APPROXIMATING FIXED POINTS OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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We introduce a new class of asymptotically nonexpansive mappings and study approximating methods for finding their fixed points. We deal with the Krasnosel'skii-Mann-type iterative process. The strong and weak convergence results for self-mappings in normed spaces are presented. We also consider the asymptotically weakly contractive mappings.

1. Introduction

Let K be a nonempty subset of a real linear normed space E. Let T be a self-mapping of K. Then $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$ the following inequality holds:

$$\left\|T^n x - T^n y\right\| \leq k_n \|x - y\|, \quad \forall n \geq 1. \tag{1.2}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [18] as a generalization of the class of nonexpansive maps. They proved that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Alber and Guerre-Delabriere have studied in [3–5] weakly contractive mappings of the class C_ψ.

Definition 1.1. An operator T is called weakly contractive of the class C_ψ on a closed convex set K of the normed space E if there exists a continuous and increasing function $\psi(t)$ defined on R^+ such that ψ is positive on $R^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to+\infty} \psi(t) = \infty$ and

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for all $x, y \in K$,

$$\|Tx - Ty\| \le \|x - y\| - \psi(\|x - y\|). \tag{1.3}$$

The class C_ψ of weakly contractive maps contains the class of strongly contractive maps and it is contained in the class of nonexpansive maps. In [3–5], in fact, there is also the concept of the asymptotically weakly contractive mappings of the class C_ψ.

Definition 1.2. The operator T is called asymptotically weakly contractive of the class C_ψ if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and strictly increasing function $\psi : R^+ \to R^+$ with $\psi(0) = 0$ such that for all $x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \le k_n \|x - y\| - \psi(\|x - y\|), \quad \forall n \ge 1. \tag{1.4}$$

Bruck et al. have introduced in [11] asymptotically nonexpansive in the intermediate sense mappings.

Definition 1.3. An operator T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.5}$$

Observe that if

$$a_n := \sup_{x,y\in K} \left(\|T^n x - T^n y\| - \|x - y\|\right), \tag{1.6}$$

then (1.5) reduces to the relation

$$\|T^n x - T^n y\| \le \|x - y\| + a_n, \quad \forall x, y \in K. \tag{1.7}$$

It is known [23] that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive maps (see, e.g., [22]).

Iterative techniques are the main tool for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings, and it has been studied by various authors using Krasnosel'skii-Mann and Ishikawa schemes (see, e.g., [12, 13, 15, 20, 21, 25, 27–37]).

Bose in [10] proved that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E satisfying Opial's condition [26] and $T : K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, that is, the limit equality

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0 \tag{1.8}$$

holds. Passty [28] and also Xu [38] showed that the requirement of the Opial's condition can be replaced by the Fréchet differentiability of the space norm. Furthermore, Tan and Xu established in [34, 35] that the asymptotic regularity of T at a point x can be weakened to the so-called weakly asymptotic regularity of T at x, defined as follows:

$$\omega - \lim_{n\to\infty} \left(T^n x - T^{n+1}x\right) = 0. \tag{1.9}$$

In [31, 32], Schu introduced a modified Krasnosel'skii-Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a uniformly convex Banach space E. In particular, he proved that the iterative sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.10}$$

converges weakly to some fixed point of T if the Opial's condition holds, $\{k_n\}_{n\geq1} \subset [1,\infty)$ for all $n \geq 1$, $\lim k_n = 1$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\{\alpha_n\}_{n\geq1}$ is a real sequence satisfying the inequalities $0 < \bar{\alpha} \leq \alpha_n \leq \tilde{\alpha} < 1$, $n \geq 1$, for some positive constants $\bar{\alpha}$ and $\tilde{\alpha}$. However, Schu's result does not apply, for instance, to L^p spaces with $p \neq 2$ because none of these spaces satisfy the Opial's condition.

In [30], Rhoades obtained strong convergence theorem for asymptotically nonexpansive mappings in uniformly convex Banach spaces using a modified Ishikawa iteration method. Osilike and Aniagbosor proved in [27] that the results of [30–32] still remain true without the boundedness requirement imposed on K, provided that $\mathcal{N}(T) = \{x \in K : Tx = x\} \neq \emptyset$. In [37], Tan and Xu extended Schu's theorem [32] to uniformly convex spaces with a Fréchet differentiable norm. Therefore, their result covers L^p spaces with $1 < p < \infty$.

Chang et al. [12] established convergence theorems for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces without assuming any of the following properties: (i) E satisfies the Opial's condition; (ii) T is asymptotically regular or weakly asymptotically regular; (iii) K is bounded. Their results improve and generalize the corresponding results of [10, 19, 28, 29, 32, 34, 35, 37, 38] and others.

Recently, Kim and Kim [22] studied the strong convergence of the Krasnosel'skii-Mann and Ishikawa iterations with errors for asymptotically nonexpansive in the intermediate sense operators in Banach spaces.

In all the above papers, the operator T remains a self-mapping of nonempty closed convex subset K in a uniformly convex Banach space. If, however, domain $D(T)$ of T is a proper subset of E (and this is indeed the case for several applications), and T maps $D(T)$ into E, then the Krasnosel'skii-Mann and Ishikawa iterative processes and Schu's modifications of type (1.10) may fail to be well-defined.

More recently, Chidume et al. [14] proved the convergence theorems for asymptotically nonexpansive nonself-mappings in Banach spaces by having extended the corresponding results of [12, 27, 30].

The purpose of this paper is to introduce more general classes of asymptotically nonexpansive mappings and to study approximating methods for finding their fixed points.

We deal with self- and nonself-mappings and the Krasnosel'skii-Mann-type iterative process (1.10). The Ishikawa iteration scheme is beyond the scope of this paper.

Definition 1.4. A mapping $T : E \to E$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n \geq 1$, with $k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)}\phi(\|x - y\|) + k_n^{(2)}. \tag{1.11}$$

Remark 1.5. If $\phi(\lambda) = \lambda$, then (1.11) takes the form

$$\|T^n x - T^n y\| \leq (1 + k_n^{(1)})\|x - y\| + k_n^{(2)}. \tag{1.12}$$

In addition, if $k_n^{(2)} = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$ for all $n \geq 1$, then we obtain from (1.11) the class of nonexpansive mappings.

