ALGEBRAIC PERIODS OF SELF-MAPS OF A RATIONAL EXTERIOR SPACE OF RANK 2

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The paper presents a complete description of the set of algebraic periods for self-maps of a rational exterior space which has rank 2.

1. Introduction

A natural number m is called a *minimal period* of a map f if f^m has a fixed point which is not fixed by any earlier iterates. One important device for studying minimal periods are the integers $i_m(f) = \sum_{k/m}\mu(m/k)L(f^k)$, where $L(f^k)$ denotes the Lefschetz number of f^k and μ is the classical Möbius function. If $i_m(f) \neq 0$, then we say that m is an *algebraic period* of f. In many cases the fact that m is an algebraic period provides information about the existence of minimal periods that are less then or equal to m. For example, let us consider f, a self-map of a compact manifold. If f is a transversal map and odd m is an algebraic period, then m is a minimal period (cf. [10, 12]). If f is a nonconstant holomorphic map, then there exists $M > 0$ such that for each prime number $m > M$, m is a minimal period of f if and only if m is an algebraic period of f (cf. [3]). Further relations between algebraic and minimal periods may be found in [8].

Sometimes the structure of the set of algebraic periods is a property of the space and may be deduced from the form of its homology groups. In [11] there is a description of algebraic periods for self-maps of a space M with three nonzero (reduced) homology groups, each of which is equal to $\mathbb{Q}$, in [6] the authors consider a space M with nonzero homology groups $H_0(M;\mathbb{Q}) = \mathbb{Q}$, $H_1(M;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$. The main difficulty in giving the overall description in the latter case is that for a map f_* induced by f on homology, for each m there are complex eigenvalues for which m is not an algebraic period. Rational exterior spaces are a wide class of spaces (e.g., Lie groups) which do not have this disadvantage, namely under the natural assumption of essentiality of f there is a constant m_X and computable set T_M, such that if $m > m_X$, $m \notin T_M$, then m is an algebraic period of f (cf. [5]). The aim of this paper is to provide a full characterization of algebraic periods

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in the case when homology spaces of X are small dimensional, namely when X is of the rank 2. Our work is based on [1, 9], where the description of the so-called "homotopical minimal periods" of self-maps of, respectively the two-, and three-dimensional torus are given using Nielsen numbers. We follow the algebraical framework of [9], the final description is similar to the one obtained in [1]. The differences result from the fact that the coefficients $i_m(f)$ are a sum of Lefschetz numbers, which unlike Nielsen numbers, do not have to be positive.

2. Rational exterior spaces

For a given space X and an integer $r \geq 0$ let $H^r(X;\mathbb{Q})$ be the rth singular cohomology space with rational coefficients. Let $H^*(X;\mathbb{Q}) = \bigoplus_{r=0}^{s} H^r(X;\mathbb{Q})$ be the cohomology algebra with multiplication given by the cup product. An element $x \in H^r(X;\mathbb{Q})$ is *decomposable* if there are pairs $(x_i, y_i) \in H^{p_i}(X;\mathbb{Q}) \times H^{q_i}(X;\mathbb{Q})$ with $p_i, q_i > 0$, $p_i + q_i = r > 0$ so that $x = \sum x_i \cup y_i$. Let $A^r(X) = H^r(X)/D^r(X)$, where D^r is the linear subspace of all decomposable elements.

Definition 2.1. By $A(f)$ we denote the induced homomorphism on $A(X) = \bigoplus_{r=0}^{s} A^r(X)$. Zeros of the characteristic polynomial of $A(f)$ on $A(X)$ will be called quotient eigenvalues of f. By $\operatorname{rank} X$ we will denote the dimension of $A(X)$ over $\mathbb{Q}$.

Definition 2.2. A connected topological space X is called a rational exterior space if there are some homogeneous elements $x_i \in H^{\text{odd}}(X;\mathbb{Q})$, $i = 1,\dots,k$, such that the inclusions $x_i \hookrightarrow H^*(X;\mathbb{Q})$ give rise to a ring isomorphism $\Lambda_{\mathbb{Q}}(x_1,\dots,x_k) = H^*(X;\mathbb{Q})$.

