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Research Article

Weak and Strong Convergence of Multistep Iteration for Finite Family of Asymptotically Nonexpansive Mappings

Balwant Singh Thakur and Jong Soo Jung

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Strong and weak convergence theorems for multistep iterative scheme with errors for finite family of asymptotically nonexpansive mappings are established in Banach spaces. Our results extend and improve the corresponding results of Chidume and Ali (2007), Cho et al. (2004), Khan and Fukhar-ud-din (2005), Plubtieng et al.(2006), Xu and Noor (2002), and many others.

1. Introduction and preliminaries

Let K be a nonempty subset of a real normed space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all x, y in K. T is said to be asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $\|T^n x - T^n y\| \le (1 + r_n)\|x - y\|$ for all x, y in K and $n \in \mathbb{N}$.

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [6]. Iteration processes for nonexpansive and asymptotically nonexpansive mappings in Banach spaces including Mann [7] and Ishikawa [8] iteration processes have been studied extensively by many authors to solve the nonlinear operators as well as variational inequalities; see [1–22, 25].

Noor [13] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and auxiliary principle. Glowinski and Le Tallec [9] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [9] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Thus, we conclude that the three-step scheme plays an important and significant role in solving various problems which arise in pure and applied sciences. Recently, Xu and Noor [5] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. Cho et al. [2] extended the work of Xu and Noor [5] to the three-step iterative scheme with errors in a Banach space and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [20] gave weak and strong convergence theorems for a new three-step iterative scheme of asymptotically nonexpansive mappings.

More recently, Plubtieng et al. [4] introduced a three-step iterative scheme with errors for three asymptotically nonexpansive mappings and established strong convergence of this scheme to common fixed point of three asymptotically nonexpansive mappings. Very recently, Chidume and Ali [1] considered multistep scheme for finite family of asymptotically nonexpansive mappings and gave weak convergence theorems for this scheme in a uniformly convex Banach space whose the dual space satisfies the Kadec-Klee property. They also proved a strong convergence theorem under some appropriate conditions on finite family of asymptotically nonexpansive mappings.

Inspired by the above facts, in this paper, a new multistep iteration scheme with errors for finite family of asymptotically nonexpansive mappings is introduced and strong and weak convergence theorems of this scheme to common fixed point of asymptotically nonexpansive mappings are proved. In particular, our weak convergence theorem is proved in a uniformly convex Banach space whose the dual has a Kadec-Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial property; however, their dual does have the Kadec-Klee property. This means that our weak convergence result can apply not only to L^p-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. Our theorems improve and generalize some previous results in [1–5, 15, 17–19]. Our iterative scheme is defined as below.

Let K be a nonempty closed subset of a normed space E, and let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings. For a given $x_1 \in K$ and a fixed $N \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the sequence $\{x_n\}$ by

$$\begin{aligned}
x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}, \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)}, \\
&\;\;\vdots \\
x_n^{(3)} &= \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)},
\end{aligned} \tag{1.1}$$

where, $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \dots, \{u_n^{(N)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, N\}$.

We now give some preliminaries and results which will be used in the rest of this paper.

A Banach space E is said to satisfy *Opial's condition* if for each sequence x_n in E, the condition, that the sequence $x_n \to x$ weakly, implies

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y|| \tag{1.2}$$

for all $y \in E$ with $y \neq x$.

A Banach space E is said to have *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply that $\|x_n - x\| \to 0$.

We will make use of the following lemmas.

Lemma 1.1 [2]. *Let E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be an asymptotically nonexpansive mapping. Then, $I - T$ is demiclosed at zero, that is, for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

Lemma 1.2 [16]. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \tag{1.3}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3 [19]. *Let E be a uniformly convex Banach space and let b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a real sequence in $[b,c]$ and $\{x_n\}$, $\{y_n\}$ are two sequences in E such that*

$$\begin{aligned} \limsup_{n\to\infty} ||x_n|| &\leq a, \\ \limsup_{n\to\infty} ||y_n|| &\leq a, \\ \lim_{n\to\infty} ||t_n x_n + (1 - t_n)y_n|| &= a. \end{aligned} \tag{1.4}$$

Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $a \geq 0$ is some constant.

