PARAMETRIC GENERAL VARIATIONAL-LIKE INEQUALITY PROBLEM IN UNIFORMLY SMOOTH BANACH SPACE

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Using the concept of P-η-proximal mapping, we study the existence and sensitivity analysis of solution of a parametric general variational-like inequality problem in uniformly smooth Banach space. The approach used may be treated as an extension and unification of approaches for studying sensitivity analysis for various important classes of variational inequalities given by many authors in this direction.

1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in mechanics, contact problems in elasticity, optimization, and control, management science, operation research, general equilibrium problems in economics and transportation, unilateral obstacle, moving boundary-valued problems, and so forth, see, for example, [3, 12, 15]. Variational inequalities have been generalized and extended in different directions using novel and innovative techniques.

In recent years, much attention has been given to develop general methods for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problems being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for variational inequalities has been studied extensively by many authors using quite different methods. By using the projection technique, Dafermos [4], Mukherjee and Verma [17], Noor [18], and Yên [23] studied the sensitivity analysis of solution of some classes of variational inequalities with single-valued mappings. By using the implicit function approach that makes use of normal mappings, Robinson [22] studied the sensitivity analysis of solutions for variational inequalities in finite-dimensional spaces. By using proximal (resolvent) mapping technique, Adly [1], M. A. Noor and K. I. Noor

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[19], and Agarwal et al. [2] studied the sensitivity analysis of solution of some classes of quasi-variational inclusions with single-valued mappings.

Recently, by using projection and proximal techniques, Ding and Luo [9], Liu et al. [16], Park and Jeong [21], and Ding [8] have studied the behaviour and sensitivity analysis of solution set for some classes of generalized variational inequalities (inclusions) with set-valued mappings. It is worth mentioning that most of the results in this direction have been obtained in the setting of *Hilbert space*.

Inspired by recent research works going on in this area, in this paper, we consider a parametric general variational-like inequality problem (PGVLIP) in *uniformly smooth Banach space*. Further, using P-η-proximal mapping, we study the existence and sensitivity analysis of the solution of PGVLIP. The method presented in this paper can be used to generalize and improve the results given by many authors, see, for example, [1–3, 7–9, 16–19, 21–23].

2. Preliminaries

Let E be a real Banach space equipped with the norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ denote the dual pair between E and its dual space E^* and let $J: E \to 2^{E^*}$ be the *normalized duality mapping* defined by

$$J(u) = \{f \in E^*,\ \langle f,u\rangle = \|u\|^2,\ \|u\| = \|f\|_{E^*}\}, \quad \forall u \in E. \tag{2.1}$$

It is well known that if E is *smooth*, then J is *single valued* and if $E \equiv H$, a *Hilbert space*, then J is an *identity mapping*.

The following concepts and results are needed in the sequel.

Definition 2.1 (see [14]). Let $P: E \to E^*$, $g: E \to E$, and $\eta: E \times E \to E$ be single-valued mappings, then

(i) P is said to be *α-strongly η-monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle P(u) - P(v), \eta(u,v)\rangle \geq \alpha\|u - v\|^2, \quad \forall u,v \in E, \tag{2.2}$$

(ii) g is said to be *k-strongly accretive*, if there exists a constant $k > 0$ and for any $u,v \in E$, $j(u-v) \in J(u-v)$ such that

$$\langle g(u) - g(v), j(u-v)\rangle \geq k\|u-v\|^2, \tag{2.3}$$

where j is a selection of set-valued mapping J,

(iii) η is said to be *τ-Lipschitz continuous*, if there exists a constant $\tau > 0$ such that

$$||\eta(u,v)|| \leq \tau\|u-v\|, \quad \forall u,v \in E. \tag{2.4}$$

Example 2.2. If $E \equiv (-\infty,+\infty)$, $P(u) \equiv -u$, $\eta(u,v) \equiv -(1/2)(u-v)$, for all $u,v \in E$, then P is 1/2-strongly η-monotone and η is 1/2-Lipschitz continuous.

Definition 2.3 (see [5]). Let $\eta: E \times E \to E$ be a single-valued mapping. A proper functional $\phi: E \to \mathbb{R} \cup \{+\infty\}$ is said to be *η-subdifferentiable* at a point $u \in E$ if there exists a point $f^* \in E^*$ such that

$$\phi(v) - \phi(u) \geq \langle f^*, \eta(v,u)\rangle, \quad \forall v \in E, \tag{2.5}$$

where f^* is called *η-subgradient* of ϕ at u. The set of all η-subgradients of ϕ at u is denoted by $\partial_\eta \phi(u)$. The mapping $\partial_\eta\phi : E \to 2^{E^*}$ defined by

$$\partial_\eta\phi(u) = \{f^* \in E^* : \phi(v) - \phi(u) \geq \langle f^*, \eta(v,u)\rangle, \ \forall v \in E\} \tag{2.6}$$

is said to be *η-subdifferential* of ϕ at u.

