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*Research Article*

# Block Iterative Methods for a Finite Family of Relatively Nonexpansive Mappings in Banach Spaces

Fumiaki Kohsaka and Wataru Takahashi

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Using the convex combination based on Bregman distances due to Censor and Reich, we define an operator from a given family of relatively nonexpansive mappings in a Banach space. We first prove that the fixed-point set of this operator is identical to the set of all common fixed points of the mappings. Next, using this operator, we construct an iterative sequence to approximate common fixed points of the family. We finally apply our results to a convex feasibility problem in Banach spaces.



## 1. Introduction

Let $H$ be a Hilbert space and let $\{C_i\}_{i=1}^m$ be a family of closed convex subsets of $H$ such that $F = \bigcap_{i=1}^m C_i$ is nonempty. Then the problem of image recovery is to find an element of $F$ using the metric projection $P_i$ from $H$ onto $C_i$ $(i = 1, 2, \dots, m)$, where

$$P_i(x) = \arg\min_{y \in C_i} \|y - x\| \tag{1.1}$$

for all $x \in H$. This problem is connected with the convex feasibility problem. In fact, if $\{g_i\}_{i=1}^m$ is a family of continuous convex functions from $H$ into $\mathbb{R}$, then the convex feasibility problem is to find an element of the feasible set

$$\bigcap_{i=1}^m \{x \in H : g_i(x) \le 0\}. \tag{1.2}$$

We know that each $P_i$ is a nonexpansive retraction from $H$ onto $C_i$, that is,

$$\left\|P_i x - P_i y\right\| \le \|x - y\| \tag{1.3}$$

for all $x,y \in H$ and $P_i^2 = P_i$. Further, it holds that $F = \bigcap_{i=1}^{m} F(P_i)$, where $F(P_i)$ denotes the set of all fixed points of $P_i$ $(i = 1,2,\dots,m)$. Thus the problem of image recovery in the setting of Hilbert spaces is a common fixed point problem for a family of nonexpansive mappings.

A well-known method for finding a solution to the problem of image recovery is the *block-iterative projection algorithm* which was proposed by Aharoni and Censor [1] in finite-dimensional spaces; see also [2–5] and the references therein. This is an iterative procedure, which generates a sequence $\{x_n\}$ by the rule $x_1 = x \in H$ and

$$x_{n+1} = \sum_{i=1}^{m} \omega_n(i)\big(\alpha_i x_n + (1-\alpha_i)P_i x_n\big) \quad (n = 1,2,\dots), \tag{1.4}$$

where $\{\omega_n(i)\}_{i=1}^{m} \subset [0,1]$ $(n \in \mathbb{N})$ with $\sum_{i=1}^{m} \omega_n(i) = 1$ $(n \in \mathbb{N})$ and $\{\alpha_i\}_{i=1}^{m} \subset (-1,1)$. In particular, Butnariu and Censor [3] studied the strong convergence of $\{x_n\}$ to an element of $F$.

Recently, Kikkawa and Takahashi [6] applied this method to the problem of finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $\{T_i\}_{i=1}^{m}$ be a finite family of nonexpansive mappings from $C$ into itself. Then the iterative scheme they dealt with is stated as follows: $x_1 = x \in C$ and

$$x_{n+1} = \sum_{i=1}^{m} \omega_n(i)\big(\alpha_{n,i} x_n + (1-\alpha_{n,i})T_i x_n\big) \quad (n = 1,2,\dots), \tag{1.5}$$

where $\{\omega_n(i)\}_{i=1}^{m} \subset [0,1]$ with $\sum_{i=1}^{m} \omega_n(i) = 1$ $(n \in \mathbb{N})$ and $\{\alpha_i\}_{i=1}^{m} \subset [0,1]$. They proved that the generated sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{m}$ under some conditions on $E$, $\{\alpha_{n,i}\}$, and $\{\omega_n(i)\}$. Then they applied their result to the problem of finding a common point of a family of nonexpansive retracts of $E$; see also [7–10] for the previous results on this subject.

Our purpose in the present paper is to obtain an analogous result for a finite family of *relatively nonexpansive mappings* in Banach spaces. This notion was originally introduced by Butnariu et al. [11]. Recently, Matsushita and Takahashi [12–14] reformulated the definition of the notion and obtained weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. It is known that if $C$ is a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space $E$, then the *generalized projection* $\Pi_C$ (see, Alber [15] or Kamimura and Takahashi [16]) from $E$ onto $C$ is relatively nonexpansive, whereas the metric projection $P_C$ from $E$ onto $C$ is not generally nonexpansive.

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $J$ be the duality mapping from $E$ into $E^*$, and let $\{T_i\}_{i=1}^{m}$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that the set of all common fixed points of $\{T_i\}_{i=1}^{m}$ is nonempty. Motivated by the convex combination based on Bregman distances [17] due to Censor and Reich [18], the iterative methods introduced by Matsushita and Takahashi [12–14], and the proximal-type algorithm due to the authors [19], we define an operator $U_n$ ($n \in \mathbb{N}$) by

$$U_n x = \Pi_C J^{-1}\left(\sum_{i=1}^{m} \omega_n(i)\left(\alpha_{n,i} Jx + (1-\alpha_{n,i}) JT_i x\right)\right) \tag{1.6}$$

for all $x \in C$, where $\{\omega_n(i)\} \subset [0,1]$ and $\{\alpha_{n,i}\} \subset [0,1]$ with $\sum_{i=1}^{m} \omega_n(i) = 1$ ($n \in \mathbb{N}$). Such a mapping $U_n$ is called a *block mapping* defined by $T_1, T_2, \dots, T_m$, $\{\alpha_{n,i}\}$ and $\{\omega_n(i)\}$. In Section 4, we show that the set of all fixed points of $U_n$ is identical to the set of all common fixed points of $\{T_i\}_{i=1}^{m}$ (Theorem 4.2). In Section 5, under some additional assumptions, we show that the sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = U_n x_n \quad (n = 1, 2, \dots) \tag{1.7}$$

converges weakly to a common fixed point of $\{T_i\}_{i=1}^{m}$ (Theorem 5.3). This result generalizes the result of Matsushita and Takahashi [12]. If $E$ is a Hilbert space and each $T_i$ is a nonexpansive mapping from $C$ into itself, then $J$ is the identity operator on $E$, and hence (1.5) and (1.7) are coincident with each other. In Section 6, we deduce some results from Theorems 4.2 and 5.3.