Lemma 1.4 [12]. *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p, q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose that $\lim_{n\to\infty} \|tx_n + (1 - t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.*

2. Main results

In this section, we prove strong and weak convergence theorems for multistep iteration with errors in Banach spaces. In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let E be a real normed space and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, $1 \leq i \leq N$. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. For any $p \in F$, we note that

$$\begin{aligned}||x_n^{(1)}-p|| &\le \alpha_n^{(1)}||T_1^n x_n - p|| + \beta_n^{(1)}||x_n - p|| + \gamma_n^{(1)}||u_n^{(1)} - p|| \\ &\le \alpha_n^{(1)}(1+r_n)||x_n - p|| + \beta_n^{(1)}||x_n - p|| + \gamma_n^{(1)}||u_n^{(1)} - p|| \\ &\le (1+r_n)||x_n - p|| + t_n^{(1)},\end{aligned} \tag{2.1}$$

where $t_n^{(1)} = \gamma_n^{(1)}\|u_n^{(1)} - p\|$. Since $\{u_n^{(1)}\}$ is bounded and $\sum_{n=1}^{\infty}\gamma_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$. It follows from (2.1) that

$$\begin{aligned}||x_n^{(2)}-p|| &\le \alpha_n^{(2)}||T_2^n x_n^{(1)} - p|| + \beta_n^{(2)}||x_n - p|| + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le \alpha_n^{(2)}(1+r_n)||x_n^{(1)} - p|| + \beta_n^{(2)}||x_n - p|| + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le \alpha_n^{(2)}(1+r_n)((1+r_n)||x_n - p|| + t_n^{(1)}) + \beta_n^{(2)}||x_n - p|| + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le \alpha_n^{(2)}(1+r_n)^2||x_n - p|| + \alpha_n^{(2)}t_n^{(1)}(1+r_n) + \beta_n^{(2)}||x_n - p|| + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le \alpha_n^{(2)}(1+r_n)^2||x_n - p|| + \alpha_n^{(2)}t_n^{(1)}(1+r_n) \\ &\quad + \beta_n^{(2)}(1+r_n)^2||x_n - p|| + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le (\alpha_n^{(2)} + \beta_n^{(2)})(1+r_n)^2||x_n - p|| + \alpha_n^{(2)}t_n^{(1)}(1+r_n) + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le (1+r_n)^2||x_n - p|| + \alpha_n^{(2)}t_n^{(1)}(1+r_n) + \gamma_n^{(2)}||u_n^{(2)} - p|| \\ &\le (1+r_n)^2||x_n - p|| + t_n^{(2)},\end{aligned} \tag{2.2}$$

where $t_n^{(2)} = \alpha_n^{(2)}t_n^{(1)}(1+r_n) + \gamma_n^{(2)}\|u_n^{(2)} - p\|$. Since $\{u_n^{(2)}\}$ is bounded and $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$. Similarly, we see that

$$\begin{aligned}||x_n^{(3)}-p|| &\le \alpha_n^{(3)}(1+r_n)((1+r_n)^2||x_n - p|| + t_n^{(2)}) + \beta_n^{(3)}||x_n - p|| + \gamma_n^{(3)}||u_n^{(3)} - p|| \\ &\le (\alpha_n^{(3)} + \beta_n^{(3)})(1+r_n)^3||x_n - p|| + \alpha_n^{(3)}t_n^{(2)}(1+r_n) + \gamma_n^{(3)}||u_n^{(3)} - p|| \\ &\le (1+r_n)^3||x_n - p|| + \alpha_n^{(3)}t_n^{(2)}(1+r_n) + \gamma_n^{(3)}||u_n^{(3)} - p|| \\ &\le (1+r_n)^3||x_n - p|| + t_n^{(3)},\end{aligned} \tag{2.3}$$

where $t_n^{(3)} = \alpha_n^{(3)}t_n^{(2)}(1+r_n) + \gamma_n^{(3)}\|u_n^{(3)} - p\|$. Since $\{u_n^{(3)}\}$ is bounded and $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(3)} < \infty$. Continuing the above process, we get

$$||x_{n+1} - p|| = ||x_n^{(N)} - p|| \le (1+r_n)^N||x_n - p|| + t_n^{(N)}, \tag{2.4}$$

where $\{t_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$. By Lemma 1.2, $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof. □

Lemma 2.2. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings* *with sequences* $\{r_n^{(i)}\}$ *such that* $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, $1 \le i \le N$ *and let* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence defined by (1.1) and some* $\alpha, \beta \in (0,1)$ *with the following restrictions:*

(i) $0 < \alpha \le \alpha_n^{(i)} \le \beta < 1$, $1 \le i \le N$, *for all* $n \ge n_0$ *for some* $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, $1 \le i \le N$.