Definition 2.4 (see [13]). A functional $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-*diagonally quasi-concave* (0-DQCV) in u, if for any finite set $\{u_1,\dots,u_n\} \subset E$ and for any $v = \sum_{i=1}^n \lambda_i u_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, $\min_{1\leq i\leq n} f(u_i, v) \leq 0$ holds.

Definition 2.5 (see [14]). Let $\eta : E \times E \to E$ be a single-valued mapping. Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable (may not be convex) and proper functional and let $P : E \to E^*$ be a nonlinear mapping. If for any given point $u^* \in E^*$ and $\rho > 0$, there exists a unique point $u \in E$ satisfying

$$\langle P(u) - u^*, \eta(v,u)\rangle + \rho\phi(v) - \rho\phi(u) \geq 0, \quad \forall v \in E, \tag{2.7}$$

then the mapping $u^* \to u$, denoted by $P_\rho^{\partial_\eta\phi}(u^*)$, is called *P-η-proximal mapping* of ϕ. Clearly, $u^* - P(u) \in \rho\partial_\eta\phi(u)$ and then it follows that

$$P_\rho^{\partial_\eta\phi}(u^*) = (P + \rho\partial_\eta\phi)^{-1}(u^*). \tag{2.8}$$

Remark 2.6 (see [14]). (i) If $\eta(v,u) \equiv v - u$ for all $u, v \in E$ and ϕ is a lower semicontinuous and proper functional on E, then the P-η-proximal mapping of ϕ reduces to the P-proximal mapping of ϕ discussed by Ding and Xia [11].

(ii) If $E \equiv H$, a Hilbert space, $\eta(v,u) \equiv v - u$ for all $u, v \in H$ and ϕ is a convex, lower semicontinuous and proper functional on E, and P is the identity mapping on H, then the P-proximal mapping of ϕ reduces to the usual proximal (resolvent) mapping of ϕ on Hilbert space.

Lemma 2.7 (see [14]). *Let E be a real reflexive Banach space; let $\eta : E \times E \to E$ be a continuous mapping such that $\eta(v,v') + \eta(v',v) = 0$ for all $v, v' \in E$; let $P : E \to E^*$ be α-strongly η-monotone continuous mapping; let, for any given $u^* \in E^*$, the function $h(v,u) = \langle u^* - P(u), \eta(v,u)\rangle$ be 0-DQCV in v and let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional on E. Then for any given constant $\rho > 0$ and $u^* \in E^*$, there exists a unique $u \in E$ such that*

$$\langle P(u) - u^*, \eta(v,u)\rangle \geq \rho\phi(u) - \rho\phi(v), \quad \forall v \in E, \tag{2.9}$$

that is, $u = P_\rho{}^{\partial_\eta\phi}(u^*)$.

Example 2.8 (see [10]). Let $E = \mathbb{R}$ be real line; let $P : \mathbb{R} \to \mathbb{R}$ be defined by $P(u) = u$, and let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\eta(u,v) = \begin{cases} u - v & \text{if } |uv| < 1, \\ |uv|(u-v) & \text{if } 1 \le |uv| < 2, \\ 2(u-v) & \text{if } 2 \le |uv|. \end{cases} \tag{2.10}$$

Then it is easy to see that

(i) $\langle \eta(u,v), u-v\rangle \ge |u-v|^2$ for all $u,v \in E$, that is, η is 1-strongly monotone,

(ii) $\eta(u,v) = -\eta(v,u)$ for all $u,v \in \mathbb{R}$,

(iii) $|\eta(u,v)| \le 2|u-v|$ for all $u,v \in \mathbb{R}$, that is, η is 2-Lipschitz continuous,

(iv) for any given $u \in E$, the function $h(v,x) = \langle u-x, \eta(v,x)\rangle = (u-x)\eta(v,x)$ is 0-DQCV in v.

If it is false, then there exist a finite set $\{v_1,\dots,v_n\}$ and $w = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that for each $i = 1,\dots,n$,

$$0 < h(v_i,w) = \begin{cases} (u-w)(v-w) & \text{if } |v_i w| < 1, \\ (u-w)|v_i w|(v-w) & \text{if } 1 \le |v_i w| < 2, \\ 2(u-w)(v-w) & \text{if } 2 \le |v_i w|. \end{cases} \tag{2.11}$$

It follows that $(u-w)(v_i-w) > 0$ for each $i = 1,\dots,n$, and hence we have

$$0 < \sum_{i=1}^n \lambda_i (u-w)(v_i-w) = (u-w)(w-w) = 0, \tag{2.12}$$

which is impossible. This proves that for any given $u \in \mathbb{R}$, the function $h(v,x)$ is 0-DQCV in v. Therefore, η satisfies all assumptions in Lemma 2.7.