Definition 1.6. A mapping T is called *total asymptotically weakly contractive* if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n \geq 1$, with $k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi, \psi : R^+ \to R^+$ with $\phi(0) = \psi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)}\phi(\|x - y\|) - \psi(\|x - y\|) + k_n^{(2)}. \tag{1.13}$$

Remark 1.7. If $\phi(\lambda) = \lambda$, then (1.13) accepts the form

$$\|T^n x - T^n y\| \leq (1 + k_n^{(1)})\|x - y\| - \psi(\|x - y\|) + k_n^{(2)}. \tag{1.14}$$

In addition, if $k_n^{(2)} = 0$ for all $n \geq 1$, then total asymptotically weakly contractive mapping coincides with the earlier known asymptotically weakly contractive mapping. If $k_n^{(2)} = 0$ and $k_n^{(1)} = 0$, then we obtain from (1.13) the class of weakly contractive mappings. If $k_n^{(1)} \equiv 0$ and $k_n^{(2)} \equiv a_n$, where $a_n := \sup_{x,y \in K}(\|T^n x - T^n y\| - \|x - y\|)$ for all $n \geq 0$, then (1.13) reduces to (1.7) which has been studied as asymptotically nonexpansive mappings in the intermediate sense.

The paper is organized in the following manner. In Section 2, we present characteristic inequalities from the standpoint of their being an important component of common theory of Banach space geometry. Section 3 is dedicated to numerical recurrent inequalities that are a crucial tool in the investigation of convergence and stability of iterative methods. In Section 4, we study the convergence of the iterative process (1.10) with total asymptotically weakly contractive mappings. The next two sections deal with total asymptotically nonexpansive mappings.

2. Banach space geometry and characteristic inequalities

Let E be a real uniformly convex and uniformly smooth Banach space (it is a reflexive space), and let E^* be a dual space with the bilinear functional of duality $\langle \phi, x \rangle$ between

$\phi \in E^*$ and $x \in E$. We denote the norms of elements in E and E^* by $\|\cdot\|$ and $\|\cdot\|_*$, respectively.

A uniform convexity of the Banach space E means that for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$, $\|x\| \le 1$, $\|y\| \le 1$, $\|x - y\| = \varepsilon$ the inequality

$$\|x + y\| \le 2(1 - \delta) \tag{2.1}$$

is satisfied. The function

$$\delta_E(\varepsilon) = \inf\left\{1 - 2^{-1}\|x + y\|,\ \|x\| = 1,\ \|y\| = 1,\ \|x - y\| = \varepsilon\right\} \tag{2.2}$$

is called to be modulus of convexity of E.

A uniform smoothness of the Banach space E means that for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$, $\|x\| = 1$, $\|y\| \le \delta$ the inequality

$$2^{-1}\left(\|x + y\| + \|x - y\|\right) - 1 \le \varepsilon\|y\| \tag{2.3}$$

holds. The function

$$\rho_E(\tau) = \sup\left\{2^{-1}\left(\|x + y\| + \|x - y\|\right) - 1,\ \|x\| = 1,\ \|y\| = \tau\right\} \tag{2.4}$$

is called to be modulus of smoothness of E.

The moduli of convexity and smoothness are the basic quantitative characteristics of a Banach space that describe its geometric properties [2, 16, 17, 24]. Let us observe that the space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$ and it is uniformly smooth if and only if $\lim_{\tau \to 0} \tau^{-1}\rho_E(\tau) = 0$.

The following properties of the functions $\delta_E(\varepsilon)$ and $\rho_E(\tau)$ are important to keep in mind throughout of this paper:

(i) $\delta_E(\varepsilon)$ is defined on the interval $[0,2]$, continuous and increasing on this interval, $\delta_E(0) = 0$,

(ii) $0 < \delta_E(\varepsilon) < 1$ if $0 < \varepsilon < 2$,

(iii) $\rho_E(\tau)$ is defined on the interval $[0,\infty)$, convex, continuous and increasing on this interval, $\rho_E(0) = 0$,

(iv) the function $g_E(\varepsilon) = \varepsilon^{-1}\delta_E(\varepsilon)$ is continuous and non-decreasing on the interval $[0,2]$, $g_E(0) = 0$,

(v) the function $h_E(\tau) = \tau^{-1}\rho_E(\tau)$ is continuous and non-decreasing on the interval $[0,\infty)$, $h_E(0) = 0$,

(vi) $\varepsilon^2\delta_E(\eta) \ge (4L)^{-1}\eta^2\delta_E(\varepsilon)$ if $\eta \ge \varepsilon > 0$ and $\tau^2\rho_E(\sigma) \le L\sigma^2\rho_E(\tau)$ if $\sigma \ge \tau > 0$. Here $1 < L < 1.7$ is the Figiel constant.

We recall that nonlinear in general operator $J : E \to E^*$ is called normalized duality mapping if

$$\|Jx\|_* = \|x\|, \qquad \langle Jx, x\rangle = \|x\|^2. \tag{2.5}$$

It is obvious that this operator is coercive because of

$$\frac{\langle Jx, x\rangle}{\|x\|} \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty \tag{2.6}$$

and monotone due to

$$\langle Jx - Jy, x - y\rangle \geq \left(\|x\| - \|y\|\right)^2. \tag{2.7}$$

In addition,

$$\langle Jx - Jy, x - y\rangle \leq \left(\|x\| + \|y\|\right)^2. \tag{2.8}$$

A normalized duality mapping $J^* : E^* \to E$ can be introduced by analogy. The properties of the operators J and J^* have been given in detail in [2].

Let us present the estimates of the normalized duality mappings used in the sequel (see [2]). Let $x, y \in E$. We denote

$$R_1 = R_1(\|x\|, \|y\|) = \sqrt{2^{-1}(\|x\|^2 + \|y\|^2)}. \tag{2.9}$$

Lemma 2.1. *In a uniformly convex Banach space E*

$$\langle Jx - Jy, x - y\rangle \geq 2R_1^2 \delta_E(\|x - y\|/2R_1). \tag{2.10}$$

If $\|x\| \leq R$ and $\|y\| \leq R$, then

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1} R^2 \delta_E(\|x - y\|/2R). \tag{2.11}$$

Lemma 2.2. *In a uniformly smooth Banach space E*

$$\langle Jx - Jy, x - y\rangle \leq 2R_1^2 \rho_E(4\|x - y\|/R_1). \tag{2.12}$$

If $\|x\| \leq R$ and $\|y\| \leq R$, then

$$\langle Jx - Jy, x - y\rangle \leq 2LR^2 \rho_E(4\|x - y\|/R). \tag{2.13}$$

Next we present the upper and lower characteristic inequalities in E (see [2]).