## 2. Preliminaries

Let $E$ be a (real) Banach space with norm $\|\cdot\|$ and let $E^*$ denote the topological dual of $E$. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to $x$ in $E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to $x^*$ in $E^*$ by $x_n^* \overset{*}{\rightharpoonup} x^*$. For all $x \in E$ and $x^* \in E^*$, we denote the value of $x^*$ at $x$ by $\langle x, x^* \rangle$. We also denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively. The *duality mapping* $J$ from $E$ into $2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.1}$$

for all $x \in E$.

A Banach space $E$ is said to be *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|(x+y)/2\| < 1$. It is also said to be *uniformly convex* if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geq \varepsilon \tag{2.2}$$

imply $\|(x+y)/2\| \leq 1 - \delta$. The space $E$ is also said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be *uniformly smooth* if the limit (2.3) exists uniformly in $x, y \in S(E)$. It is well known that $\ell^p$ and $L^p$ ($1 < p < \infty$) are uniformly convex and uniformly smooth; see Cioranescu [20] or Diestel [21]. We know that if $E$ is smooth, strictly convex, and reflexive, then the duality mapping $J$ is single-valued, one-to-one, and onto. The duality mapping from a smooth Banach space $E$ into

$E^*$ is said to be *weakly sequentially continuous* if $Jx_n \xrightarrow{*} Jx$ whenever $\{x_n\}$ is a sequence in $E$ converging weakly to $x$ in $E$; see, for instance, [20, 22].

Let $E$ be a smooth, strictly convex, and reflexive Banach space, let $J$ be the duality mapping from $E$ into $E^*$, and let $C$ be a nonempty closed convex subset of $E$. Throughout the present paper, we denote by $\phi$ the mapping defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y,Jx\rangle + \|x\|^2 \tag{2.4}$$

for all $y,x \in E$. Following Alber [15], the *generalized projection* from $E$ onto $C$ is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x) \tag{2.5}$$

for all $x \in E$; see also Kamimura and Takahashi [16]. If $E$ is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ for all $y,x \in E$, and hence $\Pi_C$ is reduced to the metric projection $P_C$. It should be noted that the mapping $\phi$ is known to be the *Bregman distance* [17] corresponding to the Bregman function $\|\cdot\|^2$, and hence the projection $\Pi_C$ is the *Bregman projection* corresponding to $\phi$. We know the following lemmas concerning generalized projections.

Lemma 2.1 (see [15]; see also [16]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$. Then*

$$\phi(x,\Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y) \tag{2.6}$$

*for all $x \in C$ and $y \in E$.*

Lemma 2.2 (see [15]; see also [16]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $x \in E$, and let $z \in C$. Then $z = \Pi_C x$ is equivalent to*

$$\langle y - z, Jx - Jz\rangle \le 0 \tag{2.7}$$

*for all $y \in C$.*

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $T$ be a mapping from $C$ into itself, and let $F(T)$ be the set of all fixed points of $T$. Then a point $z \in C$ is said to be an *asymptotic fixed point* of $T$ (see Reich [23]) if there exists a sequence $\{z_n\}$ in $C$ converging weakly to $z$ and $\lim_n \|z_n - Tz_n\| = 0$. We denote the set of all asymptotic fixed points of $T$ by $\hat{F}(T)$. Following Matsushita and Takahashi [12–14], we say that $T$ is a *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;
(R2) $\phi(u,Tx) \le \phi(u,x)$ for all $u \in F(T)$ and $x \in C$;
(R3) $\hat{F}(T) = F(T)$.

Some examples of relatively nonexpansive mappings are listed below; see Reich [23] and Matsushita and Takahashi [12] for more details.

(a) If $C$ is a nonempty closed convex subset of a Hilbert space $E$ and $T$ is a nonexpansive mapping from $C$ into itself such that $F(T)$ is nonempty, then $T$ is a relatively nonexpansive mapping from $C$ into itself.

(b) If $E$ is a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone operator such that $A^{-1}0$ is nonempty, then the resolvent $J_r = (J + rA)^{-1}J$ $(r > 0)$ is a relatively nonexpansive mapping from $E$ onto $D(A)$ (the domain of $A$) and $F(J_r) = A^{-1}0$.

(c) If $\Pi_C$ is the generalized projection from a smooth, strictly convex, and reflexive Banach space $E$ onto a nonempty closed convex subset $C$ of $E$, then $\Pi_C$ is a relatively nonexpansive mapping from $E$ onto $C$ and $F(\Pi_C) = C$.

(d) If $\{C_i\}_{i=1}^m$ is a finite family of closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ such that $\bigcap_{i=1}^m C_i$ is nonempty and $T = \Pi_1\Pi_2 \cdots \Pi_m$ is the composition of the generalized projections $\Pi_i$ from $E$ onto $C_i$ $(i = 1,2,\ldots,m)$, then $T$ is a relatively nonexpansive mapping from $E$ into itself and $F(T) = \bigcap_{i=1}^m C_i$.