Finite H-spaces including all finite dimensional Lie groups and some real Stiefel manifolds are the most common examples of rational exterior spaces. The two dimensional torus T^2, a product of two n-dimensional sphere $S^n \times S^n$, and the unitary group $U(2)$ are examples of rational exterior spaces of rank 2.

The Lefschetz number of self-maps of a rational exterior space can be expressed in terms of quotient eigenvalues.

Theorem 2.3 (cf. [7]). *Let f be a self-map of a rational exterior space, and let $\lambda_1,\dots,\lambda_k$ be the quotient eigenvalues of f. Let A denote the matrix of $A(f)$. Then* $L(f^m) = \det(I - A^m) = \prod_{i=1}^{k}(1-\lambda_i^m)$.

Remark 2.4. A basis of the space $A(X)$ may be chosen in such a way that the matrix A is integral (cf. [7]).

3. The set of algebraic periods of self-maps of rational exterior space of rank 2

Let μ denote the Möbius function, that is, the arithmetical function defined by the three following properties: $\mu(1) = 1$, $\mu(k) = (-1)^r$ if k is a product of r different primes, and $\mu(k) = 0$ otherwise. Let $\operatorname{APer}(f) = \{m \in \mathbb{N} : i_m(f) \neq 0\}$, where $i_m(f) = \sum_{k/m} \mu(m/k) L(f^k)$. We will study the form of $\operatorname{APer}(f)$ for $f : X \to X$ and X a rational exterior space of rank 2. We assume that X is not simple which means that there exists $r \geq 1$ such that $\dim A^r = 2$, otherwise, that is, if there are $i, j \geq 1$ such that $\dim A^i = \dim A^j = 1$, we get the case with

Table 3.1. The set of algebraic periods $\mathrm{APer}(f)$ for the set R.

No.	(t,d)	$\mathrm{APer}(f)$
1^0	$(-2,1)$	$\{1,2\}$
2^0	$(-1,0)$	$\{1,2\}$
3^0	$(0,0)$	$\{1\}$
4^0	$(0,1)$	$\{1,2,4\}$
5^0	$(1,1)$	$\{1,2,3,6\}$
6^0	$(-1,1)$	$\{1,3\}$

integer quotient eigenvalues (cf. [7]) for which the description of $\mathrm{APer}(f)$ easily follows from the case under consideration.

By Theorem 2.3 we see that A is a 2×2 matrix and that the Lefschetz numbers $L(f^m)$ are expressed by its two quotient eigenvalues (in short we will call them eigenvalues): $\lambda_1,\lambda_2 : L(f^m) = (1-\lambda_1^m)(1-\lambda_2^m)$. The characteristic polynomial of A has integer coefficients by Remark 2.4 and is given by the formula: $W_A(x) = x^2 - tx + d$, where $t = \lambda_1+\lambda_2$ is the trace of A and $d = \lambda_1\lambda_2$ is its determinant. The characterization of the set $\mathrm{APer}(f)$ will be given in terms of these two parameters: t and d. Let us define the set $R = \{(-2,1),(-1,0),(0,0),(0,1),(1,1),(-1,1)\}$.

Theorem 3.1. *Let f be a self-map of a rational exterior space X of rank 2, which is not simple. Then* $\mathrm{APer}(f)$ *is one of the three mutually exclusive types:*

(E) $\mathrm{APer}(f)$ is empty if and only if 1 is an eigenvalue of A, which is equivalent to $t-d=1$.

(F) $\mathrm{APer}(f)$ is nonempty but finite if and only if all the eigenvalues of A are either zero or roots of unity not equal to 1, which is equivalent to $(t,d)\in R$. The algebraic periods for the set R are given in Table 3.1.