Remark 2.9 (see [14]). Lemma 2.7 shows that for any strongly monotone continuous mapping $P : E \to E^*$ and $\rho > 0$, the P-η-proximal mapping $P_\rho^{\partial_\eta \phi} : E^* \to E$ of a lower semicontinuous, η-subdifferentiable and proper functional ϕ is well defined and for each $u^* \in E^*$, $u = P_\rho^{\partial_\eta \phi}(u^*)$ is the unique solution of the problem (2.9).

Lemma 2.10 (see [14]). *Let E be a real reflexive Banach space and let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(v,v') + \eta(v',v) = 0$ for all $v,v' \in E$; let $P : E \to E^*$ be α-strongly η-monotone continuous mapping; let, for any given $u^* \in E^*$, the function $h(v,u) = \langle u^* - P(u), \eta(v,u)\rangle$ be 0-DQCV in v; let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional on E and let $\rho > 0$ be any given constant. Then the P-η-proximal mapping $P_\rho^{\partial_\eta \phi}$ of ϕ is τ/α-Lipschitz continuous.*

Throughout the rest of the paper unless otherwise stated, let E be a real uniformly smooth Banach space with $\rho_E(t) \le ct^2$ for some $c > 0$, where ρ_E is the *modulus of smoothness* defined below.

LEMMA 2.11 (see [5]). *Let E be a real uniformly smooth Banach space and let $J : E \to E^*$ be the normalized duality mapping. Then, for all $u,v \in E$,*

(i) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, J(u+v)\rangle$,

(ii) $\langle u-v, Ju-Jv\rangle \leq 2d^2\rho_E(4\|u-v\|/d)$, *where* $d = \sqrt{(\|u\|^2+\|v\|^2)/2}$, $\rho_E(t) = \sup\{(\|u\| + \|v\|)/2 - 1 : \|u\| = 1,\ \|v\| = t\}$ *is called the modulus of smoothness of E.*

Let $T, A : E \to E^*$, $g : E \to E$, $\eta : E \times E \to E$, $N : E^* \times E^* \to E^*$ be the given nonlinear mappings and let $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable (may not be convex) and proper functional such that $g(u) \in \partial_\eta \phi(u,z)$, for all $u,z \in E$, then we consider the following general variational-like inequality problem (GVLIP): find $u \in E$ such that

$$\langle N(T(u),A(u)), \eta(v,g(u))\rangle + \phi(v,u) - \phi(g(u),u) \geq 0, \quad \forall v \in E. \tag{2.13}$$

Some special cases of GVLIP (2.13).

(i) If $N(T(u),A(u)) \equiv M(Tu,Au) - w^*$, for all $u \in E$, where $M : E^* \times E^* \to E^*$ and $w^* \in E^*$ fixed, and $g \equiv I$, identity mapping, then GVLIP (2.13) reduces to the following problem. Find $u \in E$ such that

$$\langle M(Tu,Au) - w^*, \eta(u,v)\rangle + \phi(u,v) - \phi(u,u) \geq 0, \quad \forall v \in E. \tag{2.14}$$

This problem has been studied by Ding [6].

(ii) If $N(T(u),A(u)) \equiv T(u) - A(u)$, for all $u \in E$, then GVLIP (2.13) reduces to the following problem: find $u \in E$ such that

$$\langle T(u) - A(u), \eta(v,g(u))\rangle + \phi(v,u) - \phi(g(u),u) \geq 0, \quad \forall v \in E. \tag{2.15}$$

This problem has been studied by Ding and Luo [10] in the setting of *Hilbert space*.

(iii) If $N(T(u),A(u)) \equiv S(u)$, for all $u \in E$, where $S : E \to E^*$, $g \equiv I$, and $\phi(u,v) \equiv 0$, for all $u,v \in E$, then GVLIP (2.13) reduces to the following problem: find $u \in E$ such that

$$\langle S(u), \eta(v,u)\rangle \geq 0, \quad \forall v \in E. \tag{2.16}$$

This problem has been studied by Parida et al. [20] in the setting of *Euclidean space*.

(iv) If $N(T(u),A(u)) \equiv S(u)$, for all $u \in E$, $\eta(u,v) \equiv u - v$, for all $u,v \in E$, $g \equiv I$, then GVLIP (2.13) reduces to the following problem: find $u \in E$ such that

$$\langle S(u), v-u\rangle + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in E. \tag{2.17}$$

This problem has been studied by M. A. Noor and K. I. Noor [19] in the setting of *Hilbert space*.