Then, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.

Proof. For any $p \in F(T)$, it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \ge 0$. We note that

$$||x_n^{N-1} - p|| \le (1 + r_n)^{N-1} ||x_n - p|| + t_n^{(N-1)}, \quad \forall n \ge 1, \tag{2.5}$$

where $\{t_n^{(N-1)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n^{(N-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} ||x_n^{(N-1)} - p|| \le \limsup_{n\to\infty} \left((1 + r_n)^{N-1} ||x_n - p|| + t_n^{N-1}\right) = \lim_{n\to\infty} ||x_n - p|| = a \tag{2.6}$$

and so

$$\limsup_{n\to\infty} ||T_N^n x_n^{(N-1)} - p|| \le \limsup_{n\to\infty} (1 + r_n) ||x_n^{(N-1)} - p|| = \limsup_{n\to\infty} ||x_n^{(N-1)} - p|| \le a. \tag{2.7}$$

Next, consider

$$||T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)|| \le ||T_N^n x_n^{(N-1)} - p|| + \gamma_n^{(N)} ||u_n^{(N)} - x_n||. \tag{2.8}$$

Thus,

$$\limsup_{n\to\infty} ||T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)|| \le a. \tag{2.9}$$

Also,

$$||x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)|| \le ||x_n - p|| + \gamma_n^{(N)} ||u_n^{(N)} - x_n|| \tag{2.10}$$

gives that

$$\limsup_{n\to\infty} ||x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)|| \le a, \tag{2.11}$$

and we observe that

$$\begin{aligned} x_n^{(N)} - p &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} - \alpha_n^{(N)} p + \alpha_n^{(N)} \gamma_n^{(N)} u_n^{(N)} - \alpha_n^{(N)} \gamma_n^{(N)} x_n + (1 - \alpha_n^{(N)}) x_n \\ &\quad - (1 - \alpha_n^{(N)}) p - \gamma_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} - \alpha_n^{(N)} \gamma_n^{(N)} u_n^{(N)} + \alpha_n^{(N)} \gamma_n^{(N)} x_n \\ &= \alpha_n^{(N)} (T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)) \\ &\quad + (1 - \alpha_n^{(N)}) (x_n - p) - (1 - \alpha_n^{(N)}) \gamma_n^{(N)} x_n + (1 - \alpha_n^{(N)}) \gamma_n^{(N)} u_n^{(N)} \\ &= \alpha_n^{(N)} (T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)) + (1 - \alpha_n^{(N)}) (x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)). \end{aligned} \tag{2.12}$$

Therefore,

$$a = \lim_{n\to\infty} ||x_n^{(N)} - p|| = \lim_{n\to\infty} ||\alpha_n^{(N)}(T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)}(u_n^{(N)} - x_n)) \\ + (1-\alpha_n^{(N)})(x_n - p + \gamma_n^{(N)}(u_n^{(N)} - x_n))||. \tag{2.13}$$

By (2.9), (2.14), and Lemma 1.3, we have

$$\lim_{n\to\infty} ||T_N^n x_n^{(N-1)} - x_n|| = 0. \tag{2.14}$$

Now, we will show that $\lim_{n\to\infty} \| T_{N-1}^n x_n^{(N-2)} - x_n \| = 0$. For each $n \geq 1$,

$$\begin{aligned} ||x_n - p|| &\leq ||T_N^n x_n^{(N-1)} - x_n|| + ||T_N^n x_n^{(N-1)} - p|| \\ &\leq ||T_N^n x_n^{(N-1)} - x_n|| + (1+r_n)||x_n^{(N-1)} - p||. \end{aligned} \tag{2.15}$$

Using (2.14), we have

$$a = \lim_{n\to\infty} ||x_n - p|| \leq \liminf_{n\to\infty} ||x_n^{(N-1)} - p||. \tag{2.16}$$

It follows that

$$a \leq \liminf_{n\to\infty} ||x_n^{(N-1)} - p|| \leq \limsup_{n\to\infty} ||x_n^{(N-1)} - p|| \leq a. \tag{2.17}$$