The following lemma is due to Matsushita and Takahashi [14].

Lemma 2.3 (see [14]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$ and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.*

We also know the following lemmas.

Lemma 2.4 (see [16]). *Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n \phi(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.*

Lemma 2.5 (see [16]). *Let $E$ be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \phi(x, y) \tag{2.8}$$

*for all $x, y \in B_r = \{z \in E : \|z\| \le r\}$.*

Lemma 2.6 (see [24]; see also [25, 26]). *Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|) \tag{2.9}$$

*for all $x, y \in B_r$ and $t \in [0,1]$.*

## 3. Lemmas

The following lemma is well known. For the sake of completeness, we give the proof.

Lemma 3.1. *Let $E$ be a strictly convex Banach space and let $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in $E$ such that*

$$\left\|\sum_{i=1}^m t_i x_i\right\|^2 = \sum_{i=1}^m t_i \|x_i\|^2, \tag{3.1}$$

*then $x_1 = x_2 = \cdots = x_m$.*

*Proof.* If $x_k \neq x_l$ for some $k,l \in \{1,2,\dots,m\}$, then the strict convexity of $E$ implies that

$$\left\|\frac{t_k}{t_k+t_l}x_k + \frac{t_l}{t_k+t_l}x_l\right\|^2 < \frac{t_k}{t_k+t_l}\|x_k\|^2 + \frac{t_l}{t_k+t_l}\|x_l\|^2. \tag{3.2}$$

Using this inequality, we have

$$\begin{aligned}
\left\|\sum_{i=1}^m t_i x_i\right\|^2 &= \left\|(t_k+t_l)\left(\frac{t_k}{t_k+t_l}x_k + \frac{t_l}{t_k+t_l}x_l\right) + \sum_{i\neq k,l} t_i x_i\right\|^2 \\
&\leq (t_k+t_l)\left\|\frac{t_k}{t_k+t_l}x_k + \frac{t_l}{t_k+t_l}x_l\right\|^2 + \sum_{i\neq k,l} t_i\|x_i\|^2 \\
&< (t_k+t_l)\left(\frac{t_k}{t_k+t_l}\|x\|^2 + \frac{t_l}{t_k+t_l}\|y\|^2\right) + \sum_{i\neq k,l} t_i\|x_i\|^2 \\
&= \sum_{i=1}^m t_i\|x_i\|^2.
\end{aligned} \tag{3.3}$$

This is a contradiction. □

We also need the following lemmas.

Lemma 3.2. *Let $E$ be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in $E$ such that*

$$\phi\left(z, J^{-1}\left(\sum_{j=1}^m t_j J x_j\right)\right) = \phi(z, x_i) \tag{3.4}$$

*for all $i \in \{1,2,\dots,m\}$, then $x_1 = x_2 = \cdots = x_m$.*

*Proof.* By assumption, we have

$$\phi\left(z, J^{-1}\left(\sum_{j=1}^m t_j J x_j\right)\right) = \sum_{i=1}^m t_i \phi(z, x_i). \tag{3.5}$$

This is equivalent to

$$\|z\|^2 - 2\left\langle z, \sum_{i=1}^m t_i J x_i \right\rangle + \left\| \sum_{i=1}^m t_i J x_i \right\|^2 = \sum_{i=1}^m t_i \left( \|z\|^2 - 2\langle z, J x_i\rangle + \|x_i\|^2 \right), \tag{3.6}$$

which is also equivalent to

$$\left\| \sum_{i=1}^m t_i J x_i \right\|^2 = \sum_{i=1}^m t_i \|J x_i\|^2. \tag{3.7}$$

Since $E$ is smooth and reflexive, $E^*$ is strictly convex. Thus, Lemma 3.1 implies that $Jx_1 = Jx_2 = \cdots = Jx_m$. By the strict convexity of $E$, $J$ is one-to-one. Hence we have the desired result. □

LEMMA 3.3. *Let $E$ be a smooth, strictly convex, and reflexive Banach space, let $\{x_i\}_{i=1}^m$ be a finite sequence in $E$ and let $\{t_i\}_{i=1}^m \subset [0,1]$ with $\sum_{i=1}^m t_i = 1$. Then*

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^m t_i J x_i\right)\right) \le \sum_{i=1}^m t_i \phi(z, x_i) \tag{3.8}$$

*for all $z \in E$.*

*Proof.* Let $V : E \times E^* \to \mathbb{R}$ be the function defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{3.9}$$

for all $x \in E$ and $x^* \in E^*$. In other words,

$$V(x, x^*) = \phi(x, J^{-1}x^*) \tag{3.10}$$

for all $x \in E$ and $x^* \in E^*$. We also have $\phi(x,y) = V(x, Jy)$ for all $x, y \in E$. Then we have from the convexity of $V$ in its second variable that

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^m t_i J x_i\right)\right) = V\left(z, \sum_{i=1}^m t_i J x_i\right) \le \sum_{i=1}^m t_i V(z, J x_i) = \sum_{i=1}^m t_i \phi(z, x_i). \tag{3.11}$$

This completes the proof. □

## 4. Block mappings by relatively nonexpansive mappings

Let $E$ be a smooth, strictly convex, and reflexive Banach space and let $J$ be the duality mapping from $E$ into $E^*$. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself. In this section, we study some properties of the mapping $U$ defined by

$$Ux = \Pi_C J^{-1}\left(\sum_{i=1}^m \omega_i \left(\alpha_i J x + (1-\alpha_i) J T_i x\right)\right) \tag{4.1}$$

for all $x \in C$, where $\{\alpha_i\}_{i=1}^m \subset [0,1]$ and $\{\omega_i\}_{i=1}^m \subset [0,1]$ with $\sum_{i=1}^m \omega_i = 1$. Recall that such a mapping $U$ is called a block mapping defined by $T_1, T_2, \ldots, T_m$, $\{\alpha_{n,i}\}$ and $\{\omega_n(i)\}$.