(v) If, in problem (2.17), $\phi(u,v) \equiv \phi(u)$, for all $u,v \in E$, then problem (2.17) reduces to the following problem: find $u \in E$ such that

$$\langle T(u), v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in E. \tag{2.18}$$

Problems (2.13)–(2.18) have many significant applications in physical, mathematical, pure and applied sciences, see [3, 6, 10, 12, 15, 20].

Next, we consider the parametric problem corresponding to GVLIP (2.13).

Let M be a nonempty open subset of E in which the parameter λ takes the values. Let $T,A : E \times M \to E^*$, $g : E \times M \to E$, $\eta : E \times E \to E$, $N : E^* \times E^* \times M \to E^*$ be given single-valued mappings and let $\phi : E \times E \times M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional such that $g(u,\lambda) \in \partial_\eta \phi(u,v,\lambda)$, for all $u,v \in E$, $\lambda \in M$. We consider the following parametric general variational-like inequality problem (PGVLIP): find $u \in E$ such that

$$\langle N(T(u,\lambda),A(u,\lambda),\lambda),\eta(v,g(u,\lambda))\rangle + \phi(v,u,\lambda) - \phi(g(u,\lambda),u,\lambda) \geq 0, \quad \forall v \in E. \tag{2.19}$$

3. Existence of solution and sensitivity analysis

First, we prove the following technical result.

Proposition 3.1. *$u \in E$ is the solution of PGVLIP (2.19) if and only if it satisfies the relation*

$$g(u,\lambda) = P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}[P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),\lambda)], \tag{3.1}$$

where $P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)} = (P + \rho\partial_\eta \phi(\cdot,u,\lambda))^{-1}$ is the P-η-proximal mapping of ϕ for each fixed $u \in E$, $\lambda \in M$, $P : E \to E^*$, $P \circ g(\cdot,\lambda)$ denotes P composition $g(\cdot,\lambda)$, and $\rho > 0$ is a constant.

Proof. Assume that $u \in E$ satisfies (3.1), that is,

$$g(u,\lambda) = P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}[P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),\lambda)]. \tag{3.2}$$

Since $P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)} = (P + \rho\partial_\eta \phi(\cdot,u,\lambda))^{-1}$, the above relation holds if and only if

$$P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),\lambda) \in P \circ g(u,\lambda) + \rho\partial_\eta \phi(g(u,\lambda),u,\lambda). \tag{3.3}$$

By the definition of η-subdifferential of $\phi(g(u,\lambda),u,\lambda)$, the above inclusion holds if and only if

$$\phi(v,u,\lambda) - \phi(g(u,\lambda),u,\lambda) \geq \langle N(T(u,\lambda),A(u,\lambda),\lambda),\eta(v,g(u,\lambda))\rangle, \quad \forall v \in E, \tag{3.4}$$

that is, $u \in E$ is the solution of PGVLIP (2.19). This completes the proof. □

Now, assume that for some $\bar{\lambda} \in M$, PGVLIP (2.19) has a solution $\bar{u}$ and K is a closed sphere in E centered at $\bar{u}$. We are interested in investigating those conditions under which, for each λ in a neighborhood of $\bar{\lambda}$, PGVLIP (2.19) has a unique solution $u(\lambda)$ near $\bar{u}$ and the solution $u(\lambda)$ is Lipschitz continuous.

Next, we give the following concepts.

Definition 3.2. The mapping $g : K \times M \to E$ is said to be

(i) *locally k-strongly accretive*, if there exists a constant $k > 0$ such that

$$\langle g(u,\lambda) - g(v,\lambda), J(u - v)\rangle \geq k\|u - v\|^2, \tag{3.5}$$

(ii) *locally* (σ_1,σ_2)*- Lipschitz continuous*, if there exist constants $\sigma_1,\sigma_2 > 0$ such that

$$||g(u,\lambda) - g(v,\widetilde{\lambda})|| \leq \sigma_1 \|u - v\| + \sigma_2 \|\lambda - \widetilde{\lambda}\|, \quad \forall u,v \in K,\ \lambda,\widetilde{\lambda} \in M. \tag{3.6}$$

Definition 3.3. Let $P : E \to E^*$, $g : K \times M \to E$, $T,A : K \times M \to E^*$, $N : E^* \times E^* \times M \to E^*$, then N is said to be