This implies that

$$\lim_{n\to\infty} ||x_n^{(N-1)} - p|| = a. \tag{2.18}$$

On the other hand, we have

$$||x_n^{(N-2)} - p|| \leq (1+r_n)^{N-2} ||x_n - p|| + t_n^{(N-2)}, \quad \forall n \geq 1, \tag{2.19}$$

where $\sum_{n=1}^{\infty} t_n^{(N-2)} < \infty$.Therefore,

$$\limsup_{n\to\infty} ||x_n^{(N-2)} - p|| \leq \limsup_{n\to\infty} (1+r_n)^{N-2} ||x_n - p|| + t_n^{(N-2)} = a, \tag{2.20}$$

and hence,

$$\limsup_{n\to\infty} ||T_{N-1}^n x_n^{(N-2)} - p|| \leq \limsup_{n\to\infty} (1+r_n) ||x_n^{(N-2)} - p|| \leq a. \tag{2.21}$$

Next, consider

$$||T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)|| \leq ||T_{N-1}^n x_n^{(N-2)} - p|| + \gamma_n^{(N-1)} ||u_n^{(N-1)} - x_n||. \tag{2.22}$$

Thus,

$$\limsup_{n\to\infty} ||T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)|| \leq a. \tag{2.23}$$

Also,

$$||x_n - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)|| \le ||x_n - p|| + \gamma_n^{(N-1)}||u_n^{(N-1)} - x_n|| \tag{2.24}$$

gives that

$$\limsup_{n\to\infty} ||x_n - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)|| \le a, \tag{2.25}$$

and we observe that

$$\begin{aligned} x_n^{(N-1)} - p &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)})x_n - \gamma_n^{(N-1)} x_n \\ &\quad + \gamma_n^{(N-1)} u_n^{(N-1)} - (1 - \alpha_n^{(N-1)})p - \alpha_n^{(N-1)} p \\ &= \alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(N-1)})(x_n - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)), \end{aligned} \tag{2.26}$$

and hence

$$\begin{aligned} a = \lim_{n\to\infty} ||x_n^{(N-1)} - p|| = \lim_{n\to\infty} ||\alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ + (1 - \alpha_n^{(N-1)})(x_n - p + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n))||. \end{aligned} \tag{2.27}$$

By (2.23), (2.25), and Lemma 1.3, we have

$$\lim_{n\to\infty} ||T_{N-1}^n x_n^{(N-2)} - x_n|| = 0. \tag{2.28}$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty} ||T_{N-1}^n x_n^{(N-2)} - x_n|| = 0. \tag{2.29}$$

Continuing similar process, we have

$$\lim_{n\to\infty} ||T_{N-i} x_n^{(N-i-1)} - x_n|| = 0, \quad 0 \le i \le (N-2). \tag{2.30}$$

Now,

$$||T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)|| \le ||T_1^n x_n - p|| + \gamma_n^{(1)}||u_n^{(1)} - x_n||. \tag{2.31}$$

Thus,

$$\limsup_{n\to\infty} ||T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)|| \le a. \tag{2.32}$$

Also,

$$||x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)|| \le ||x_n - p|| + \gamma_n^{(1)}||u_n^{(1)} - x_n|| \tag{2.33}$$

gives that

$$\limsup_{n\to\infty} ||x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)|| \le a, \tag{2.34}$$

and hence,

$$\begin{aligned} a = \lim_{n\to\infty} ||x_n^{(1)} - p|| = \lim_{n\to\infty} ||\alpha_n^{(1)}(T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \\ + (1 - \alpha_n^{(1)})(x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n))||. \end{aligned} \tag{2.35}$$

By (2.32), (2.34), and Lemma 1.3, we have

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = 0, \tag{2.36}$$

and this implies that

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||\alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1 - \alpha_n^{(N)} - \gamma_n^{(N)}) x_n + \gamma_n^{(N)} u_n^{(N)} - x_n|| \\ &\le \alpha_n^{(N)} ||T_N^n x_n^{(N-1)} - x_n|| + \gamma_n^{(N)} ||u_n^{(N)} - x_n|| \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.37}$$