LEMMA 4.1. *Let $E$ be a smooth, strictly convex, and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and let $U$ be the block mapping defined by (4.1), where $\{\alpha_i\} \subset [0,1]$ and $\{\omega_i\} \subset [0,1]$ with $\sum_{i=1}^m \omega_i = 1$. Then*

$$\phi(u, Ux) \leq \phi(u, x) \tag{4.2}$$

*for all $u \in \bigcap_{i=1}^m F(T_i)$ and $x \in C$.*

*Proof.* Let $u \in \bigcap_{i=1}^m F(T_i)$ and $x \in C$. Then it holds from Lemmas 2.1 and 3.3 that

$$\begin{aligned}
\phi(u, Ux) &= \phi\left(u, \Pi_C J^{-1}\left(\sum_{i=1}^m \omega_i\left(\alpha_i Jx + (1-\alpha_i) JT_i x\right)\right)\right) \\
&\leq \phi\left(u, J^{-1}\left(\sum_{i=1}^m \omega_i\left(\alpha_i Jx + (1-\alpha_i) JT_i x\right)\right)\right) \\
&\leq \sum_{i=1}^m \omega_i\left(\alpha_i \phi(u,x) + (1-\alpha_i)\phi(u, T_i x)\right) \leq \phi(u,x).
\end{aligned} \tag{4.3}$$

This completes the proof. □

THEOREM 4.2. *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and let $U$ be the block mapping defined by (4.1), where $\{\alpha_i\} \subset [0,1)$ and $\{\omega_i\} \subset (0,1]$ with $\sum_{i=1}^m \omega_i = 1$. Then*

$$F(U) = \bigcap_{i=1}^m F(T_i). \tag{4.4}$$

*Proof.* Since the inclusion $F(U) \supset \bigcap_{i=1}^m F(T_i)$ is obvious, it suffices to show the inverse inclusion $F(U) \subset \bigcap_{i=1}^m F(T_i)$. Let $z \in F(U)$ be given and fix $u \in \bigcap_{i=1}^m F(T_i)$. Let $V : E \times E^* \to \mathbb{R}$ be the function defined by (3.9). Then, as in the proof of Lemma 4.1, we have

$$\begin{aligned}
\phi(u,z) = \phi(u, Uz) &\leq \phi\left(u, J^{-1}\left(\sum_{i=1}^m \omega_i\left(\alpha_i Jz + (1-\alpha_i) JT_i z\right)\right)\right) \\
&\leq \sum_{i=1}^m \omega_i\left(\alpha_i \phi(u,z) + (1-\alpha_i)\phi(u, T_i z)\right) \leq \phi(u,z).
\end{aligned} \tag{4.5}$$

If $k \in \{1, 2, \ldots, m\}$, then we have

$$\begin{aligned}
\phi(u,z) &= \sum_{i=1}^m \omega_i\left(\alpha_i \phi(u,z) + (1-\alpha_i)\phi(u, T_i z)\right) \\
&\leq \sum_{i \neq k} \omega_i \phi(u,z) + \omega_k\left(\alpha_k \phi(u,z) + (1-\alpha_k)\phi(u, T_k z)\right).
\end{aligned} \tag{4.6}$$

Using (4.6), we have

$$\omega_k\phi(u,z) = \left(1-\sum_{i\neq k}\omega_i\right)\phi(u,z) \leq \omega_k\left(\alpha_k\phi(u,z)+(1-\alpha_k)\phi(u,T_kz)\right). \tag{4.7}$$

Hence we have

$$\omega_k(1-\alpha_k)\phi(u,z) \leq \omega_k(1-\alpha_k)\phi(u,T_kz). \tag{4.8}$$

Since $\omega_k > 0$, $\alpha_k < 1$, and $u \in F(T_k)$, we have

$$\phi(u,z) \leq \phi(u,T_kz) \leq \phi(u,z). \tag{4.9}$$

Thus

$$\phi\left(u,J^{-1}\left(\sum_{i=1}^{m}\omega_i(\alpha_iJz+(1-\alpha_i)JT_iz)\right)\right) = \phi(u,T_jz) = \phi(u,z) \tag{4.10}$$

for all $j \in \{1,2,\dots,m\}$.

If $m = 1$, then $\omega_1 = 1$. In this case,

$$Ux = \Pi_C J^{-1}(\alpha_1Jx+(1-\alpha_1)JT_1x) \tag{4.11}$$

for all $x \in C$. If $\alpha_1 = 0$, then $U = T_1$, and hence the conclusion obviously holds. If $\alpha_1 > 0$, then we have from (4.10) that

$$\phi(u,J^{-1}(\alpha_1Jz+(1-\alpha_1)JT_1z)) = \phi(u,T_1z) = \phi(u,z). \tag{4.12}$$

Then, using Lemma 3.2, we have $z = T_1z$.