(i) *locally α-strongly $P \circ g$-accretive with respect to T and A*, if there exists a constant $\alpha > 0$ such that

$$\begin{aligned} &\langle N(T(u,\lambda),A(u,\lambda),\lambda) - N(T(v,\lambda),A(v,\lambda),\lambda), \\ &J^*(P \circ g(u,\lambda) - P \circ g(v,\lambda))\rangle \geq \alpha \|u - v\|^2, \quad \forall u,v \in K,\ \lambda \in M, \end{aligned} \tag{3.7}$$

where $J^* : E^* \to E$ is a normalized duality mapping,

(ii) *locally* $(\beta_1,\beta_2,\beta_3)$*- Lipschitz continuous*, if there exist constants $\beta_1,\beta_2,\beta_3 > 0$ such that

$$\begin{aligned} &||N(u_1,v_1,\lambda) - N(u_2,v_2,\widetilde{\lambda})|| \\ &\leq \beta_1 ||u_1 - u_2|| + \beta_2 ||v_1 - v_2|| + \beta_3 \|\lambda - \widetilde{\lambda}\|, \quad \forall u_1,u_2,v_1,v_2 \in K,\ \lambda,\widetilde{\lambda} \in M. \end{aligned} \tag{3.8}$$

Using the technique of Daformos [4], we consider the mapping $F(\cdot,\lambda) : K \times M \to E$ defined by

$$F(u,\lambda) := u - g(u,\lambda) + P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)} [P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),\lambda)]. \tag{3.9}$$

Remark 3.4. It follows from Proposition 3.1 that the fixed point of the mapping F defined by (3.9) is the solution of PGVLIP (2.19).

Now, we show that the mapping $F(u,\lambda)$ defined by (3.9) is a contraction mapping with respect to u uniformly in $\lambda \in M$.

Theorem 3.5. *Let the mapping $g : K \times M \to E$ be locally k-strongly accretive and locally (σ_1,σ_2)-Lipschitz continuous; let $T,A : K \times M \to E^*$ be locally ϵ-Lipschitz continuous and locally ξ-Lipschitz continuous, respectively; let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(u,v) + \eta(v,u) = 0$, for all $u,v \in E$ and let $P : E \to E^*$ be δ-strongly η-monotone continuous mapping; the fuction $h(v,u) = \langle u^* - P(u),\eta(u,v)\rangle$ be 0-DQCV in v. Let ϕ : $E \times E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional such that $g(u,\lambda) \in \partial_\eta \phi(u,v,\lambda)$, for all $u,v \in E$, $\lambda \in M$; let $P \circ g : K \times M \to E^*$ be locally (γ_1,γ_2)-Lipschitz continuous and let $N : E^* \times E^* \times M \to E^*$ be locally α-strongly $P \circ g$-accretive with respect to T and A and locally $(\beta_1,\beta_2,\beta_3)$-Lipschitz continuous. If there are* *some real constants* $\nu_1 > 0$ *and* $\rho > 0$ *such that*

$$\left\|P_\rho^{\partial_\eta\phi(\cdot,u,\lambda)}(z) - P_\rho^{\partial_\eta\phi(\cdot,v,\lambda)}(z)\right\| \le \nu_1\|u-v\|, \quad \forall u,v \in E,\ z \in E^*,\ \lambda \in M, \tag{3.10}$$

$$\begin{gathered}\left|\rho - \frac{\alpha}{64c(\beta_1\epsilon+\beta_2\xi)^2}\right| < \frac{\sqrt{\alpha^2 - 64c(\beta_1\epsilon+\beta_2\xi)^2[\gamma_1^2 - (\delta^2/\tau^2)(1-l^2)]}}{64c(\beta_1\epsilon+\beta_2\xi)^2}, \\ \alpha > (\beta_1\epsilon+\beta_2\xi)\sqrt{64c\left[\gamma_1^2 - \frac{\tau^2}{\delta^2}(1-l^2)\right]}, \quad \gamma_1 > \frac{\tau}{\delta}\sqrt{1-l^2},\ l<1,\end{gathered} \tag{3.11}$$

where $l = \nu_1 + \sqrt{1-2k+64c\sigma_1^2}$. *Then, for each* $u_1, u_2 \in E$, $\lambda \in M$,

$$||F(u_1,\lambda) - F(u_2,\lambda)|| \le \theta||u_1-u_2||, \tag{3.12}$$

where $\theta = l + (\tau/\delta)t(\rho) \in (0,1)$, $t(\rho) = \sqrt{\gamma_1^2 - 2\rho\alpha + \rho^2 64c(\beta_1\epsilon+\beta_2\xi)^2}$, *that is, F is* θ*-contraction uniformly in* $\lambda \in M$.