Lemma 2.3. *Let E be uniformly convex Banach space. Then for all $x, y \in E$ and for all $0 \leq \lambda \leq 1$*

$$\left\|\lambda x + (1-\lambda) y\right\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - 2\lambda(1-\lambda) R_1^2 \delta_E(\|x - y\|/2R_1). \tag{2.14}$$

If $\|x\| \leq R$ and $\|y\| \leq R$, then

$$\left\|\lambda x + (1-\lambda) y\right\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - L^{-1}\lambda(1-\lambda) R^2 \delta_E(\|x - y\|/2R). \tag{2.15}$$

Lemma 2.4. *Let E be uniformly smooth Banach space. Then for all $x, y \in E$ and for all $0 \leq \lambda \leq 1$*

$$\left\|\lambda x + (1-\lambda) y\right\|^2 \geq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - 8\lambda(1-\lambda) R_1^2 \rho_E(4\|x - y\|/R_1). \tag{2.16}$$

If $\|x\| \leq R$ and $\|y\| \leq R$, then

$$||\lambda x + (1-\lambda)y||^2 \geq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - 16L\lambda(1-\lambda)R^2\rho_E(4\|x-y\|/R). \qquad (2.17)$$

3. Recurrent numerical inequalities

Lemma 3.1 (see, e.g., [7]). *Let $\{\lambda_n\}_{n\geq 1}$, $\{\kappa_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be sequences of nonnegative real numbers such that for all $n \geq 1$*

$$\lambda_{n+1} \leq (1+\kappa_n)\lambda_n + \gamma_n. \qquad (3.1)$$

Let $\sum_1^\infty \kappa_n < \infty$ and $\sum_1^\infty \gamma_n < \infty$. Then $\lim_{n\to\infty}\lambda_n$ exists.

Lemma 3.2 [1, 8]. *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions*

$$\sum_1^\infty \alpha_n = \infty, \qquad \lim_{n\to\infty}\frac{\gamma_n}{\alpha_n} \longrightarrow 0. \qquad (3.2)$$

Let the recursive inequality

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\psi(\lambda_n) + \gamma_n, \quad n = 1,2,\ldots, \qquad (3.3)$$

be given, where $\psi(\lambda)$ is a continuous and nondecreasing function from R^+ to R^+ such that it is positive on $R^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty}\psi(t) > 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

We present more general statement.

Lemma 3.3. *Let $\{\lambda_k\}$, $\{\kappa_n\}_{n\geq 1}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions*

$$\sum_1^\infty \alpha_n = \infty, \quad \sum_1^\infty \kappa_n < \infty, \quad \frac{\gamma_n}{\alpha_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \qquad (3.4)$$

Let the recursive inequality

$$\lambda_{n+1} \leq (1+\kappa_n)\lambda_n - \alpha_n\psi(\lambda_n) + \gamma_n, \quad n = 1,2,\ldots, \qquad (3.5)$$

be given, where $\psi(\lambda)$ is the same as in Lemma 3.2. Then $\lambda_n \to 0$ as $n \to \infty$.

Proof. We produce in (3.5) the following replacement:

$$\lambda_n = \mu_n \Pi_{j=1}^{n-1}(1+\kappa_n). \qquad (3.6)$$

Then

$$\mu_{n+1} \leq \mu_n - \alpha_n\Big(\Pi_{j=1}^{n-1}(1+\kappa_n)\Big)^{-1}\psi\Big(\mu_n\Pi_{j=1}^{n-1}(1+\kappa_n)\Big) + \Big(\Pi_{j=1}^{n-1}(1+\kappa_n)\Big)^{-1}\gamma_n. \qquad (3.7)$$

Since $\sum_1^\infty \kappa_n < \infty$, we conclude that there exists a constant $C > 0$ such that

$$1 \leq \Pi_{j=1}^{n-1}(1+\kappa_n) \leq C. \qquad (3.8)$$

Therefore, taking into account nondecreasing property of ψ, we have

$$\mu_{n+1} \leq \mu_n - \alpha_n C^{-1}\psi(\mu_n) + \gamma_n. \tag{3.9}$$

Consequently, by Lemma 3.2, $\mu_n \to 0$ as $n \to \infty$ and this implies $\lim_{n\to\infty} \lambda_n = 0$. □

Lemma 3.4. *Let $\{\lambda_n\}_{n\geq 1}$, $\{\kappa_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be nonnegative, $\{\alpha_n\}_{n\geq 1}$ be positive real numbers such that*

$$\lambda_{n+1} \leq \lambda_n + \kappa_n\phi(\lambda_n) - \alpha_n\psi(\lambda_n) + \gamma_n, \quad \forall n \geq 1, \tag{3.10}$$

where $\phi, \psi : R^+ \to R^+$ are strictly increasing and continuous functions such that $\phi(0) = \psi(0) = 0$. Let for all $n > 1$

$$\frac{\gamma_n}{\alpha_n} \leq c_1, \quad \frac{\kappa_n}{\alpha_n} \leq c_2, \quad \alpha_n \leq \alpha < \infty, \tag{3.11}$$

where $0 \leq c_1$, $c_2 < \infty$. Assume that the equation $\psi(\lambda) = c_1 + c_2\phi(\lambda)$ has the unique root λ_* on the interval $(0,\infty)$ and

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\phi(\lambda)} > c_2. \tag{3.12}$$

Then $\lambda_n \leq \max\{\lambda_1, K_*\}$, where $K_* = \lambda_* + \alpha(c_1 + c_2\phi(\lambda_*))$. In addition, if

$$\sum_1^\infty \alpha_n = \infty, \qquad \frac{\gamma_n + \kappa_n}{\alpha_n} \longrightarrow 0, \tag{3.13}$$

then $\lambda_n \to 0$ as $n \to \infty$.

Proof. For each $n \in I = \{1,2,\ldots\}$, just one alternative can happen: either

$$H_1 : \kappa_n\phi(\lambda_n) - \alpha_n\psi(\lambda_n) + \gamma_n > 0, \tag{3.14}$$

or

$$H_2 : \kappa_n\phi(\lambda_n) - \alpha_n\psi(\lambda_n) + \gamma_n \leq 0. \tag{3.15}$$

Denote $I_1 = \{n \in I \mid H_1 \text{ is true}\}$ and $I_2 = \{n \in I \mid H_2 \text{ is true}\}$. It is clear that $I_1 \cup I_2 = I$.

(i) Let $c_1 > 0$. Since $\psi(0) = 0$, we see that hypothesis H_1 is valid on the interval $(0,\lambda_*)$ and H_2 is valid on $[\lambda_*,\infty)$. Therefore, the following result is obtained:

$$\begin{gathered} \lambda_n \leq \lambda_*, \quad \forall n \in I_1 = \{1,2,\ldots,N\}, \\ \lambda_{N+1} \leq \lambda_N + \gamma_N + \kappa_N\phi(\lambda_N) \leq \lambda_* + \gamma_N + \kappa_N\phi(\lambda_*) \leq K_*, \\ \lambda_n \leq \lambda_{N+1} \leq K_*, \quad \forall n \geq N+2. \end{gathered} \tag{3.16}$$

Thus, $\lambda_n \leq K_*$ for all $n \geq 1$.