We next consider the case where $m \geq 2$. In this case, it holds that $0 < \omega_i < 1$ for all $i \in \{1,2,\dots,m\}$. Let $I = \{i \in \{1,2,\dots,m\} : \alpha_i \neq 0\}$. If $I$ is empty, then we have from (4.10) that

$$\phi\left(u,J^{-1}\left(\sum_{i=1}^{m}\omega_iJT_iz\right)\right) = \phi(u,T_iz) \tag{4.13}$$

for all $i \in \{1,2,\dots,m\}$. Using Lemma 3.2, we have $T_1z = T_2z = \cdots = T_mz$. Hence we have

$$z = Uz = \Pi_C J^{-1}\left(\sum_{i=1}^{m}\omega_iJT_iz\right) = \Pi_C J^{-1}\left(\sum_{i=1}^{m}\omega_iJT_jz\right) = \Pi_C T_jz = T_jz \tag{4.14}$$

for all $j \in \{1,2,\dots,m\}$. Thus $z \in \bigcap_{i=1}^{m} F(T_i)$.

On the other hand, if $I$ is nonempty, then we have from (4.10) that

$$\phi\left(u,J^{-1}\left(\sum_{i\in I}\omega_i\alpha_iJz+\sum_{i=1}^{m}\omega_i(1-\alpha_i)JT_iz\right)\right) = \phi(u,T_iz) = \phi(u,z) \tag{4.15}$$

for all $i \in \{1,2,\dots,m\}$. Then, from Lemma 3.2, we have $z = T_1z = T_2z = \cdots = T_mz$. Thus $z \in \bigcap_{i=1}^{m} F(T_i)$. This completes the proof. □

## 5. Weak and strong convergence theorems

Let $E$ be a smooth, strictly convex, and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and let $U_n$ be a block mapping from $C$ into itself defined by

$$U_n x = \Pi_C J^{-1}\left(\sum_{i=1}^{m} \omega_n(i)\big(\alpha_{n,i} Jx + (1-\alpha_{n,i}) JT_i x\big)\right) \tag{5.1}$$

for all $x \in C$, where $\{\omega_n(i)\} \subset [0,1]$ and $\{\alpha_{n,i}\} \subset [0,1]$ with $\sum_{i=1}^m \omega_n(i) = 1$ for all $n \in \mathbb{N}$. In this section, we study the asymptotic behavior of $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = U_n x_n \quad (n = 1,2,\dots). \tag{5.2}$$

Lemma 5.1. *Let $E$ be a smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $F = \bigcap_{i=1}^m F(T_i)$ is nonempty and let $\{\alpha_{n,i} : n,i \in \mathbb{N}, 1 \le i \le m\}$ and $\{\omega_n(i) : n,i \in \mathbb{N}, 1 \le i \le m\}$ be sequences in $[0,1]$ such that $\sum_{i=1}^m \omega_n(i) = 1$ for all $n \in \mathbb{N}$. Let $\{U_n\}$ be a sequence of block mappings defined by (5.1) and let $\{x_n\}$ be a sequence generated by (5.2). Then $\{\Pi_F x_n\}$ converges strongly to the unique element $z$ of $F$ such that*

$$\lim_{n\to\infty} \phi(z,x_n) = \min\left\{\lim_{n\to\infty} \phi(y,x_n) : y \in F\right\}. \tag{5.3}$$

*Proof.* If $u \in F$, then we have from Lemma 4.1 that

$$\phi(u,x_{n+1}) \le \phi(u,x_n) \tag{5.4}$$

for all $n \in \mathbb{N}$. Thus the limit of $\phi(u,x_n)$ exists. Since $\phi(u,x_n) \ge (\|u\| - \|x_n\|)^2$ for all $u \in F$ and $n \in \mathbb{N}$, the sequence $\{x_n\}$ is bounded. By Lemma 2.1, we have $\phi(u,\Pi_F x_n) \le \phi(u,x_n)$. So, the sequence $\{\Pi_F x_n\}$ is also bounded. By the definition of $\Pi_F$ and (5.4), we have

$$\phi(\Pi_F x_{n+1}, x_{n+1}) \le \phi(\Pi_F x_n, x_{n+1}) \le \phi(\Pi_F x_n, x_n). \tag{5.5}$$

Thus $\lim_n \phi(\Pi_F x_n, x_n)$ exists. We next show that $\{\Pi_F x_n\}$ is a Cauchy sequence. Take $r > 0$ such that $\{\Pi_F x_n\} \subset B_r$. Then, by Lemma 2.5 , we have a strictly increasing, continuous and convex function $g : [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$g(\|\Pi_F x_m - \Pi_F x_n\|) \le \phi(\Pi_F x_m, \Pi_F x_n) \tag{5.6}$$

for all $m,n \in \mathbb{N}$. If $m > n$, then it follows from Lemma 2.1 that

$$\phi(\Pi_F x_n, \Pi_F x_m) \le \phi(\Pi_F x_n, x_m) - \phi(\Pi_F x_m, x_m) \le \phi(\Pi_F x_n, x_n) - \phi(\Pi_F x_m, x_m). \quad (5.7)$$

Thus, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m > n \ge N$ implies that

$$g(\|\Pi_F x_m - \Pi_F x_n\|) \le \phi(\Pi_F x_n, x_n) - \phi(\Pi_F x_m, x_m) \le \varepsilon. \quad (5.8)$$

Therefore, $\{\Pi_F x_n\}$ is a Cauchy sequence in $F$, and hence it converges strongly to an element $z$ of $F$.