(G) $\mathrm{APer}(f)$ is infinite. Assume that (t,d) is not covered by the types (E) and (F), then,

(1) for $(t,d) = (-2,2)$, $\mathrm{APer}(f) = \mathbb{N}\setminus\{2,3\}$;
(2) for $(t,d) = (-1,2)$, $\mathrm{APer}(f) = \mathbb{N}\setminus\{3\}$;
(3) for $(t,d) = (0,2)$, $\mathrm{APer}(f) = \mathbb{N}\setminus\{4\}$;
(4) for $t=-d$ and $(t,d)\neq(-2,2)$, $\mathrm{APer}(f) = \mathbb{N}\setminus\{2\}$;
(5) for $t+d=-1$, $\mathrm{APer}(f) = \mathbb{N}\setminus\{n\in\mathbb{N} : n\equiv 0 \pmod 4\}$;
(6) if (t,d) is not covered by any of the cases 1–5, then $\mathrm{APer}(f) = \mathbb{N}$.

Remark 3.2. The letters E, F, G are chosen to represent empty, finite and generic case, respectively, which corresponds to the notation used in [9].

The rest of the paper consists of the proof of Theorem 3.1 and is organized in the following way: in the first part we describe the conditions equivalent to the fact that $m\in\{1,2,3\}$ is not an algebraic period. In the second part we analyze the situation when $m>3$ and none of eigenvalues is a root of unity. This is done by considering two cases: we will study the behaviour of $i_m(f)$ separately for real and complex eigenvalues. In the third stage we consider the case when $m>3$ and one of eigenvalues is a root of unity.

3.1. Algebraic periods in $\{1,2,3\}$

(A) *Conditions for* $1 \notin \mathrm{APer}(f)$. We have: $i_1(f) = L(f) = (1-\lambda_1)(1-\lambda_2) = 0$. This may happen if and only if one of the eigenvalues is equal to 1, that is, $t-d=1$.

(B) *Conditions for* $2 \notin \mathrm{APer}(f)$. We have: $i_2(f) = L(f^2) - L(f) = 0$, which is equivalent to: $(1-\lambda_1^2)(1-\lambda_2^2) - (1-\lambda_1)(1-\lambda_2) = 0$. This gives: $(1-\lambda_1)(1-\lambda_2)[(1+\lambda_1)(1+\lambda_2)-1] = 0$, so again $t-d=1$ or:

$$\lambda_1\lambda_2 + \lambda_1 + \lambda_2 = 0, \tag{3.1}$$

which gives $d+t=0$. The conditions for $2 \notin \mathrm{APer}(f)$ are: $t-d=1$ or $t=-d$.

(C) *Conditions for* $3 \notin \mathrm{APer}(f)$. We have: $i_3(f) = L(f^3) - L(f) = 0$, which is equivalent to: $(1-\lambda_1^3)(1-\lambda_2^3) - (1-\lambda_1)(1-\lambda_2) = 0$. We obtain the following equation: $(1-\lambda_1)(1-\lambda_2)[(1+\lambda_1+\lambda_1^2)(1+\lambda_2+\lambda_2^2)-1] = 0$. Again $t-d=1$ if one of the eigenvalues is equal to 1, otherwise:

$$\lambda_1 + \lambda_2 + \lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2(\lambda_1+\lambda_2) + (\lambda_1\lambda_2)^2 = 0. \tag{3.2}$$

In parameters t and d this gives:

$$t^2 + t - d + dt + d^2 = 0. \tag{3.3}$$

The last equality may be written as:

$$\left(d - \frac{1-t}{2}\right)^2 + \frac{3}{4}(1+t)^2 = 1, \tag{3.4}$$

which leads to the following alternatives.

If $t=0$, then $d \in \{0,1\}$, which corresponds to characteristic polynomials $x^2=0$ ($\lambda_1 = \lambda_2 = 0$) and $x^2+1=0$ ($\lambda_{1,2} = \pm i$).

If $t=-1$, then $d \in \{0,2\}$, which corresponds to characteristic polynomials $x^2+x=0$ ($\lambda_1 = 0, \lambda_2 = -1$) and $x^2+x+2=0$ ($\lambda_{1,2} = -(1/2) \pm i(\sqrt{7}/2)$).