(ii) Let $c_1 = 0$. This takes place if $\gamma_n = 0$ for all $n > 1$. In this case, along with situation described above it is possible $I_2 = I$ and then $\lambda_n < \lambda_1$ for all $n \geq 1$. Hence, $\lambda_n \leq \max\{\lambda_1, K_*\} = \bar{C}$. The second assertion follows from Lemma 3.2 because

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \kappa_n \phi(\bar{C}) + \gamma_n, \quad n = 1,2,\dots. \tag{3.17}$$

□

Lemma 3.5. *Suppose that the conditions of the previous lemma are fulfilled with positive κ_n for $n \geq 1$, $0 < c_1 < \infty$, and the equation $\psi(\lambda) = c_1 + c_2\phi(\lambda)$ has a finite number of solutions $\lambda_*^{(1)}, \lambda_*^{(2)}, \dots, \lambda_*^{(l)}$, $l \geq 1$. Then there exists a constant $\bar{C} > 0$ such that all the conclusions of Lemma 3.4 hold.*

Proof. It is sufficiently to consider the following two cases.

(i) If there is no points of contact among $\lambda_*^{(l)}$, $i = 1,2,\dots,l$, then

$$I = I_1^{(1)} \cup I_2^{(1)} \cup I_1^{(2)} \cup I_2^{(2)} \cup I_1^{(3)} \cup I_2^{(3)} \cup \cdots \cup I_1^{(l)} \cup I_2^{(l)}, \tag{3.18}$$

where $I_1^{(k)} \subset I_1$ and $I_2^{(k)} \subset I_2$, $k = 1,2,\dots,l$. It is not difficult to see that $\lambda_n \leq \lambda_*$ on the interval $I_1^{(1)}$. Denote $N_1^{(1)} = \max\{n \mid n \in I_1^{(1)}\}$. Then $N_1^{(1)} + 1 = \min\{n \mid n \in I_2^{(1)}\}$ and this yields the inequality

$$\lambda_{N_1^{(1)}+1} \leq \lambda_{N_1^{(1)}} + \gamma_{N_1^{(1)}} + \kappa_{N_1^{(1)}} \phi(\lambda_{N_1^{(1)}}) \leq \lambda_* + \gamma_{N_1^{(1)}} + \kappa_{N_1^{(1)}} \phi(\lambda_*) \leq K_*. \tag{3.19}$$

By the hypothesis H_2, for the rest $n \in I_2^{(1)}$, we have $\lambda_n \leq \lambda_{N_1^{(1)}+1} \leq K_*$. The same situation arrises on the intervals $I_1^{(2)} \cup I_2^{(2)}$, $I_1^{(3)} \cup I_1^{(3)}$, and so forth. Thus, $\lambda_n \leq K_*$ for all $n \in I$.

(ii) If some $\lambda_*^{(i)}$ is a point of contact, then either $I^i \subset I_2$ and $I^{i+1} \subset I_2$ or $I^i \subset I_1$ and $I^{i+1} \subset I_1$. We presume, respectively, $I^i \cup I^{i+1} \subset I_2$ and $I^i \cup I^{i+1} \subset I_1$ and after this number intervals again. It is easy to verify that the proof coincides with the case (i). □

Remark 3.6. Lemma 3.4 remains still valid if the equation $\psi(\lambda) = c_1 + c_2\phi(\lambda)$ has a manifold of solutions on the interval $(0, \infty)$.

Lemma 3.7 (see [6]). *Let $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences of non-negative real numbers satisfying the recurrence inequality*

$$\mu_{n+1} \leq \mu_n - \alpha_n \beta_n + \gamma_n. \tag{3.20}$$

Assume that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty. \tag{3.21}$$

Then

(i) *there exists an infinite subsequence $\{\beta_{\ell_n}\} \subset \{\beta_n\}$ such that*

$$\beta_{\ell_n} \leq \frac{1}{\sum_{j=1}^{\ell_n} \alpha_j}, \tag{3.22}$$

and, consequently, $\lim_{n\to\infty} \beta_{\ell_n} = 0$;

(ii) *if* $\lim_{n\to\infty}\alpha_n = 0$ *and there exists a constant* $\kappa > 0$ *such that*

$$|\beta_{n+1} - \beta_n| \le \kappa\alpha_n \tag{3.23}$$

for all $n \ge 1$, *then* $\lim_{n\to\infty}\beta_n = 0$.

4. Convergence analysis of the iterations (1.10) with total asymptotically weakly contractive mappings

In this section, we are going to prove the strong convergence of approximations generated by the iterative process (1.10) to fixed points of the total asymptotically weakly contractive mappings $T: K \to K$, where $K \subseteq E$ is a nonempty closed convex subset. In the sequal, we denote a fixed point set of T by $\mathcal{N}(T)$, that is, $\mathcal{N}(T) := \{x \in K : Tx = x\}$.

Theorem 4.1. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T: K \to K$ be a mapping which is total asymptotically weakly contractive. Suppose that $\mathcal{N}(T) \neq \emptyset$ and $x^* \in \mathcal{N}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by the iterative scheme (1.10), where $\{\alpha_n\}_{n\ge1} \subset (0,1)$ such that $\sum \alpha_n = \infty$. Suppose that there exist constants $m_1, m_2 > 0$ such that $k_n^{(1)} \le m_1$, $k_n^{(2)} \le m_2$,*

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\phi(\lambda)} > m_1 \tag{4.1}$$

and the equation $\psi(\lambda) = m_1\phi(\lambda) + m_2$ has the unique root λ_*. Then $\{x_n\}$ converges strongly to x^*.

Proof. Since K is closed convex subset of E, $T: K \to K$ and $\{\alpha_n\}_{n\ge1} \subset (0,1)$, we conclude that $\{x_n\} \subset K$. We first show that the sequence $\{x_n\}$ is bounded. From (1.10) and (1.13) one gets

$$\begin{aligned}
||x_{n+1} - x^*|| &\le ||(1-\alpha_n)x_n + \alpha_n T^n x_n - x^*|| \\
&\le (1-\alpha_n)||x_n - x^*|| + \alpha_n||T^n x_n - T^n x^*|| \\
&\le ||x_n - x^*|| + \alpha_n k_n^{(1)}\phi(||x_n - x^*||) - \alpha_n\psi(||x_n - x^*||) + \alpha_n k_n^{(2)}.
\end{aligned} \tag{4.2}$$

By Lemma 3.4, we obtain that $\{x_n - x^*\}$ is bounded, namely, $\|x_n - x^*\| \le \bar{C}$, where

$$\bar{C} = \max\{||x_1 - x^*||, \lambda_* + m_1\phi(\lambda_*) + m_2\}. \tag{4.3}$$

