhood of ∂K (so it has the same degree that f in K). One can deduce from [2, Theorem 4.1.9] as in Section 1, that

$$\tilde{f}(x) \in \operatorname{int}\big(C_K(x)\big), \quad \forall x \in \partial K \backslash K_s(f). \tag{4.3}$$

So the sets $K_s(\tilde{f})$, $K_e(\tilde{f})$, and $K_s(f)$ are equal. (This point is crucial. It allows to prove as in [16] that functions τ_K, τ_m are continuous.) One can deduce using same arguments that in Section 3 that

$$\deg(-f,K,0) = \deg\big(-\tilde{f},K,0\big) = \chi(K) - \chi(K_s(\tilde{f})). \tag{4.4}$$

□

Proposition 4.2. *Let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous set valued map with convex compact nonempty values. Suppose that $K_s(F)$ is closed and equal to $K_e(F)$. If $\chi(K_s(F)) \neq \chi(K)$, then there exists an equilibrium point x of F in K (namely $x \in K$ with $0 \in F(x)$).*

Proof. For any continuous selection f of F, we have easily $K_s(f) = K_s(F)$. By Proposition 4.1, f has an equilibrium in K, consequently so does F. □

Note that the above theorem is false without the assumption $K_s(F) = K_e(F)$ as shown in the following.

Example 4.3. In $\mathbb{R}^2$ we consider the *constant* set-valued map $F(x,y) = \{1\} \times [-1,+1]$. Consider

$$K = \big\{(x,y) \in \mathbb{R}^2 \,|\, 0 \le y \le 4,\ |x| \le y \le |x|+1\big\}. \tag{4.5}$$

Then one easily obtains $K_s(F) = ([-4,-3] \times \{4\}) \cup ([3,4] \times \{4\})$. So $\chi(K_s) = 2 \neq \chi(K) = 1$. But obviously there is no equilibria of F in K.

Remark 4.4. When $K_s(F) \neq K_e(F)$ one could expect to find a selection f of F with $K_s(f) = K_e(F)$. This seems be difficult without assuming more regularity assumptions on the boundary of K, moreover this is out of the scope of the present paper devoted to epilipschitz compact sets. We refer the reader to [7] for a detailed study of this question for very smooth sets.

Remark 4.5. Surprisingly, we do not need the epilipschitz assumption on K when $K_s = \partial K$; but in this case the approach is rather different (this case is studied in [17]).

References

[1] J.-P. Aubin, *Viability Theory*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Massachusetts, 1991.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Systems & Control: Foundations & Applications, vol. 2, Birkhäuser Boston, Massachusetts, 1990.

[3] H. Ben-El-Mechaiekh and W. Kryszewski, *Equilibria of set-valued maps on nonconvex domains*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4159–4179.

[4] J.-M. Bonnisseau and B. Cornet, *Fixed-point theorems and Morse's lemma for Lipschitzian functions*, J. Math. Anal. Appl. **146** (1990), no. 2, 318–332.

[5] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[6] P. Cardaliaguet, *Conditions suffisantes de non-vacuité du noyau de viabilité* [*Sufficient conditions for nonemptiness of the viability kernel*], C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), no. 11, 797–800 (French).

[7] P. Cardaliaguet, G. Gabor, and M. Quincampoix, *Equilibria and strict equilibria of multivalued maps on noninvariant sets*, Ann. Polon. Math. **82** (2003), no. 1, 19–37.

[8] A. Cellina, *A theorem on the approximation of compact multivalued mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **47** (1970), 429–433.

[9] A. Cellina and A. Lasota, *A new approach to the definition of topological degree for multi-valued mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **47** (1970), 434–440.

[10] B. Cornet, *Paris avec handicaps et théorèmes de surjectivité de correspondances*, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), no. 12, A479–A482 (French).

[11] ———, *Euler characteristic and fixed-point theorems*, Positivity **6** (2002), no. 3, 243–260.

[12] B. Cornet and M.-O. Czarnecki, *Smooth representations of epi-Lipschitzian subsets of* $\mathbf{R}^n$, Nonlinear Anal. Ser. A: Theory Methods **37** (1999), no. 2, 139–160.

[13] ———, *Existence of (generalized) equilibria: necessary and sufficient conditions*, Comm. Appl. Nonlinear Anal. **7** (2000), no. 1, 21–53.

[14] A. Dold, *Lectures on Algebraic Topology*, Grundlehren der Mathematischen Wissenschaften, vol. 200, Springer-Verlag, Berlin, 1980.

[15] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.

[16] G. Gabor and M. Quincampoix, *On existence of solutions to differential equations or inclusions remaining in a prescribed closed subset of a finite-dimensional space*, J. Differential Equations **185** (2002), no. 2, 483–512.

[17] ———, *On equilibria of Lipschitz maps in finite-dimensional spaces*, Boll. Un. Mat. Ital. **8** (2003), no. 6-B, 308–321.

[18] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and Its Applications, vol. 495, Kluwer Academic, Dordrecht, 1999.

[19] A. Granas, *Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 191–194 (French).

[20] J. W. Milnor, *Topology from the Differentiable Viewpoint*, The University Press of Virginia, Virginia, 1965.

[21] M. Mrozek, *Periodic and stationary trajectories of flows and ordinary differential equations*, Univ. Iagel. Acta Math. (1988), no. 27, 29–37.

[22] S. Plaskacz, *Periodic solutions of differential inclusions on compact subsets of* $\mathbf{R}^n$, J. Math. Anal. Appl. **148** (1990), no. 1, 202–212.

[23] ———, *On the solution sets for differential inclusions*, Boll. Un. Mat. Ital. A (7) **6** (1992), no. 3, 387–394.

[24] M. Quincampoix and V. Veliov, *Viability with a target: theory and applications*, Applications of Mathematics in Engineering (Sozopol, 1997) (B. Cheshankov and M. Todorov, eds.), Heron Press, Sofia, 1998, pp. 47–54.

[25] R. T. Rockafellar, *Clarke's tangent cones and the boundaries of closed sets in* $\mathbf{R}^{\mathbf{n}}$, Nonlinear Anal. **3** (1979), no. 1, 145–154.

Mikhail Kamenskii: Voronezh State University, Universitetskaya pl.1, 394063 Voronezh, Russia
E-mail address: kmi@kma.vsu.ru

Marc Quincampoix: Laboratoire de Mathématiques, Unité CNRS UMR 6205, Université de Bretagne Occidentale, 6 Avenue Victor Le Gorgeu, 29200 Brest, France
E-mail address: marc.quincampoix@univ-brest.fr