If $t=-2$, then $d \in \{1,2\}$, which corresponds to characteristic polynomials $x^2+2x+1=0$ ($\lambda_{1,2} = -1$) and $x^2+2x+2=0$ ($\lambda_{1,2} = -1 \pm i$).

The conditions for $3 \notin \mathrm{APer}(f)$ are: $t-d=1$ or $(t,d) \in \{(0,0),(0,1),(-1,0),(-1,2),(-2,1),(-2,2)\}$.

3.2. Algebraic periods in the set $m > 3$ **in the case when none of the two eigenvalues is a root of unity.** Let for the rest of the paper $|\lambda_1| = \max\{|\lambda_1|,|\lambda_2|\}$. We will need the following lemma.

Lemma 3.3. *If for some* m *and each* $n|m$, $n \neq m$ *we have* $|L(f^m)|/|L(f^n)| > 2\sqrt{m}-1$, *then* m *is an algebraic period.*

Proof. Let $|L(f^s)| = \max\{|L(f^l)| : l|m,\ l \neq m\}$. We have

$$\begin{aligned} |i_m(f)| &= \left|\sum_{l|m} \mu\left(\frac{m}{l}\right) L(f^l)\right| \geq |L(f^m)| - \left|\sum_{l|m,\, l\neq m} \mu\left(\frac{m}{l}\right) L(f^l)\right| \\ &\geq |L(f^m)| - (2\sqrt{m}-1)|L(f^s)|. \end{aligned} \tag{3.5}$$

The last inequality is a consequence of the fact that the number of different divisors of m is not greater than $2\sqrt{m}$ (cf. [2]), by the assumption we get $|i_m(f)| > 0$, which is the desired assertion. □

Now, using the algebraic arguments of [9] in a case of two eigenvalues, we find the bound for the ratio $|L(f^m)|/|L(f^n)|$. We have

$$\frac{|L(f^m)|}{|L(f^n)|} = \frac{|1-\lambda_1^m||1-\lambda_2^m|}{|1-\lambda_1^n||1-\lambda_2^n|} \geq \frac{|\lambda_1|^m - 1}{|\lambda_1|^n + 1} \frac{|\lambda_2|^m - 1}{|\lambda_2|^n + 1}. \tag{3.6}$$

Let us consider two cases.

Case 1. λ_1, λ_2 are complex conjugates, then $|\lambda_1| = |\lambda_2|$. Notice that $|\lambda_1| = \sqrt{d}$, so if we exclude three pairs $(t,d) \in \{(0,1),(-1,1),(1,1)\}$, which correspond to some roots of unity, we obtain: $|\lambda_1| > 1.4$.

Let $n|m$, for Lefschetz numbers in this case we have

$$\frac{|L(f^m)|}{|L(f^n)|} \geq \left(|\lambda_1|^{m/2} - 1\right)\left(|\lambda_2|^{m/2} - 1\right) = \left(|\lambda_1|^{m/2} - 1\right)^2. \tag{3.7}$$

Case 2. λ_1, λ_2 are real. Then $|\lambda_1| = (|t| + \sqrt{t^2 - 4d})/2$. If $(t,d) = (0,0)$ then we immediately have $\mathrm{APer}(f) = \{1\}$. Cases $t = 0$, $d = -1$ and $t = \pm 1$, $d = 0$ and $t = \pm 2$, $d = 1$ give some roots of unity. In the rest of the cases: $|\lambda_1| \geq 1.4$.

In order to obtain the estimation for Lefschetz numbers we use the following inequality for the moduli of eigenvalues (cf. [9, Lemma 5.2]).

Lemma 3.4. *Let $\lambda_i \neq \pm 1$, $i = 1,2$, then*

$$\big|1 - |\lambda_2|\big| \geq \frac{1}{1 + |\lambda_1|}. \tag{3.8}$$

Proof. $|(\pm 1 - \lambda_1)(\pm 1 - \lambda_2)| = |W_A(\pm 1)| \geq 1$, because both eigenvalues are different from ± 1. We obtain $|1 \pm \lambda_2| \geq 1/|1 \pm \lambda_1| \geq 1/(1 + |\lambda_1|)$, which gives the needed inequality. □

We have by Lemma 3.4: $|\lambda_2| - 1 \geq (|\lambda_1| + 1)^{-1}$ for $|\lambda_2| > 1$ and $1 - |\lambda_2| \geq (|\lambda_1| + 1)^{-1}$ for $|\lambda_2| < 1$.