Next the convergence $x_n \to x^*$ is shown by the relation

$$||x_{n+1} - x^*|| \le ||x_n - x^*|| - \alpha_n\psi(||x_n - x^*||) + \alpha_n k_n^{(1)}\phi(\bar{C}) + \alpha_n k_n^{(2)}, \tag{4.4}$$

applying Lemma 3.2 to the recurrent inequality (3.5) with $\lambda_n = \|x_n - x^*\|$. □

In particular, if $\psi(t)$ is convex, continuous and non-decreasing, $\phi(t) = t$, $k_n^{(2)} = 0$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \alpha_n k_n^{(1)} < \infty$, then there holds the estimate

$$||x_n - x^*|| \leq \bar{R}\Phi^{-1}\left(\Phi(||x_1 - x^*||) - (1+a)^{-1}\sum_{i=1}^{n-1}\alpha_i\right), \tag{4.5}$$

where $\alpha k_n^{(1)} \leq a$ and $\Pi_{i=1}^{\infty}(1+\alpha_n k_n^{(1)}) \leq \bar{R} < \infty$, Φ is defined by the formula $\Phi(t) = \int (dt/\psi(t))$ and Φ^{-1} is the inverse function to Φ. Observe that a and $\bar{R}$ exists because the series $\sum_{n=1}^{\infty} \alpha_n k_n^{(1)}$ is convergent.

Theorem 4.2. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a mapping which is total asymptotically weakly contractive. Suppose that $\mathcal{N}(T) \neq \emptyset$ and $x^* \in \mathcal{N}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (1.10), where $\{\alpha_n\}_{n\geq 1} \subset (0,c]$ with some $c > 0$ such that $\sum \alpha_n = \infty$. Suppose that $k_n^{(1)} \leq 1$, and there exists $M > 0$ such that $\phi(\lambda) \leq \psi(\lambda)$ for all $\lambda \geq M$. Then $\{x_n\}$ converges strongly to x^*.*

Proof. Since ϕ and ψ are increasing functions, we have

$$\phi(\lambda) \leq \phi(M) + \psi(\lambda). \tag{4.6}$$

Then

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| - \alpha_n(1 - k_n^{(1)})\psi(||x_n - x^*||) + \alpha_n k_n^{(1)}\phi(M) + \alpha_n k_n^{(2)}, \tag{4.7}$$

and the result follows from Lemma 3.2 again. □

The following theorem gives the sufficient convergence condition of the scheme (1.10) which includes $\phi(\lambda) = \lambda^p$, $0 < p \leq 1$, regardless of what ψ is.

Theorem 4.3. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a mapping which is total asymptotically weakly contractive. Suppose that $\mathcal{N}(T) \neq \emptyset$ and there exist positive constants M_0 and $M > 0$ such that $\phi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ as (1.10), where $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ such that $\sum_1^{\infty} \alpha_n = \infty$. Suppose that $\sum_1^{\infty} \alpha_n k^{(1)} < \infty$. Then $\{x_n\}$ converges strongly to x^*.*

Proof. We follow the proof scheme of Theorem 4.1 to show that $\{x_n\}$ is bounded. Since $\phi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$, one can deduce from (4.2) the inequality

$$||x_{n+1} - x^*|| \leq (1 + M_0\alpha_n k_n^{(1)})||x_n - x^*|| - \alpha_n\psi(||x_n - x^*||) + MM_0\alpha_n k_n^{(1)} + \alpha_n k_n^{(2)}. \tag{4.8}$$

Then Lemma 3.3 implies the assertion. □

We now combine Theorems 4.2 and 4.3 and establish the following theorem.

Theorem 4.4. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a mapping which is total asymptotically weakly contractive. Suppose that $\mathcal{N}(T) \neq \emptyset$ and $x^* \in \mathcal{N}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by the* *formula (1.10), where* $\{\alpha_n\}_{n\geq 1}\subset(0,1)$ *such that* $\sum_1^\infty \alpha_n=\infty$*. Suppose that* $\sum_1^\infty \alpha_n k_n^{(1)}<\infty$, $\sum_1^\infty \alpha_n k_n^{(2)}<\infty$*, and there exists* $M>0$ *such that* $\phi(\lambda)\leq m^{-1}\psi(\lambda)+M_0\lambda$ *for all* $\lambda\geq M$*, where* $m:=\max\{k_n^{(1)}\}_{n\geq 1}$*. Then* $\{x_n\}$ *converges strongly to* x^**.*

Proof. Since $\phi(\lambda)\leq m^{-1}\psi(\lambda)+M_0\lambda$ for all $\lambda\geq M$, we have

$$k_n^{(1)}\phi(\lambda)-\psi(\lambda)\leq k_n^{(1)}\phi(M)+M_0k_n^{(1)}\lambda. \tag{4.9}$$

Then from (4.2) one gets

$$||x_{n+1}-x^*||\leq\left(1+M_0\alpha_nk_n^{(1)}\right)||x_n-x^*||+\alpha_nk_n^{(1)}\phi(M)+\alpha_nk_n^{(2)}. \tag{4.10}$$

Due to Lemma 3.1, the sequence $\{x_n\}$ is bounded because $\sum_1^\infty \alpha_n k_n^{(1)}<\infty$ and $\sum_1^\infty \alpha_n k_n^{(2)}<\infty$. Therefore, using (4.2) again, we derive the inequality

$$||x_{n+1}-x^*||\leq||x_n-x^*||-\alpha_n\psi\left(||x_n-x^*||\right)+\alpha_nk_n^{(1)}\phi(C)+\alpha_nk_n^{(2)}. \tag{4.11}$$

By Lemma 3.2, $\|x_n-x^*\|\to 0$ as $n\to\infty$, and the theorem follows. □

If in Theorems 4.1–4.4, the sequence $\{x_n\}$ is assumed to be bounded, in particular, if K is bounded, then the following corollary appears.

Corollary 4.5. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T:K\to K$ be a mapping which is total asymptotically weakly contractive. Suppose* $\mathcal{N}(T)\neq\emptyset$ *and* $x^*\in\mathcal{N}(T)$*. Let* $\{\alpha_n\}_{n\geq 1}\subset(0,1)$ *be such that* $\sum_1^\infty\alpha_n=\infty$*. Starting from arbitrary* $x_1\in K$ *define the sequence* $\{x_n\}$ *by (1.10). Suppose that* $\{x_n\}$ *is bounded. Then* $\{x_n\}$ *converges strongly to* x^**.*

Remark 4.6. The estimates of convergence rate are calculated as in [4].