Thus, we have

$$\begin{aligned} ||T_N^n x_n - x_n|| &\le ||T_N^n x_n - T_N^n x_n^{(N-1)}|| + ||T_N^n x_n^{(N-1)} - x_n|| \\ &\le (1 + r_n)||x_n - x_n^{(N-1)}|| + ||T_N^n x_n^{(N-1)} - x_n|| \\ &= (1 + r_n)||x_n - \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)} - \gamma_n^{(N-1)}) x_n \\ &\qquad + \gamma_n^{(N-1)} u_n^{(N-1)}|| + ||T_N^n x_n^{(N-1)} - x_n|| \\ &\le (1 + r_n)[\alpha_n^{(N-1)}||x_n - T_{N-1}^n x_n^{(N-2)}|| + \gamma_n^{(N-1)}||u_n^{(N-1)} - x_n||] \\ &\quad + ||T_N^n x_n^{(N-1)} - x_n|| \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{2.38}$$

and we have

$$\begin{aligned} ||T_N x_n - x_n|| &\le ||x_{n+1} - x_n|| + ||x_{n+1} - T_N^{n+1} x_{n+1}|| \\ &\quad + ||T_N^{n+1} x_{n+1} - T_N^{n+1} x_n|| + ||T_N^{n+1} x_n - T_N x_n|| \\ &\le ||x_{n+1} - x_n|| + ||x_{n+1} - T_N^{n+1} x_{n+1}|| \\ &\quad + (1 + r_{n+1})||x_{n+1} - x_n|| + (1 + r_1)||T_N^n x_n - x_n||. \end{aligned} \tag{2.39}$$

It follows from (2.37), (2.38), and (2.39) that

$$\lim_{n\to\infty} ||T_N x_n - x_n|| = 0. \tag{2.40}$$

Next, we consider

$$
\begin{aligned}
\|T_{N-1}^n x_n - x_n\| &\le \|T_{N-1}^n x_n - T_{N-1}^n x_n^{(N-2)}\| + \|T_{N-1}^n x_n^{(N-2)} - x_n\| \\
&\le (1+r_n)\|x_n - x_n^{(N-2)}\| + \|T_{N-1}^n x_n^{(N-2)} - x_n\| \\
&\le (1+r_n)\big[\alpha_n^{(N-2)}\|x_n - T_{N-2}^n x_n^{(N-3)}\| + \gamma_n^{(N-2)}\|u_n^{(N-2)} - x_n\|\big] \\
&\quad + \|T_{N-1}^n x_n^{(N-2)} - x_n\| \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty,
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
\|T_{N-1} x_n - x_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_{N-1}^{n+1} x_{n+1}\| \\
&\quad + \|T_{N-1}^{n+1} x_{n+1} - T_{N-1}^{n+1} x_n\| + \|T_{N-1}^{n+1} x_n - T_{N-1} x_n\| \\
&\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_{N-1}^{n+1} x_{n+1}\| \\
&\quad + (1+r_{n+1})\|x_{n+1} - x_n\| + (1+r_1)\|T_{N-1}^n x_n - x_n\|.
\end{aligned}
\tag{2.42}
$$

It follows from (2.37), (2.41) and the above inequality that

$$
\lim_{n\to\infty} \|T_{N-1} x_n - x_n\| = 0. \tag{2.43}
$$

Continuing similar process, we have

$$
\lim_{n\to\infty} \|T_{N-i} x_n - x_n\| = 0, \quad 0 \le i \le (N-2). \tag{2.44}
$$

Now,

$$
\begin{aligned}
\|T_1 x_n - x_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\
&\quad + \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1 x_n\| \\
&\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\
&\quad + (1+r_{n+1})\|x_{n+1} - x_n\| + (1+r_1)\|T_1^n x_n - x_n\|.
\end{aligned}
\tag{2.45}
$$

It follows from (2.36), (2.37) and the above inequality that

$$
\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0, \tag{2.46}
$$

and hence,

$$
\lim_{n\to\infty} \|T_{N-i} x_n - x_n\| = 0, \quad 0 \le i \le (N-1). \tag{2.47}
$$

This completes the proof. □

We recall the following definitions:

(i) A mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [21] on K if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$ $\|x - Tx\| \ge f(d(x,F))$, where $d(x,F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

(ii) A finite family $\{T_1,\dots,T_N\}$ of N self mappings of K with $F = \bigcap_{i=1^N} F(T_i) \neq \emptyset$ is said to satisfy *condition* (B) on K [1] if there exist f and d as in (i) such that $\max_{1\le i\le N} \|x - T_i x\| \ge f(d(x,F))$ for all $x \in K$.