Let $h(x) = (x^m - 1)/(x^n + 1)$, notice that $h(x)$ is an increasing and $-h(x)$ is a decreasing function for $m > n > 0$ and $x > 0$.

Taking into account the two facts mentioned above we obtain:

$$\frac{|1-\lambda_2^m|}{|1-\lambda_2^n|} \geq \min\left\{ \frac{\left[1 + (|\lambda_1| + 1)^{-1}\right]^m - 1}{\left[1 + (|\lambda_1| + 1)^{-1}\right]^n + 1}, \frac{1 - \left[1 - (|\lambda_1| + 1)^{-1}\right]^m}{1 + \left[1 - (|\lambda_1| + 1)^{-1}\right]^n} \right\}. \tag{3.9}$$

As $n|m$ we get

$$\frac{|L(f^m)|}{|L(f^n)|} \geq \left(|\lambda_1|^{m/2}-1\right)\min\left\{\left[1+(|\lambda_1|+1)^{-1}\right]^{m/2}-1, 1-\left[1-(|\lambda_1|+1)^{-1}\right]^{m/2}\right\}. \tag{3.10}$$

Let $\bar{f}_C(|\lambda_1|,m)$, $\bar{f}_R(|\lambda_1|,m)$ be the functions equal to the right-hand side of the formulas (3.7) and (3.10), respectively. We define functions $f_C(|\lambda_1|,m) = \bar{f}_C(|\lambda_1|,m) - (2\sqrt{m}-1)$ and $f_R(|\lambda_1|,m) = \bar{f}_R(|\lambda_1|,m) - (2\sqrt{m}-1)$. Notice that the inequalities:

$$f_C(|\lambda_1|,m) > 0, \tag{3.11}$$

$$f_R(|\lambda_1|,m) > 0, \tag{3.12}$$

imply that $|L(f^m)|/|L(f^n)| > 2\sqrt{m}-1$ for $n|m$.

It is not difficult to verify the following statement by calculation and estimation of appropriate partial derivatives.

Remark 3.5. $f_C(\cdot,m)$ and $f_C(|\lambda_1|,\cdot)$ are increasing functions for $|\lambda_1| > 1.4$, $m \geq 4$.

$f_R(\cdot,m)$ and $f_R(|\lambda_1|,\cdot)$ are increasing functions for $|\lambda_1| > 1.4$, $m \geq 6$ and for $|\lambda_1| \geq 3$, $m \geq 4$.

If one of the inequalities (3.11), (3.12) is satisfied for given values $|\lambda_1^0|$ and m_0, then, by Remark 3.5, it is valid for each $|\lambda_1| > |\lambda_1^0|$ and $m > m_0$ and by Lemma 3.3 all such $m > m_0$ are algebraic periods.

Lemma 3.6. *Let us assume that both eigenvalues are complex*

(a) *if $m \geq 7$, then m is an algebraic period,*

(b) *if $|\lambda_1| \geq 2$ and $m \geq 4$, then m is an algebraic period.*

Proof. We take the minimal modulus of the eigenvalue which may appear and put it in the formula (3.11): (a) $f_C(1.4,7) > 0.75$, (b) $f_C(2,4) = 6$, which gives the result by Remark 3.5. □

Lemma 3.7. *Let us assume that both eigenvalues are real*

(a) *if $m \geq 12$, then m is an algebraic period,*

(b) *if $|\lambda_1| \geq 3$ and $m \geq 6$, then m is an algebraic period.*

Proof. We put in the formula (3.12) the minimal modulus of the greater eigenvalue: (a) $f_R(1.4,12) > 0.59$, (b) $f_R(3,6) > 17.47$, which implies the result by Remark 3.5. □