5. Auxiliary assertions for total asymptotically nonexpansive mappings

Lemma 5.1. *Let E be a real linear normed space and K a nonempty closed convex subset of E. Let $T:K\to K$ be a mapping which is total asymptotically nonexpansive and there exist constants* $M_0,M>0$ *such that* $\phi(\lambda)\leq M_0\lambda$ *for all* $\lambda\geq M$*. Let* $x^*\in\mathcal{N}(T):=\{x\in K: Tx=x\}$ *and* $\{\alpha_n\}_{n\geq 1}\subset(0,1)$ *for all* $n\geq 1$*. Starting from arbitrary* $x_1\in K$ *define the sequence* $\{x_n\}$ *generated by (1.10). Suppose that* $\sum_1^\infty\alpha_nk_n^{(1)}<\infty$ *and* $\sum_1^\infty\alpha_nk_n^{(2)}<\infty$*. Then* $\lim_{n\to\infty}\|x_n-x^*\|$ *exists.*

Proof. We first show that the sequence $\{x_n\}$ is bounded. From (4.2) one has

$$\begin{aligned}||x_{n+1}-x^*||&\leq||(1-\alpha_n)x_n+\alpha_nT^nx_n-x^*||\\&\leq(1-\alpha_n)||x_n-x^*||+\alpha_n||T^nx_n-T^nx^*||\\&\leq||x_n-x^*||+\alpha_nk_n^{(1)}\phi\left(||x_n-x^*||\right)+\alpha_nk_n^{(2)}.\end{aligned} \tag{5.1}$$

Since ϕ is increasing function, it results that $\phi(\lambda) \le \phi(M)$ if $\lambda \le M$ and $\phi(\lambda) \le M_0\lambda$ if $\lambda \ge M$. In either case we obtain

$$\phi(\|x_n - x^*\|) \le \phi(M) + M_0\|x_n - x^*\| \quad \forall n \ge 1. \tag{5.2}$$

Thus, (5.1) yields the following inequality:

$$\|x_{n+1} - x^*\| \le (1 + M_0\alpha_n k_n^{(1)})\|x_n - x^*\| + \alpha_n k_n^{(1)}\psi(M) + \alpha_n k_n^{(2)}. \tag{5.3}$$

However, $\sum_{k=1}^{\infty}\alpha_n k_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty}\alpha_n k_n^{(2)} < \infty$, therefore, due to Lemma 3.1, the sequence $\{\|x_n - x^*\|\}$ is bounded and it has a limit. This completes the proof. □

Lemma 5.2. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly continuous mapping which is total asymptotically nonexpansive. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the algorithm (1.10), where $\{\alpha_n\}_{n\ge1} \in (0,1]$. Then the condition $\|T^n x_n - x_n\| \to 0$ as $n \to \infty$ implies that*

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \tag{5.4}$$

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{5.5}$$

Proof. We have from (1.10) that

$$\|x_{n+1} - x_n\| = \|(1-\alpha_n)x_n + \alpha_n T^n x_n - x_n\| = \alpha_n\|T^n x_n - x_n\|. \tag{5.6}$$

Therefore, (5.4) holds. Also

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\le 2\|x_n - x_{n+1}\| + k_n^{(1)}\phi(\|x_n - x_{n+1}\|) + k_n^{(2)} \\ &\quad + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_n - Tx_n\|.\end{aligned} \tag{5.7}$$

Since T is uniformly continuous, there exists a continuous increasing function $\omega : R \to R$ with $\omega(0) = 0$ satisfying the inequality

$$\|T^{n+1}x_n - Tx_n\| = \|T(T^n x_n) - Tx_n\| \le \omega(\|T^n x_n - x_n\|). \tag{5.8}$$

The hypotheses $\|T^n x_n - x_n\| \to 0$ as $n \to \infty$ implies that

$$\|T^{n+1}x_n - Tx_n\| \longrightarrow 0, \qquad \|x_{n+1} - T^{n+1}x_{n+1}\| \longrightarrow 0. \tag{5.9}$$

The result (5.4) and conditions on $k_n^{(1)}$ and $k_n^{(2)}$ allow us to conclude from (5.7) that (5.5) follows. □

Next we assume that E is a Banach space.

LEMMA 5.3. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly continuous mapping which is total asymptotically nonexpansive and there exist $M_0, M > 0$ such that $\phi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Suppose $\mathcal{N}(T) \neq \emptyset$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the algorithm (1.10), where $\{\alpha_n\}_{n\geq 1}$ is such that $\eta_1 \leq \alpha_n \leq 1-\eta_2$ with some $\eta_1, \eta_2 > 0$. Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Then $\|Tx_n - x_n\| \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.*

Proof. Let $x^* \in \mathcal{N}(T)$. By making use of Lemma 5.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. If $\lim_{n\to\infty}\|x_n - x^*\| = 0$, by continuity of T, we are done. Let $\lim_{n\to\infty}\|x_n - x^*\| = r > 0$. Observe that $\{x_n\}$ is bounded. Therefore, there exists $R > 0$ such that $\|x_n\| \leq R$ for all $n \geq 1$.

We claim that

$$\lim_{n\to\infty} ||T^n x_n - x_n|| = 0. \tag{5.10}$$

Indeed, due to Lemma 2.3, one gets

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1-\alpha_n)x_n + \alpha_n T^n x_n - x^*||^2 \\
&= ||(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n x_n - x^*)||^2 \\
&\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n\left(||x_n - x^*|| + M' k_n^{(1)} + k_n^{(2)}\right)^2 \\
&\quad - (2L)^{-1}R^2\alpha_n(1-\alpha_n)\delta_E(||T^n x_n - x_n||/2R),
\end{aligned} \tag{5.11}$$

where $M' = \phi(R + \|x^*\|)$. We deduce from this that there exists a constant $M'' > 0$ such that

$$\begin{aligned}
&(2L)^{-1}R^2\epsilon_1\epsilon_2\delta_E(||T^n x_n - x_n||/2R) \\
&\quad \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + M''(1-\epsilon_2)\left(M' k_n^{(1)} + k_n^{(2)}\right).
\end{aligned} \tag{5.12}$$

Since $\sum_1^\infty k_n^{(1)} < \infty$, $\sum_1^\infty k_n^{(2)} < \infty$ and

$$\sum_1^\infty \left(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2\right) = ||x_1 - x^*||^2 - r^2, \tag{5.13}$$

we have

$$\sum_1^\infty \delta_E(||T^n x_n - x_n||/2R) < \infty. \tag{5.14}$$

This implies

$$\lim_{n\to\infty} \delta_E(||T^n x_n - x_n||/2R) = 0. \tag{5.15}$$

Hence, (5.10) holds because of the properties of $\delta_E(\epsilon)$. Lemma 5.2 yields now the conclusions of the lemma. □

Remark 5.4. If in the inequality (1.11) $k_n^{(2)} = 0$, then the operator $T : K \to K$ is uniformly continuous.

6. Convergence analysis of the iterations (1.10) with total asymptotically nonexpansive mappings

In this section, we study the weak and strong convergence of approximations generated by the iterative process (1.10) to fixed points of the total asymptotically nonexpansive mappings $T : K \to K$. As before, we denote $\mathcal{N}(T) = \{x \in K : Tx = x\}$.