(iii) A finite family $\{T_1,\dots,T_N\}$ of N self mappings of K with $F=\bigcap_{i=1^N} F(T_i)\neq\emptyset$ is said to satisfy *condition* (C) on K [1] if there exist f and d as in (i) such that $(1/N)\sum_{i=1}^N \|x-T_ix\|\geq f(d(x,F))$ for all $x\in K$.

Note that condition (B) reduces to condition (A) when $T_i=T$, for all $i=1,2,\dots,N$.

It is well known that every continuous and demicompact mapping must satisfy condition (A) (see [21]). Since every completely continuous mapping $T:K\to K$ is continuous and demicompact, it satisfies condition (A). Therefore, to study strong convergence of $\{x_n\}$ defined by (1.1), we use condition (B) instead of the complete continuity of mappings $T_1,T_2,\dots,T_N$.

Theorem 2.3. *Let E be a real uniformly convex Banach space and K let be a nonempty closed convex subset of E. Let $\{T_1,\dots,T_N\}:K\to K$ be N asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^\infty r_n^{(i)}<\infty$ for all $1\leq i\leq N$ and $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$. Suppose that $\{T_1,T_2,\dots,T_N\}$ satisfies condition (B). Let $\{x_n\}$ be the sequence defined by (1.1) and some $\alpha,\beta\in(0,1)$ with the following restrictions:*

(i) $0<\alpha\leq\alpha_n^{(i)}\leq\beta<1$, $1\leq i\leq N$ $\forall$ $n\geq n_0$ *for some* $n_0\in\mathbb{N}$;

(ii) $\sum_{n=1}^\infty \gamma_n^i<\infty$, $1\leq i\leq N$.

Then, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1,\dots,T_N\}$.

Proof. By Lemma 2.1, we see that $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$. Let $\lim_{n\to\infty}\|x_n-p\|=a$ for some $a\geq 0$. Without loss of generality, if $a=0$, there is nothing to prove. Assume that $a>0$, as proved in Lemma 2.1, we have

$$||x_{n+1}-p||=||x_n^{(N)}-p||\leq\left(1+r_n\right)^N||x_n-p||+t_n^{(N)},\tag{2.48}$$

where $\{t_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^\infty t_n^{(N)}<\infty$. This gives that

$$d(x_{n+1},F)\leq\left(1+r_n\right)^N d(x_n,F)+t_n^{(N)}.\tag{2.49}$$

Applying Lemma 1.2 to the above inequality, we obtain that $\lim_{n\to\infty} d(x_n,F)$ exists. Also, by Lemma 2.2, $\lim_{n\to\infty}\|x_n-T_ix_n\|=0$, for all $i=1,2,\dots,N$. Since $\{T_1,T_2,\dots,T_N\}$ satisfies condition (B), we conclude that $\lim_{n\to\infty} d(x_n,F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n,F)=0$, given any $\varepsilon>0$, there exists a natural number n_0 such that $d(x_n,F)<\varepsilon/3$ for all $n\geq n_0$. So, we can find $p^*\in F$ such that $\|x_{n_0}-p^*\|<\varepsilon/2$. For all $n\geq n_0$ and $m\geq 1$, we have

$$||x_{n+m}-x_n||\leq||x_{n+m}-p^*||+||x_n-p^*||\leq||x_{n_0}-p^*||+||x_{n_0}-p^*||<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\tag{2.50}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n=q^*$. Then $q^*\in K$. It remains to show that $q^*\in F$. Let $\varepsilon_1>0$ be given. Then, there exists a natural number n_1 such that $\|x_n-q^*\|<\varepsilon_1/4$ for all $n\geq n_1$. Since $\lim_{n\to\infty} d(x_n,F)=0$, there exists a natural number $n_2\geq n_1$ such that for all $n\geq n_2$ we have $d(x_n,F)<\varepsilon_1/5$ and in particular we have $d(x_{n_2},F)\leq\varepsilon_1/5$. Therefore, there exists $w^*\in F$

such that $\|x_{n_2} - w^*\| < \varepsilon_1/4$. For any $i \in I$ and $n \geq n_2$, we have

$$\begin{aligned} ||T_i q^* - q^*|| &\leq ||T_i q^* - w^*|| + ||w^* - q^*|| \\ &\leq 2||q^* - w^*|| \leq 2(||q^* - x_{n_2}|| + ||x_{n_2} - q^*||) < \varepsilon_1. \end{aligned} \tag{2.51}$$

This implies that $T_i q^* = q^*$. Hence, $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in F = \bigcap_{n=1}^{N} F(T_i)$. This completes the proof. □

Remark 2.4. Theorem 2.3 holds true if we replace condition (B) with condition (C).