We next show that $z$ is the unique element of $F$ such that

$$\lim_{n\to\infty} \phi(z, x_n) = \min\left\{ \lim_{n\to\infty} \phi(y, x_n) : y \in F \right\}. \quad (5.9)$$

We define a function $h : F \to \mathbb{R}$ by

$$h(y) = \lim_{n\to\infty} \phi(y, x_n) \quad (5.10)$$

for all $y \in F$. Then we can show that $h$ is a continuous convex function. In fact, if $y_1, y_2 \in F$ and $t \in (0,1)$, then

$$\phi(ty_1 + (1-t)y_2, x_n) \le t\phi(y_1, x_n) + (1-t)\phi(y_2, x_n) \quad (5.11)$$

for all $n \in \mathbb{N}$. Tending $n \to \infty$, we have the convexity of $h$. We next show the continuity of $h$. Let $y_1, y_2 \in F$ and take $M > 0$ such that $\{x_n\}, \{y_1, y_2\} \subset B_M$. Then we have

$$\begin{aligned}
\phi(y_1, x_n) - \phi(y_2, x_n) &= \|y_1\|^2 - \|y_2\|^2 + 2\langle y_2 - y_1, Jx_n\rangle \\
&\le (\|y_1\| + \|y_2\|)(\|y_1\| - \|y_2\|) + 2\|x_n\|\|y_1 - y_2\| \\
&\le 4M\|y_1 - y_2\|
\end{aligned} \quad (5.12)$$

for all $n \in \mathbb{N}$. Tending $n \to \infty$, we have $h(y_1) - h(y_2) \le 4M\|y_1 - y_2\|$. Similarly, we have $h(y_2) - h(y_1) \le 4M\|y_1 - y_2\|$. Thus $h$ is continuous. We can also show that $\|z_n\| \to \infty$ implies that $h(z_n) \to \infty$. Since $E$ is reflexive and $F$ is closed and convex by Lemma 2.3, the set

$$A = \left\{ p \in F : h(p) = \inf_{y \in F} h(y) \right\} \quad (5.13)$$

is nonempty; see Takahashi [27, 28] for more details.

On the other hand, if $y \in F$, then we have

$$h(\Pi_F x_n) = \lim_{m\to\infty} \phi(\Pi_F x_n, x_m) \le \phi(\Pi_F x_n, x_n) \le \phi(y, x_n) \tag{5.14}$$

for all $n \in \mathbb{N}$. Tending $n \to \infty$, we have $h(z) \le h(y)$, and hence $z \in A$. We finally show that $A$ is singleton. Suppose that there exist $z_1, z_2 \in A$ such that $z_1 \neq z_2$. Take $s > 0$ such that $\{z_1, z_2\} \subset B_s$. Then, by Lemma 2.6, we have a strictly increasing, continuous, and convex function $\hat{g} : [0, 2s] \to \mathbb{R}$ such that $\hat{g}(0) = 0$ and

$$\left\|\frac{z_1 + z_2}{2}\right\|^2 \le \frac{1}{2}\|z_1\|^2 + \frac{1}{2}\|z_2\|^2 - \frac{1}{4}\hat{g}(\|z_1 - z_2\|). \tag{5.15}$$

Using this inequality, we have

$$\begin{aligned} h\left(\frac{z_1+z_2}{2}\right) &= \lim_{n\to\infty}\left\{\left\|\frac{z_1+z_2}{2}\right\|^2 - 2\left\langle\frac{z_1+z_2}{2}, Jx_n\right\rangle + \|x_n\|^2\right\} \\ &\le \lim_{n\to\infty}\left\{\frac{\phi(z_1,x_n)}{2} + \frac{\phi(z_2,x_n)}{2} - \frac{\hat{g}(\|z_1-z_2\|)}{4}\right\} \\ &= \frac{h(z_1)}{2} + \frac{h(z_2)}{2} - \frac{\hat{g}(\|z_1-z_2\|)}{4} \\ &< \frac{h(z_1)}{2} + \frac{h(z_2)}{2} = \min_{y\in F} h(y). \end{aligned} \tag{5.16}$$

This is a contradiction. □

Following an idea due to Matsushita and Takahashi [12], we prove the following lemma.

Lemma 5.2. *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $F = \bigcap_{i=1}^m F(T_i)$ is nonempty and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ and $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ be sequences such that $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\liminf_n \omega_n(i) > 0$ for all $i \in \{1,2,\ldots,m\}$ and $\sum_{i=1}^m \omega_n(i) = 1$ for all $n \in \mathbb{N}$. Let $\{U_n\}$ be a sequence of block mappings defined by (5.1) and let $\{z_n\}$ be a bounded sequence in $C$ such that $\lim_n\{\phi(u, z_n) - \phi(u, U_n z_n)\} = 0$ for some $u \in F$ and $z_{n_k} \rightharpoonup z$. Then $z \in F$.*

*Proof.* Since $\{z_n\}$ is bounded and $\phi(u, T_i z_n) \le \phi(u, z_n)$ for all $n \in \mathbb{N}$, $\{T_i z_n\}$ is also bounded. It follows from the uniform smoothness of $E$ that $E^*$ is uniformly convex; see Takahashi [27, 28]. Take $r > 0$ such that $\{z_n\}, \{T_i z_n\} \subset B_r$ $(i = 1, 2, \ldots, m)$. Then, Lemma 2.6 ensures the existence of a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$\|tJz_n + (1-t)JT_i z_n\|^2 \le t\|z_n\|^2 + (1-t)\|T_i z_n\|^2 - t(1-t)g(\|Jz_n - JT_i z_n\|) \tag{5.17}$$

for all $t \in [0,1]$, $n \in \mathbb{N}$, and $i \in \{1,2,\ldots,m\}$. Since $u$ is an element of $F$, we can show from Lemma 2.1 that

$$\begin{aligned}
&\phi(u, U_n z_n)\\
&\quad\le \phi\left(u, J^{-1}\left(\sum_{i=1}^{m} \omega_n(i)(\alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n)\right)\right)\\
&\quad= V\left(u, \sum_{i=1}^{m} \omega_n(i)(\alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n)\right)\\
&\quad\le \sum_{i=1}^{m} \omega_n(i) V(u, \alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n)\\
&\quad= \sum_{i=1}^{m} \omega_n(i)(\|u\|^2 - 2\langle u, \alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n\rangle + \|\alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n\|^2).
\end{aligned} \tag{5.18}$$