Remark 3.8. We must only check the cases when $|\lambda_1| < 3$ and $4 \leq m \leq 11$. Notice that for the coefficients t, d of the characteristic polynomial $W_A(x)$ we have the following estimates: $|t| \leq 2|\lambda_1|$, $|d| \leq |\lambda_1|^2$. This gives the bound: $|t| < 6$, $|d| < 9$, thus there are at most $11 \times 17 \times 8 = 1496$ cases which should be checked. This is done by numerical computation. If we exclude $(t,d) = (0,0)$ and the pairs which give the eigenvalues being roots of unity, we find in the range under consideration that only for $(t,d) = (0,2)$, $m = 4$ is not an algebraic period.

3.3. Algebraic periods in the set $m > 3$ in the case when one of the two eigenvalues is a root of unity. If both eigenvalues are real, then one of them is equal ± 1. If they are complex conjugates, then $\lambda_1\lambda_2 = \lambda_1\bar{\lambda}_1 = 1$, thus $d = 1$. On the other hand $0 \le |\lambda_1 + \lambda_2| \le |\lambda_1| + |\lambda_2| = 2$, thus $|t| \le 2$. This gives three pairs of complex eigenvalues: $\pm i$ $(t = 0, d = 1)$ and $(1/2) \pm i(\sqrt{3}/2)$ $(t = 1, d = 1)$ and $-(1/2) \pm i(\sqrt{3}/2)$ $(t = -1, d = 1)$. Each of these five cases we consider separately.

(1) 1 *is one of eigenvalues* $(t - d = 1)$. Then $L(f^m) = 0$ for all m and consequently $i_m(f) = 0$ for all m. Thus $\text{APer}(f) = \emptyset$.

(2) -1 *is one of eigenvalues* $(t + d = -1)$. We have to consider the subcases.

(2a) If $d = -1$, then $t = 0$, so we are in case 1.

(2b) If $d = 0$, then $t = -1$, so $W_A(x) = x^2 + x$ and the second eigenvalue is equal to 0. $L(f^m) = 1 - (-1)^m$, thus $L(f^m) = 0$ for m even and $L(f^m) = 2$ for m odd. We get: $i_m(f) = \sum_{k:2|k|m} \mu(m/k)L(f^k) + \sum_{k:2\nmid k|m} \mu(m/k)L(f^k) = 2\sum_{k:2\nmid k|m} \mu(m/k)$. It is easy to find (see the calculation of $i_m(f)$ in (2d)) that $i_1(f) = 2$, $i_2(f) = -2$, $i_m(f) = 0$ for $m \ge 3$. As a consequence: $\text{APer}(f) = \{1,2\}$.

(2c) If $d = 1$, then $t = -2$, so $W_A(x) = x^2 + 2x + 1$ and the second eigenvalue is equal to -1. $L(f^m) = (1 - (-1)^m)^2$, thus $L(f^m) = 0$ for m even and $L(f^m) = 4$ for m odd. We check in the same way as above that $i_1(f) = 4$, $i_2(f) = -4$, $i_m(f) = 0$ for $m \ge 3$, so $\text{APer}(f) = \{1,2\}$.

(2d) If $d \in \mathbb{Z} \setminus \{-1,0,1\}$, then for each $m : |L(f^m)| = |(1 - (-1)^m)||1 - \lambda_1^m|$. Notice that in the case under consideration $\{1,2,3\} \subset \text{APer}(f)$, which follows from Section 3.1.

As $|d| = |\lambda_1||\lambda_2|$ and -1 is one of eigenvalues we obtain for k odd : $|L(f^k)| \ge 2(|\lambda_1^k| - 1) = 2(|d|^k - 1)$, $|L(f^k)| \le 2(|\lambda_1^k| + 1) = 2(|d|^k + 1)$. Thus, for m odd, estimating in the same way as in Lemma 3.3, we get:

$$\left| i_m(f) \right| \ge 2(|d|^m - 1) - (2\sqrt{m} - 1)2(|d|^{m/3} + 1). \tag{3.13}$$

The right-hand side of the above formula is greater then zero for $|d| \ge 2$, $m > 3$, so all odd $m > 3$ are algebraic periods.