Proof. For all $u_1, u_2 \in E$, $\lambda \in M$, using condition (3.10), locally (γ_1,γ_2)-Lipschitz continuity of $P \circ g$ and locally ϵ-Lipschitz continuity of T, we have

$$\begin{aligned}
&||F(u_1,\lambda) - F(u_2,\lambda)|| \\
&\quad = \Big\|u_1 - g(u_1,\lambda) + P_\rho^{\partial_\eta\phi(\cdot,u_1,\lambda)}[P\circ g(u_1,\lambda) - \rho N(T(u_1,\lambda),A(u_1,\lambda),\lambda)] \\
&\qquad\quad - \Big[u_2 - g(u_2,\lambda) + P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_2,\lambda) - \rho N(T(u_2,\lambda),A(u_2,\lambda),\lambda)]\Big]\Big\| \\
&\quad \le ||u_1 - u_2 - (g(u_1,\lambda) - g(u_2,\lambda))|| \\
&\qquad + \Big\|P_\rho^{\partial_\eta\phi(\cdot,u_1,\lambda)}[P\circ g(u_1,\lambda) - \rho N(T(u_1,\lambda),A(u_1,\lambda),\lambda)] \\
&\qquad\quad - P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_1,\lambda) - \rho N(T(u_1,\lambda),A(u_1,\lambda),\lambda)]\Big\| \\
&\qquad + \Big\|P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_1,\lambda) - \rho N(T(u_1,\lambda),A(u_1,\lambda),\lambda)] \\
&\qquad\quad - P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_2,\lambda) - \rho N(T(u_2,\lambda),A(u_2,\lambda),\lambda)]\Big\| \\
&\quad \le ||u_1 - u_2 - (g(u_1,\lambda) - g(u_2,\lambda))|| + \nu_1||u_1-u_2|| \\
&\qquad + \frac{\tau}{\delta}[||P\circ g(u_1,\lambda) - P\circ g(u_2,\lambda) \\
&\qquad\quad - \rho[N(T(u_1,\lambda),A(u_1,\lambda),\lambda) - N(T(u_2,\lambda),A(u_2,\lambda),\lambda)]||].
\end{aligned} \tag{3.13}$$

Using Lemma 2.11, locally k-strongly accretiveness and locally (σ_1,σ_2)-Lipschitz continuity of g, we have

$$\begin{aligned}
&\|u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda))\|^2 \\
&\quad\leq \|u_1-u_2\|^2-2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda)))\rangle \\
&\quad\leq \|u_1-u_2\|^2-2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2)\rangle \\
&\qquad +2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2)-J(u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda)))\rangle \\
&\quad\leq (1-2k)\|u_1-u_2\|+64c\|g(u_1,\lambda)-g(u_2,\lambda)\|^2 \leq (1-2k+64c\sigma_1^2)\|u_1-u_2\|^2.
\end{aligned} \tag{3.14}$$

Since N is locally α-strongly accretive and locally $(\beta_1,\beta_2,\beta_3)$-Lipschitz continuous, and T and A are locally ϵ-Lipschitz continuous and locally ξ-Lipschitz continuous, respectively, we have

$$\begin{aligned}
&\|N(T(u_1,\lambda),A(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),\lambda)\| \\
&\quad\leq \beta_1\|T(u_1,\lambda)-T(u_2,\lambda)\|+\beta_2\|A(u_1,\lambda)-A(u_2,\lambda)\| \\
&\quad\leq (\beta_1\epsilon+\beta_2\xi)\|u_1-u_2\|.
\end{aligned} \tag{3.15}$$

Moreover, since $P\circ g$ is locally (γ_1,γ_2)-Lipschitz continuous, then using Lemma 2.11, we have

$$\begin{aligned}
&\|P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)-\rho[N(T(u_1,\lambda),A(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),\lambda)]\|^2 \\
&\quad\leq \|P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)\|^2 \\
&\qquad -2\rho\langle N(T(u_1,\lambda),A(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),\lambda), \\
&\qquad\qquad J^*(P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda))\rangle \\
&\qquad +2\rho\langle N(T(u_1,\lambda),A(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),\lambda),J^*(P\circ g(u_1,\lambda) \\
&\qquad\qquad -P\circ g(u_2,\lambda))-J^*(P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)-\rho[N(T(u_1,\lambda),A(u_1,\lambda),\lambda) \\
&\qquad\qquad -N(T(u_2,\lambda),A(u_2,\lambda),\lambda)])\rangle \\
&\quad\leq (\gamma_1^2-2\rho\alpha)\|u_1-u_2\|^2+64c\rho^2\|N(T(u_1,\lambda),A(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),\lambda)\|^2.
\end{aligned} \tag{3.16}$$

Combining (3.13), (3.14), (3.15), and (3.16), we have

$$\|F(u_1,\lambda)-F(u_2,\lambda)\|\leq\theta\|u_1-u_2\|, \tag{3.17}$$

where

$$\theta:=l+\frac{\tau}{\delta}t(\rho),\quad l=\sqrt{1-2k+64c\sigma_1{}^2}+\nu_1,\ t(\rho)=\sqrt{\gamma_1^2-2\rho\alpha+64c\rho^2(\beta_1\epsilon+\beta_2\xi)^2}. \tag{3.18}$$

Next, we have to show that $\theta < 1$. It is clear that $t(\rho)$ assumes its minimum value for $\bar{\rho} = \alpha/64c(\beta_1\epsilon + \beta_2\xi)^2$ with $t(\bar{\rho}) = \sqrt{\gamma_1^2 - \alpha^2/64c(\beta_1\epsilon + \beta_2\xi)^2}$.