Using (5.17) and $\phi(u, T_i z_n) \le \phi(u, z_n)$, we have

$$\begin{aligned}
\phi(u, U_n z_n) &\le \sum_{i=1}^{m} \omega_n(i)(\|u\|^2 - 2\langle u, \alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n\rangle + \|\alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n\|^2)\\
&\le \sum_{i=1}^{m} \omega_n(i)(\|u\|^2 - 2\langle u, \alpha_{n,i}Jz_n + (1-\alpha_{n,i})JT_i z_n\rangle + \alpha_{n,i}\|z_n\|^2 + (1-\alpha_{n,i})\|T_i z_n\|^2\\
&\qquad\qquad - \alpha_{n,i}(1-\alpha_{n,i})g(\|Jz_n - JT_i z_n\|))\\
&= \sum_{i=1}^{m} \omega_n(i)(\alpha_{n,i}\phi(u, z_n) + (1-\alpha_{n,i})\phi(u, T_i z_n) - \alpha_{n,i}(1-\alpha_{n,i})g(\|Jz_n - JT_i z_n\|))\\
&\le \phi(u, z_n) - \sum_{i=1}^{m} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|Jz_n - JT_i z_n\|).
\end{aligned} \tag{5.19}$$

Thus we have

$$\sum_{i=1}^{m} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|Jz_n - JT_i z_n\|) \le \phi(u, z_n) - \phi(u, U_n z_n) \tag{5.20}$$

for all $n \in \mathbb{N}$. Then it follows from $\lim_n\{\phi(u, z_n) - \phi(u, U_n z_n)\} = 0$ that

$$\lim_{n\to\infty} \sum_{i=1}^{m} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|Jz_n - JT_i z_n\|) = 0. \tag{5.21}$$

Since $\liminf_n \omega_n(i) > 0$ and $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$ for all $i \in \{1,2,\ldots,m\}$, we have

$$\lim_{n\to\infty} g(\|Jz_n - JT_i z_n\|) = 0 \tag{5.22}$$

for all $i \in \{1,2,\dots,m\}$. Then, the properties of $g$ yield

$$\lim_{n\to\infty} \|Jz_n - JT_iz_n\| = 0 \tag{5.23}$$

for all $i \in \{1,2,\dots,m\}$. Since $E$ is uniformly convex, the duality mapping $J^{-1}$ from $E^*$ into $E$ is uniformly norm-to-norm continuous on every bounded subset of $E^*$; see Takahashi [27, 28]. Hence, we have

$$\lim_{n\to\infty} \|z_n - T_iz_n\| = \lim_{n\to\infty} \|J^{-1}(Jz_n) - J^{-1}(JT_iz_n)\| = 0 \tag{5.24}$$

for all $i \in \{1,2,\dots,m\}$. Thus $z \in \hat{F}(T_i)$ for all $i \in \{1,2,\dots,m\}$. Since each $T_i$ is relatively nonexpansive, we have $\hat{F}(T_i) = F(T_i)$ for all $i \in \{1,2,\dots,m\}$, and hence $z \in F$. This completes the proof. □

Using Lemmas 5.1 and 5.2, we study the asymptotic behavior of $\{x_n\}$ generated by (5.2).

Theorem 5.3. *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from $C$ into itself such that $F = \bigcap_{i=1}^m F(T_i)$ is nonempty and let $\{\alpha_{n,i} : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ and $\{\omega_n(i) : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ be sequences such that $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\liminf_n \omega_n(i) > 0$ for all $i \in \{1,2,\dots,m\}$ and $\sum_{i=1}^m \omega_n(i) = 1$ for all $n \in \mathbb{N}$. Let $\{U_n\}$ be a sequence of block mappings defined by (5.1) and let $\{x_n\}$ be a sequence generated by (5.2). Then the following hold:*

(a) *the sequence $\{x_n\}$ is bounded and each weak subsequential limit of $\{x_n\}$ belongs to $\bigcap_{i=1}^m F(T_i)$;*

(b) *if the duality mapping $J$ from $E$ into $E^*$ is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_F x_n\}$.*

*Proof.* We first prove part (a). Let $u \in F$. As in the proof of Lemma 5.1, we can show that $\{\phi(u,x_n)\}$ is nonincreasing and $\{x_n\}$, $\{T_ix_n\}$ are bounded. It also holds that

$$\phi(u,x_n) - \phi(u,U_nx_n) = \phi(u,x_n) - \phi(u,x_{n+1}) \longrightarrow 0 \tag{5.25}$$

as $n \to 0$. Using Lemma 5.2, we know that every weak subsequential limit of $\{x_n\}$ belongs to $F$.

We next prove part (b). Suppose that $J$ is weakly sequentially continuous. If $x_{n_k} \rightharpoonup z$, then $z \in F$ by part (a). It follows from Lemma 2.2 that

$$\langle z - \Pi_F x_n, Jx_n - J\Pi_F x_n\rangle \le 0 \tag{5.26}$$

for all $n \in \mathbb{N}$. By Lemma 5.1, $\Pi_F x_n \to w \in F$. Tending $n_k \to \infty$, we have

$$\langle z - w, Jz - Jw\rangle \le 0. \tag{5.27}$$

Since $J$ is a monotone operator, we have $\langle z - w, Jz - Jw\rangle = 0$. Then the strict convexity of $E$ implies that $z = w$; see Takahashi [27, 28]. This completes the proof. □

## 6. Deduced results

As direct consequences of Theorem 4.2, we have the following two corollaries.