Theorem 6.1. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly continuous and compact mapping which is total asymptotically nonexpansive and there exist constants $M_0, M > 0$ such that $\phi(\lambda) \le M_0\lambda$ for all $\lambda \ge M$. Suppose that $\mathcal{N}(T) \ne \emptyset$. Let $\{\alpha_n\}_{n\ge1}$ be such that $\eta_1 \le \alpha_n \le 1 - \eta_2$ for all $n \ge 1$ with some $\eta_1, \eta_2 > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by (1.10). Suppose that $\sum_1^\infty k_n^{(1)} < \infty$, $\sum_1^\infty k_n^{(2)} < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since T is continuous and compact on K, it is completely continuous. Moreover, by Lemma 5.1, $\{x_n\}$ is bounded, say, $\|x_n\| \le C$. Consequently, if $x^* \in \mathcal{N}(T)$, then the sequence $\{T^n x_n\}$ is also bounded, in view of the relations

$$||T^n x_n - x^*|| = ||T^n x_n - T^n x^*|| \le ||x_n - x^*|| + M k_n^{(1)} + k_n^{(2)}, \tag{6.1}$$

where $M = \phi(C + \|x^*\|)$. Then we conclude that there exists a subsequence $\{T^{n_j} x_{n_j}\}$ of $\{T^n x_n\}$ such that $T^{n_j} x_{n_j} \to y^*$ as $j \to \infty$. Furthermore, by (5.10), one gets

$$||T^{n_j} x_{n_j} - x_{n_j}|| \longrightarrow 0 \tag{6.2}$$

which implies that $x_{n_j} \to y^*$ as $j \to \infty$. By Lemma 5.3, we also conclude that

$$||T x_{n_j} - x_{n_j}|| \longrightarrow 0. \tag{6.3}$$

Therefore, the continuity of T yields the equality $Ty^* = y^*$. Finally, the limit of $\|x_n - y^*\|$ exists as $n \to \infty$ because of Lemma 5.1. Therefore, the strong convergence of $\{x_n\}$ to some point of $\mathcal{N}(T)$ holds. This accomplishes the proof. □

Theorem 6.2. *Let E be a real uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly continuous and compact mapping which is total asymptotically nonexpansive and there exist constants $M_0, M > 0$ such that $\phi(\lambda) \le M_0\lambda$ for all $\lambda \ge M$. Suppose that $\mathcal{N}(T) \ne \emptyset$ and $x^* \in \mathcal{N}(T)$. Let $\{\alpha_n\}_{n\ge1} \subset (0,1)$ be such that $\sum_1^\infty \alpha_n = \infty$. Taking an arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (1.10). Suppose that $\sum_1^\infty \alpha_n k_n^{(1)} < \infty$, $\sum_1^\infty \alpha_n k_n^{(2)} < \infty$, $\sum_1^\infty \rho_B(\alpha_n) < \infty$ and $k_n^{(1)} \le D_1\alpha_n$ and $k_n^{(2)} \le D_2\alpha_n$. Assume that there exists a positive differentiable function $\widetilde{\delta}(\epsilon) : [0,2] \to [0,1]$ and positive constants $c > 0$ and $D_0 > 0$, such that $\delta_E(\epsilon) \ge c\widetilde{\delta}(\epsilon)$, and $|\widetilde{\delta}'(\epsilon)| \le D_0$ for all $0 \le \epsilon \le 2$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We denote $F^n = I - T^n$. Since T is total asymptotically nonexpansive, one can consider without loss of generality that $k_n^{(1)} \le c_1$ and $k_n^{(2)} \le c_2$. Consequently, by Lemma 2.3, if $\|x\| \le \bar{R}$ and $\|y\| \le \bar{R}$, then

$$\begin{aligned} \|T^n x - T^n y\|^2 &= \|(x-y) - (F^n x - F^n y)\|^2 \\ &\ge \|x-y\|^2 - 2\langle J(x-y), F^n x - F^n y\rangle \\ &\quad + (2L)^{-1} R^2 \delta_E(\|F^n x - F^n y\|/2R), \end{aligned} \tag{6.4}$$

where $R = 5\bar{R} + c_1\phi(2\bar{R}) + c_2$, because $\|x-y\| \le 2\bar{R}$ and

$$\|(x-y) - (F^n x - F^n y)\| \le 5\bar{R} + c_1\phi(2\bar{R}) + c_2. \tag{6.5}$$

This means that

$$\langle J(x_n - x^*), F^n x_n\rangle \ge (4L)^{-1} R^2 \delta_E(\|F^n x_n\|/2R) - 2^{-1}\left(\|T^n x_n - T^n x^*\|^2 - \|x_n - x^*\|^2\right), \tag{6.6}$$

where $x^* \in \mathcal{N}(T)$. Let us evaluate the difference

$$\|T^n x_n - T^n x^*\|^2 - \|x_n - x^*\|^2. \tag{6.7}$$

By Lemma 5.1, the sequence $\{x_n\}$ is bounded, say, $\|x_n\| \le C$. Therefore, $\|x_n - x^*\| \le C + \|x^*\| = R_1$. Now it is not difficult to verify that

$$\begin{aligned} \|F^n x_n\| = \|F^n x_n - F^n x^*\| &\le 2\|x_n - x^*\| + k_n^{(1)}\phi(\|x_n - x^*\|) + k_n^{(2)} \\ &\le 2R_1 + c_1\phi(R_1) + c_2 = R_2. \end{aligned} \tag{6.8}$$

Then

$$\|x_{n+1} - x_n\| = \alpha_n \|F^n x_n\| \le R_2 \alpha_n \longrightarrow 0. \tag{6.9}$$

In addition, since $\phi(\lambda) \le MM_0 + M_0\lambda$, we have

$$\begin{aligned} \|T^n x_n - T^n x^*\|^2 &\le (1 + M_0 k_n^{(1)})^2 \|x_n - x^*\|^2 \\ &\quad + (1 + M_0 k_n^{(1)})(MM_0 k_n^{(1)} + k_n^{(2)})\|x_n - x^*\| + \left(MM_0 k_n^{(1)} + k_n^{(2)}\right)^2. \end{aligned} \tag{6.10}$$

This implies the estimate

$$\|T^n x_n - T^n x^*\|^2 - \|x_n - x^*\|^2 \le \gamma_n, \tag{6.11}$$

where

$$\gamma_n = 2R_1^2 M_0 k_n^{(1)} + R_1^2 M_0^2 (k_n^{(1)})^2 + R_1(1 + M_0 k_n^{(1)})(MM_0 k_n^{(1)} + k_n^{(2)}) + \left(MM_0 k_n^{(1)} + k_n^{(2)}\right)^2. \tag{6.12}$$