For $\rho = \bar{\rho}$, $l + (\tau/\delta)t(\rho) < 1 \to l < 1$, then it follows that $\theta < 1$ for all ρ satisfying (3.11). Hence, it follows that F defined by (3.9) is a θ-contraction mapping uniformly in $\lambda \in M$. Therefore, invoking Banach contraction principle, F admits a unique fixed point, say $u(\lambda)$, which in turn is a solution of PGVLIP (2.19). This completes the proof. □

Remark 3.6. From Theorem 3.5, it is clear that the mapping F defined by (3.9) has a unique fixed point $u(\lambda)$, that is, $u(\lambda) = F(u,\lambda)$.

It also follows from our assumption that the function $\bar{u}$ for $\lambda = \bar{\lambda}$ is a solution of PGVLIP (2.19). Again, using Theorem 3.5, we observe that for $\lambda = \bar{\lambda}$, $\bar{u}$ is a fixed point of $F(u,\lambda)$ and it is a fixed point of $F(u,\bar{\lambda})$. Consequently, we conclude that

$$u(\bar{\lambda}) = \bar{u} = F(u(\bar{\lambda}),\bar{\lambda}). \tag{3.19}$$

Finally, using Theorem 3.5, we show the Lipschitz continuity of the solution of $u(\lambda)$ of PGVLIP (2.19).

Theorem 3.7. *Let the mappings T, P, g, η, h, $P \circ g$ be the same as in Theorem 3.5 and let conditions (3.10)-(3.11) of Theorem 3.5 hold. Suppose that $\lambda \to P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}$ is γ_2-Lipschitz continuous at $\lambda = \bar{\lambda}$, then the function $u(\lambda)$ is Lipschitz continuous at $\lambda = \bar{\lambda}$.*

Proof. For all $\lambda \in M$, using Theorem 3.5, we have

$$\begin{aligned}
||u(\lambda) - u(\bar{\lambda})|| &= ||F(u(\lambda),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})|| \\
&\leq ||F(u(\lambda),\lambda) - F(u(\bar{\lambda}),\lambda)|| + ||F(u(\bar{\lambda}),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})|| \\
&\leq \theta||u(\lambda) - u(\bar{\lambda})|| + ||F(u(\bar{\lambda}),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})||,
\end{aligned} \tag{3.20}$$

where θ is given by (3.18). Using (3.9) and using the conditions on the mappings T, P, g, η, $P \circ g$, and $P_\phi^{\partial_\eta \phi}(\cdot,u,\lambda)$, we have

$$\begin{aligned}
&||F(u(\bar{\lambda}),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})|| \\
&\quad = \Big|\Big| u(\bar{\lambda}) - g(u(\bar{\lambda}),\lambda) + P_\rho^{\partial_\eta \phi(\cdot,u(\bar{\lambda}),\lambda)}[P \circ g(u(\bar{\lambda}),\lambda) - \rho N(T(u(\bar{\lambda})),A(u(\bar{\lambda})),\lambda)] \\
&\qquad - \Big[u(\bar{\lambda}) - g(u(\bar{\lambda}),\bar{\lambda}) + P_\rho^{\partial_\eta \phi(\cdot,u(\bar{\lambda}),\bar{\lambda})}[P \circ g(u(\bar{\lambda}),\bar{\lambda}) - \rho N(T(u(\bar{\lambda})),A(u(\bar{\lambda})),\bar{\lambda})]\Big]\Big|\Big| \\
&\quad \leq \sigma_2||\lambda - \bar{\lambda}|| + \nu_2||\lambda - \bar{\lambda}|| \\
&\qquad + \frac{\tau}{\delta}[||P \circ g(u(\bar{\lambda}),\lambda) - P \circ g(u(\bar{\lambda}),\bar{\lambda})|| \\
&\qquad\qquad + \rho||N(T(u(\bar{\lambda})),A(u(\bar{\lambda})),\lambda) - N(T(u(\bar{\lambda})),A(u(\bar{\lambda})),\bar{\lambda})||] \\
&\quad \leq (\sigma_2 + \nu_2)||\lambda - \bar{\lambda}|| + \frac{\tau}{\delta}[\gamma_2||\lambda - \bar{\lambda}|| + \rho\beta_3||\lambda - \bar{\lambda}||] \leq \left[\sigma_2 + \nu_2 + \frac{(\gamma_2 + \rho\beta_3)\tau}{\delta}\right]||\lambda - \bar{\lambda}||.
\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$||u(\lambda)-u(\bar{\lambda})|| \leq \theta||u(\lambda)-u(\bar{\lambda})|| + \left[\sigma_2+\nu_2+\frac{(\gamma_2+\rho\beta_3)\tau}{\delta}\right]\|\lambda-\bar{\lambda}\|, \tag{3.22}$$

which implies

$$||u(\lambda)-u(\bar{\lambda})|| \leq \left[\frac{(\sigma_2+\nu_2)\delta+(\gamma_2+\rho\beta_3)\tau}{\delta(1-\theta)}\right]\|\lambda-\bar{\lambda}\|. \tag{3.23}$$