Remark 2.5. (1) Theorem 2.3 extends [3, Theorem 2], [4, Theorem 2.4], [17, Theorem], [18, Theorem 1.5], and [5, Theorems 2.1–2.3] to the case of finite family of nonexpansive mappings and multistep iteration considered here and no boundedness condition imposed on K.

(2) Theorem 2.3 also generalizes [1, Theorem 3.5] to the case of the iteration with errors in the sense of Xu [23].

We recall that a mapping $T : K \to K$ is called *semicompact* (or *hemicompact*) if any sequence $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 2.6. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $1 \leq i \leq N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Let $\{x_n\}$ be the sequence defined by (1.1) and some $\alpha, \beta \in (0,1)$ with the following restrictions:*

(i) *$0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$;*

(ii) *$\sum_{n=1}^{\infty} \gamma_n^i < \infty$, $1 \leq i \leq N$.*

Then, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, \ldots, T_N\}$.

Proof. Suppose that T_{i_0} is semicompact for some $i_0 \in \{1, 2, \ldots, N\}$. By Lemma 2.2, we have $\lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j\to\infty} x_{n_j} = p \in K$. Now, Lemma 2.2 guarantees that $\lim_{n_j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i \in \{1, 2, \ldots, N\}$ and so $\|p - T_i p\| = 0$ for all $i \in \{1, 2, \ldots, N\}$. This implies that $p \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 2.3, that $\{x_n\}$ converges strongly to some common fixed point in F. This completes the proof. □

Remark 2.7. Theorem 2.6 extends [15, Theorem 2] and [19, Theorem 2.2] to the case of finite family of nonexpansive mappings and multistep iteration considered here and no boundedness condition imposed on K.

Next, we give the weak convergence.

Lemma 2.8. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, $1 \leq i \leq N$, and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$*

be the sequence defined by (1.1) and some $\alpha,\beta \in (0,1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, *for all* $n \geq n_0$ *for some* $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, $1 \leq i \leq N$.

Then for all $u, v \in F$, $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ *exists for all* $t \in [0,1]$.

Proof. Since $\{x_n\}$ is bounded, there exist $R > 0$ such that $\{x_n\} \subset C := B_R(0) \cap K$. Then, C is a nonempty closed convex bounded subset of E. Basically, we follow the idea of [22]. Let $a_n(t) = \|tx_n + (1-t)u - v\|$. Then, $\lim_{n\to\infty} a_n(0) = \|u - v\|$, and from Lemma 2.1, $\lim_{n\to\infty} a_n(1) = \|x_n - v\|$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - u\| = r > 0$ and $t \in (0,1)$. Define $U_n : C \to C$ by

$$\begin{aligned}
U_n x &= \alpha_n^{(N)} T_N^n x^{(N-1)} + \beta_n^{(N)} x + \gamma_n^{(N)} u_n^{(N)}, \\
x^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x^{(N-2)} + \beta_n^{(N-1)} x + \gamma_n^{(N-1)} u_n^{(N-1)}, \\
&\vdots \\
x^{(3)} &= \alpha_n^{(3)} T_3^n x^{(2)} + \beta_n^{(3)} x + \gamma_n^{(3)} u_n^{(3)}, \\
x^{(2)} &= \alpha_n^{(2)} T_2^n x^{(1)} + \beta_n^{(2)} x + \gamma_n^{(2)} u_n^{(2)}, \\
x^{(1)} &= \alpha_n^{(1)} T_1^n x + \beta_n^{(1)} x + \gamma_n^{(1)} u_n^{(1)}, \quad x \in C.
\end{aligned} \tag{2.52}$$

Then,

$$||U_n x - U_n y|| \leq (1 + r_n)^N \|x - y\|. \tag{2.53}$$

Set

$$\begin{aligned}
S_{n,m} &:= U_{n+m-1} U_{n+m-2} \cdots U_n, \quad m \geq 1, \\
b_{n,m} &= ||S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)S_{n,m}u)||.
\end{aligned} \tag{2.54}$$