Corollary 6.1. *Let $E$ be a smooth, strictly convex, and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $T$ be a relatively nonexpansive mapping from $C$ into itself and let $U$ be the mapping defined by*

$$Ux = \Pi_C J^{-1}(\alpha Jx + (1-\alpha)JTx) \tag{6.1}$$

*for all $x \in C$, where $\alpha \in [0,1)$. Then*

$$F(U) = F(T). \tag{6.2}$$

Corollary 6.2. *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\{T_i\}_{i=1}^m$ be a finite family of nonexpansive mappings from $C$ into itself such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and let $U$ be the mapping defined by*

$$Ux = \sum_{i=1}^m \omega_i(\alpha_i x + (1-\alpha_i)T_i x) \tag{6.3}$$

*for all $x \in C$, where $\{\alpha_i\} \subset [0,1)$, $\{\omega_i\} \subset (0,1]$ and $\sum_{i=1}^m \omega_i = 1$. Then*

$$F(U) = \bigcap_{i=1}^m F(T_i). \tag{6.4}$$

As a direct consequence of Theorem 5.3, we obtain the weak convergence theorem according to Matsushita and Takahashi [12].

Corollary 6.3 (see [12]). *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $T$ be a relatively nonexpansive mapping from $C$ into itself and let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \quad (n = 1,2,\dots), \tag{6.5}$$

*where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_n \alpha_n(1-\alpha_n) > 0$. Then the following hold:*

(a) *the sequence $\{x_n\}$ is bounded and each weak subsequential limit of $\{x_n\}$ belongs to $F(T)$;*

(b) *if the duality mapping $J$ from $E$ into $E^*$ is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_F x_n\}$.*

If $E$ is a Hilbert space and each $T_i$ is a nonexpansive mapping from $C$ into itself, then Theorem 5.3 is reduced to the following.

Corollary 6.4. *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\{T_i\}_{i=1}^m$ be a finite family of nonexpansive mappings from $C$ into itself such that $F = \bigcap_{i=1}^m F(T_i)$ is nonempty and let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \sum_{i=1}^m \omega_n(i)(\alpha_{n,i}x_n + (1-\alpha_{n,i})T_i x_n) \quad (n = 1,2,\dots), \tag{6.6}$$

*where* $\{\alpha_{n,i} : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ *and* $\{\omega_n(i) : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ *satisfy* $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$ *and* $\liminf_n \omega_n(i) > 0$ *for all* $i \in \{1,2,\ldots,m\}$ *and* $\sum_{i=1}^m \omega_n(i) = 1$ *for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges weakly to the strong limit of* $\{P_F x_n\}$, *where* $P_F$ *is the metric projection from* $H$ *onto* $F$.

Using Theorems 4.2 and 5.3, we can deal with the image recovery problem in Banach spaces as follows.

Corollary 6.5. *Let* $E$ *be a smooth, strictly convex, and reflexive Banach space, let* $\{C_i\}_{i=1}^m$ *be a finite family of closed convex subsets of* $E$ *such that* $\bigcap_{i=1}^m C_i$ *is nonempty, and let* $\Pi_i$ *be the generalized projection from* $E$ *onto* $C_i$ *for all* $i \in \{1,2,\ldots,m\}$. *Let* $U$ *be the mapping defined by*

$$Ux = J^{-1}\left(\sum_{i=1}^m \omega_i(\alpha_i Jx + (1-\alpha_i)J\Pi_i x)\right), \tag{6.7}$$

*where* $\{\alpha_i\} \subset [0,1)$ *and* $\{\omega_i\} \subset (0,1]$ *with* $\sum_{i=1}^m \omega_i = 1$. *Then*

$$F(U) = \bigcap_{i=1}^m C_i. \tag{6.8}$$

Corollary 6.6. *Let* $E$ *be a uniformly smooth and uniformly convex Banach space, let* $\{C_i\}_{i=1}^m$ *be a finite family of closed convex subsets of* $E$ *such that* $\bigcap_{i=1}^m C_i$ *is nonempty, and let* $\Pi_i$ *be the generalized projection from* $E$ *onto* $C_i$ *for all* $i \in \{1,2,\ldots,m\}$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$ *and*

$$x_{n+1} = J^{-1}\left(\sum_{i=1}^m \omega_n(i)(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})J\Pi_i x_n)\right) \quad (n = 1,2,\ldots), \tag{6.9}$$

*where* $\{\alpha_{n,i} : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ *and* $\{\omega_n(i) : n,i \in \mathbb{N}, 1 \le i \le m\} \subset [0,1]$ *satisfy* $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$ *and* $\liminf_n \omega_n(i) > 0$ *for all* $i \in \{1,2,\ldots,m\}$ *and* $\sum_{i=1}^m \omega_n(i) = 1$ *for all* $n \in \mathbb{N}$. *Then the following hold:*

(a) *the sequence* $\{x_n\}$ *is bounded and each weak subsequential limit of* $\{x_n\}$ *belongs to* $\bigcap_{i=1}^m C_i$;

(b) *if the duality mapping* $J$ *from* $E$ *into* $E^*$ *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to the strong limit of* $\{\Pi_{\bigcap_{i=1}^m C_i} x_n\}$.

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Fumiaki Kohsaka: Department of Information Environment, Tokyo Denki University,
Muzai Gakuendai, Inzai 270-1382, Chiba, Japan
*Email address*: kohsaka@sie.dendai.ac.jp

Wataru Takahashi: Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku 152-8552, Tokyo, Japan
*Email address*: wataru@is.titech.ac.jp