Since $\theta \in (0,1)$, by (3.11), $a := ((\sigma_2+\nu_2)\delta+(\gamma_2+\rho\beta_3)\tau)/\delta(1-\theta) > 0$. Hence, it follows from (3.23) that $u(\lambda)$ is a-Lipschitz continuous at $\lambda=\bar{\lambda}$. This completes the proof. □

Lemma 3.8. *If the assumptions of Theorem 3.7 hold, then there exists a neighborhood $N \subset M$ of $\bar{\lambda}$ such that for $\lambda \in N, u(\lambda)$ is the unique solution of PGVLIP (2.19) in the interior of K.*

Proof. It follows by using similar arguments as given in the proof of Theorem 3.7. □

Theorem 3.9. *Let $\bar{u}$ be the solution of PGVLIP (2.19). Let the mappings η, h be the same as in Theorem 3.5; let g be locally k-strongly accretive and locally (σ_1,σ_2)-Lipschitz continuous at $\lambda=\bar{\lambda}$; let T, A be locally ϵ-Lipschitz continuous and locally ξ-continuous, respectively; let P be δ-strongly η-monotone continuous mapping; let $P\circ g$ be locally (γ_1,γ_2)-Lipschitz continuous at $\lambda=\bar{\lambda}$. Let N be locally α-strongly accretive with respect to T and A, and locally $(\beta_1,\beta_2,\beta_3)$-Lipschitz continuous at $\lambda=\bar{\lambda}$, and let ϕ be a lower semicontinuous, η-subdifferentiable functional such that $g(u,\lambda)\in\partial_\eta\phi(u,v,\lambda)$, for all $u,v\in E$, $\lambda\in M$. If conditions (3.10)-(3.11) of Theorem 3.5 hold and $\lambda \to P_\rho^{\partial_\eta\phi(\cdot,u,\lambda)}$ is γ_2-Lipschitz continuous at $\lambda=\bar{\lambda}$, then there exists a neighborhood $N\subset M$ of $\bar{\lambda}$ such that for $\lambda\in N$, $u(\lambda)$ is the unique solution of PGVLIP (2.19) in the interior of K, $u(\bar{\lambda})=\bar{u}$, and $u(\lambda)$ is Lipschitz continuous at $\lambda=\bar{\lambda}$.*

Proof. It follows from Theorems 3.5–3.7, Lemma 3.8, and Remark 3.6. □

Example 3.10. If $E\equiv\mathbb{R}$, $g(u,\lambda)\equiv 2u+\lambda$, $P(u)\equiv u$, $T(u,\lambda)\equiv u+2\lambda$, $A(u,\lambda)\equiv 3u+\lambda$, $N(u,v,\lambda)\equiv 2u+v+\lambda$, $\eta(u,v)\equiv u-v$, for all $u,v\in\mathbb{R}$, $\lambda\in M$. Then

(i) $g(u,\lambda)$ is 2-strongly monotone and $(2,1)$-Lipschitz continuous, that is, $k=2$, $\sigma_1=2$, $\sigma_2=1$;

(ii) P is 1-strongly η-monotone and η is 1-Lipschitz continuous, that is, $\delta=1$, $\tau=1$;

(iii) $P\circ g$ is $(2,1)$-Lipschitz continuous, that is, $\gamma_1=2$, $\gamma_2=1$;

(iv) T and A are $(1,2)$-Lipschitz continuous and $(3,1)$-Lipschitz continuous, that is, $\epsilon=1$, $\xi=3$;

(v) N is 10-strongly $P\circ g$-monotone with respect to T and A, and $(2,1,1)$-Lipschitz continuous, that is, $\alpha=10$, $\beta_1=2$, $\beta_2=\beta_3=1$.

If $\nu_1=\nu_2=0.1$, then after simple calculation, we have $|\rho-1.6/3|<1/3 \Rightarrow \rho\in(0.2,0.8)$. For $\rho=0.75$, $\theta\approx 0.66$.

Further, it is easily observed that $a=((\sigma_2+\nu_2)\delta+(\gamma_2+\rho\beta_3)\tau)/\delta(1-\theta)>0$.

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