It follows from (6.4) that the inequality

$$\langle J(x_n - x^*), F^n x_n\rangle \geq (4L)^{-1} R_3^2 \delta_E(||F^n x_n||/2R_3) - 2^{-1}\gamma_n \tag{6.13}$$

holds, where $R_3 = 3R_1 + c_1\phi(R_1) + c_2$. Further,

$$\begin{aligned}
&||x_{n+1} - x^*||^2 - ||x_n - x^*||^2 \\
&\quad \leq 2\langle J(x_n - x^*), x_{n+1} - x_n\rangle + 2\langle J(x_{n+1} - x^*) - J(x_n - x^*), x_{n+1} - x_n\rangle \\
&\quad \leq -2\alpha_n\langle J(x_n - x^*), F^n x_n\rangle + 2R_1^2\rho_E(4R_1^{-1}\alpha_n||F^n x_n||) \\
&\quad \leq -(2L)^{-1} c R_3^2 \alpha_n \widetilde{\delta}(||F^n x_n||/2R_3) + 2R_1^2 \rho_E(4R_1^{-1} R_2 \alpha_n) + \alpha_n \gamma_n.
\end{aligned} \tag{6.14}$$

Let $\mu_n = \|x_n - x^*\|$ and $\beta_n = \widetilde{\delta}(\|F^n x_n\|/2R)$. Then the previous inequality gives

$$\mu_{n+1} \leq \mu_n - (2L)^{-1} c R_3^2 \alpha_n \beta_n + 2R_1^2 \rho_E(4R_1^{-1} R_2 \alpha_n) + \alpha_n \gamma_n. \tag{6.15}$$

Since $\widetilde{\delta}(\epsilon)$ is differentiable, we derive for some $0 \leq \eta \leq 2$ the following estimate:

$$\begin{aligned}
|\beta_{n+1} - \beta_n| &\leq c(2R_3)^{-1} |\widetilde{\delta}'(\eta)| \, \big| ||F^n x_{n+1}|| - ||F^n x_n|| \big| \\
&\leq cD_0(2R_3)^{-1}(||F^n x_{n+1} - F^n x_n||) \\
&\leq cD_0(2R_3)^{-1}\Big(2||x_{n+1} - x_n|| + k_n^{(1)}\phi(||x_{n+1} - x_n||) + k_n^{(2)}\Big) \\
&\leq cD_0(2R_3)^{-1}(2R_2 + D_1\phi(R_2\alpha_n) + D_2)\alpha_n \leq \bar{C}\alpha_n,
\end{aligned} \tag{6.16}$$

where

$$\bar{C} = cD_0(2R_3)^{-1}(2R_2 + D_1\phi(R_2) + D_2) > 0. \tag{6.17}$$

Due to Lemma 3.7,

$$\lim_{n\to\infty} \widetilde{\delta}(||F^n x_n||/2R_3) = 0 \tag{6.18}$$

because of $\sum_1^\infty \rho_B(\alpha_n) < \infty$, $\sum_1^\infty \alpha_n k_n^{(1)} < \infty$ and $\sum_1^\infty \alpha_n k_n^{(2)} < \infty$. By the properties of $\widetilde{\delta}(\epsilon)$,

$$\lim_{n\to\infty} ||x_n - T^n x_n|| = 0. \tag{6.19}$$

Since $\sum_1^\infty \rho_E(\alpha_n) < \infty$, we obtain that $\alpha_n \to 0$ and then, by (6.9),

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{6.20}$$

As it was shown by Lemma 5.2, the relations (6.19) and (6.20) yield (5.5). The rest of the proof follows the pattern of Theorem 6.1. □

Remark 6.3. It is known that $\delta_E(\epsilon) \geq \epsilon^s$, $s \geq 2$, in spaces l^p, L^p and W_m^p, $1 < p < \infty$, that is, $\widetilde{\delta}_E(\epsilon) = \epsilon^s$.

Remark 6.4. If $\delta_E(\epsilon)$ is differentiable, then there is no need to introduce $\widetilde{\delta}(\epsilon)$. Moreover, in this case, $\delta'(\epsilon)$ is positive and bounded on $[0,2]$.

If K is bounded, then Theorems 6.1 and 6.2 do not need constants M_0 and M satisfying the inequality $\phi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. In particular, we have the following corollary.

Corollary 6.5. *Let E be a real uniformly convex Banach space and K a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be a uniformly continuous and compact mapping which is total asymptotically nonexpansive. Suppose $\mathcal{N}(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\geq 1}$ be such that $\eta_1 \leq \alpha_n \leq 1 - \eta_2$ for all $n \geq 1$ and with some $\eta_1, \eta_2 > 0$. Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Taken an arbitrary $x_1 \in K$, we define the sequence $\{x_n\}$ by (1.10). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Further we omit the compactness property of T and study weak convergence of the iterations (1.10).

Theorem 6.6. *Let E be a real uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly continuous mapping which is total asymptotically nonexpansive and there exist constants $M_0, M > 0$ such that $\phi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Let $\mathcal{N}(T) \neq \emptyset$ and $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ be such that $\sum_1^\infty \alpha_n = \infty$. Taking an arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (1.10). Assume that*

$$\sum_1^\infty \alpha_n k_n^{(1)} < \infty, \qquad \sum_1^\infty \alpha_n k_n^{(2)} < \infty, \qquad \sum_1^\infty \rho_B(\alpha_n) < \infty, \tag{6.21}$$

and there exist a positive differentiable function $\widetilde{\delta}(\epsilon) : [0,2] \to [0,1]$ and positive constants c, D, D_1 and D_2 such that $\delta_E(\epsilon) \geq c\widetilde{\delta}(\epsilon)$, $|\delta_E'(\epsilon)| \leq D$ for all $0 \leq \epsilon \leq 2$, $k_n^{(1)} \leq D_1\alpha_n$ and $k_n^{(2)} \leq D_2\alpha_n$. If the operator $F = I - T$ is demi-closed, then $\{x_n\}$ weakly converges to a fixed point of T.

Proof. In Theorem 6.2, we have established that $\|x_n\| \leq C$ and $\lim_{n\to\infty} Fx_n = 0$. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ that weakly converges to a limit point $\widetilde{x}$. Since K is closed and convex, it is also weakly closed. Therefore $\widetilde{x} \in K$. Since $F = I - T$ is demi-closed, $\widetilde{x} \in \mathcal{N}(T)$. Thus, all weak accumulation points of $\{x_n\}$ belong to $\mathcal{N}(T)$. If $\mathcal{N}(T)$ is a singleton, then the whole sequence $\{x_n\}$ converges weakly to $\widetilde{x}$. Otherwise, we will prove the claim by contradiction (see [9]). □

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