If $m > 3$ is even, then $m = 2^n q$, where q is odd. By the fact that $L(f^r) = 0$ if $2|r$, we get $L(f^{2^i q}) = 0$, for $1 \le i \le n$, thus

$$i_m(f) = \sum_{l|2^n q} \mu\left(2^n \frac{q}{l}\right) L(f^l) = \sum_{l|q} \mu\left(2^n \frac{q}{l}\right) L(f^l). \tag{3.14}$$

As μ is multiplicative and $\mu(2^n) = -1$ for $n = 1$ and $\mu(2^n) = 0$ for $n > 1$, we get

$$i_m(f) = \begin{cases} -i_q(f) & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{3.15}$$

This leads to the conclusion that even m is an algebraic period if and only if $m = 2q$ where q is odd. Finally in the case (2d) we obtain

$$\text{APer}(f) = \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 0 \ (\text{mod}\, 4)\}. \tag{3.16}$$

Before we consider complex cases let us state the following fact (cf. [4]). Let g_*, generated by g on homology, have as its only eigenvalues $\varepsilon_1,\dots,\varepsilon_{\phi(d)}$ which are all the dth primitive roots of unity ($\phi(d)$ denotes the Euler function). Then the Lefschetz numbers of iterations of g are the sum of powers of these roots: $L(g^m) = \sum_{i=1}^{\phi(d)} \varepsilon_i^m$. We have the formula for $i_m(g)$ in such a case:

$$i_m(g) = \begin{cases} 0 & \text{if } m \nmid d, \\ \displaystyle\sum_{k|m} \mu\left(\frac{d}{k}\right)\mu\left(\frac{m}{k}\right)\frac{\phi(d)}{\phi(d/k)} & \text{if } m \mid d. \end{cases} \tag{3.17}$$

Let now $\lambda_{1,2}$ be complex conjugates eigenvalues, then

$$L(f^m) = 1 - \lambda_1^m - \lambda_2^m + (\lambda_1\lambda_2)^m = 2 - (\lambda_1^m + \lambda_2^m). \tag{3.18}$$

We may rewrite formula for $L(f^m)$ in the following way: $L(f^m) = 2 - L(g^m)$, where g is described above. As $\sum_{k|m}\mu(m/k)2 = 2$ for $m = 1$ and 0 for $m > 1$; we get

$$i_m(f) = \begin{cases} 2 - i_m(g) & \text{if } m = 1, \\ -i_m(g) & \text{if } m > 1. \end{cases} \tag{3.19}$$

(3) $\lambda_{1,2} = \pm i$ $(t = 0, d = 1)$ are all primitive roots of unity of degree 4. Thus, applying formula (3.17) and (3.19), we get $i_1(f) = 2$, $i_2(f) = 2$, $i_3(f) = 0$, $i_4(f) = -4$, and $i_m(f) = 0$ for $m > 4$. Summing it up: $\text{APer}(f) = \{1,2,4\}$.

(4) $\lambda_{1,2} = -1/2 \pm i(\sqrt{3}/2)$ $(t = 1, d = 1)$ are all the primitive roots of unity of degree 6. Again by formulas (3.17) and (3.19) we calculate the values of $i_m(f)$ and get: $i_1(f) = 1$, $i_2(f) = 2$, $i_3(f) = 3$, $i_4(f) = 0$, $i_5(f) = 0$, $i_6(f) = -6$ and $i_m(f) = 0$ for $m > 6$, so $\text{APer}(f) = \{1,2,3,6\}$.

(5) $\lambda_{1,2} = (1/2) \pm i(\sqrt{3}/2)$ $(t = -1, d = 1)$ are all the primitive roots of unity of degree 3. By (3.17) and (3.19) we have: $i_1(f) = 3$, $i_2(f) = 0$, $i_3(f) = -3$, $i_m(f) = 0$ for $m > 3$, so $\text{APer}(f) = \{1,3\}$.

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