Then, observing $S_{n+m}x_n = x_{n+m}$, we get

$$\begin{aligned}
a_{n+m}(t) &= ||tx_{n+m} + (1-t)u - v|| \leq b_{n,m} + ||S_{n,m}(tx_n + (1-t)u) - v|| \\
&\leq b_{n,m} + \left(\prod_{j=n}^{n+m-1} (1 + r_j)^N\right) a_n \leq b_{n,m} + H_n a_n,
\end{aligned} \tag{2.55}$$

where $H_n = \prod_{j=n}^{\infty} (1 + r_j)^N$. By a result of Bruck [24] we have

$$\begin{aligned}
b_{n,m} &\leq H_n g^{-1}(||x_n - u|| - H_n^{-1} ||S_{n,m} - u||) \\
&\leq H_n g^{-1}(||x_n - u|| - ||x_{n+m} - u|| + (1 - H_n^{-1}) d),
\end{aligned} \tag{2.56}$$

where $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, is a strictly increasing continuous function depending only on d, the diameter of C. Since $\lim_{n\to\infty} H_n = 1$, it follows from Lemma 2.1 that $\lim_{n,m\to\infty} b_{n,m} = 0$. Therefore,

$$\limsup_{m\to\infty} a_m \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} H_n a_n = \liminf_{n\to\infty} a_n. \tag{2.57}$$

This completes the proof. □

Theorem 2.9. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\}$: $K \to K$ be N asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, $1 \le i \le N$, and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.1) and some $\alpha, \beta \in (0,1)$ with the following restrictions:*

(i) $0 < \alpha \le \alpha_n^{(i)} \le \beta < 1$, $1 \le i \le N$, *for all* $n \ge n_0$ *for some* $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, $1 \le i \le N$.

Then, $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$.

Proof. Let $p \in F$. Then by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Since E is reflexive and $\{x_n\}$ is a bounded sequence in K, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q \in K$. Moreover $\lim_{n_j\to\infty} \|x_{n_j} - T_i x_{n_j}\| \to 0$ for all $i \in \{1, 2, \ldots, N\}$, by Lemma 2.2. By Lemma 1.2, we have that $(I - T_i)q = 0$, that is, $q \in F(T_i)$. By arbitrariness of $i \in \{1, 2, \ldots, N\}$, we have $q \in F = \bigcap_{i=1}^{N} F(T_i)$.

Now, we show that $\{x_n\}$ converges weakly to q. Suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ which converges weakly to some $q^* \in K$ and $q \neq q^*$. By the similar method as above, we have $q^* \in F = \bigcap_{i=1}^{N} F(T_i)$ and so $p, q \in \omega_w(x_n) \cap F$. Then by Lemma 2.8,

$$\lim_{n\to\infty} \|tx_n + (1-t)q - q^*\| \tag{2.58}$$

exists for all $t \in [0,1]$. Now, Lemma 1.4 guarantees that $q = q^*$. As a result, $\omega_w(x_n)$ is a singleton, this implies that $\{x_n\}$ converges weakly to a point in $F = \bigcap_{i=1}^{N} F(T_i)$. This completes the proof. □

Remark 2.10. (1) Since the duals of reflexive Banach spaces with Fréchet differentiable norms have the Kadec-Klee property, Theorem 2.9 extends [2, Theorem 2.1], [3, Theorem 1], [15, Theorem 1], [4, Theorem 2.9], [19, Theorem 2.1], and [22, Theorems 3.1-3.2] to the case of finite family of asymptotically nonexpansive mappings and multistep iteration and also to other Banach spaces which do not satisfy Opial's condition or have Fréchet differentiable norm. Moreover, we do not impose boundedness condition on K.

(2) Theorem 3.4 in [1] is also a special case of Theorem 2.9 with $\gamma_n^{(i)} = 0$ for $i = 1, 2, \ldots, N$.

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Balwant Singh Thakur: School of Studies in Mathematics, Pondit Ravishankar Shukla University, Raipur 492010 CG, India
Email address: balwantst@gmail.com

Jong Soo Jung: Department of Mathematics, Dong-A University, Busan 604-714, South Korea
Email address: jungjs@mail.donga.ac.kr