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Research Article

Fixed Points and Hyers-Ulam-Rassias Stability of Cauchy-Jensen Functional Equations in Banach Algebras

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We prove the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras and of generalized derivations on real Banach algebras for the following Cauchy-Jensen functional equations: $f(x+y/2+z)+f(x-y/2+z)=f(x)+2f(z)$, $2f(x+y/2+z)=f(x)+f(y)+2f(z)$, which were introduced and investigated by Baak (2006). The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper (1978).

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [2] concerning the stability of group homomorphisms: let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta \tag{1.1}$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon \tag{1.2}$$

for all $x\in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y)=H(x)\diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that "how do the solutions of the inequality differ from those of the given functional equation"?

Hyers [3] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon \tag{1.3}$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \le \varepsilon \tag{1.4}$$

for all $x \in X$.

Rassias [4] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from anormed vector space E into a Banach space E' subject to the inequality*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.5}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then, the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.6}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - L(x)|| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.7}$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.5) has provided a lot of influence in the development of what is now known as a *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruţa [5] generalized Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6–17]).

Rassias [18], following the spirit of the innovative approach of Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \ne 1$ (see also [19] for a number of other new results).

Theorem 1.2 [18–20]. *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot \|x\|^{p/2} \cdot \|y\|^{p/2} \tag{1.8}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p \tag{1.9}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

We recall two fundamental results in fixed point theory.

Theorem 1.3 [21]. *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, that is,*

$$d(Jx, Jy) \leq Lf(x, y), \quad \forall x, y \in X \tag{1.10}$$

for some Lipschitz constant $L < 1$. Then,
(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*
(2) *the fixed point x^* is globally attractive, that is,*

$$\lim_{n\to\infty} J^n x = x^* \tag{1.11}$$

for any starting point $x \in X$;
(3) *one has the following estimation inequalities:*

$$\begin{aligned} d(J^n x, x^*) &\leq L^n d(x, x^*), \\ d(J^n x, x^*) &\leq \frac{1}{1-L} d(J^n x, J^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Jx) \end{aligned} \tag{1.12}$$

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following:
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + f(y, z)$ for all $x, y, z \in X$.

Theorem 1.4 [22]. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.13}$$

for all nonnegative integers n or there exists a positive integer n_0 *such that*

(1) $d(J^n x, J^{n+1}x) < \infty$, $\forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ *for all* $y \in Y$.

This paper is organized as follows. In Section 2, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras for the Cauchy-Jensen functional equations.

In Section 3, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of generalized derivations on real Banach algebras for the Cauchy-Jensen functional equations.

2. Stability of homomorphisms in real Banach algebras

Throughout this section, assume that A is a real Banach algebra with norm $\|\cdot\|_A$ and that B is a real Banach algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$Cf(x,y,z) := f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) \tag{2.1}$$

for all $x, y, z \in A$.

We prove the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras for the functional equation $Cf(x,y,z) = 0$.

Theorem 2.1. *Let* $f : A \to B$ *be a mapping for which there exists a function* $\varphi : A^3 \to [0,\infty)$ *such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty, \tag{2.2}$$

$$||Cf(x,y,z)||_B \leq \varphi(x,y,z), \tag{2.3}$$

$$||f(xy) - f(x)f(y)||_B \leq \varphi(x,y,0) \tag{2.4}$$

for all $x, y, z \in A$. *If there exists an* $L < 1$ *such that* $\varphi(x,x,x) \leq 2L\varphi(x/2,x/2,x/2)$ *for all* $x \in A$ *and if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique homomorphism* $H : A \to B$ *such that*

$$||f(x) - H(x)||_B \leq \frac{1}{2-2L}\varphi(x,x,x) \tag{2.5}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\} \tag{2.6}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x,x,x),\ \forall x \in A\}. \tag{2.7}$$

It is easy to show that (X,d) is complete.

Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{2.8}$$

for all $x \in A$.

By [21, Theorem 3.1],

$$d(Jg,Jh) \leq Ld(g,h) \tag{2.9}$$

for all $g,h \in X$.

Letting $y = z = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\|_B \leq \varphi(x,x,x) \tag{2.10}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \leq \frac{1}{2}\varphi(x,x,x) \tag{2.11}$$

for all $x \in A$. Hence $d(f,Jf) \leq 1/2$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that the following hold.

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.12}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}. \tag{2.13}$$

This implies that H is a unique mapping satisfying (2.12) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x,x,x) \tag{2.14}$$

for all $x \in A$.

(2) $d(J^n f,H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.15}$$

for all $x \in A$.

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{1}{2-2L}. \tag{2.16}$$

This implies that the inequality (2.5) holds.

It follows from (2.2), (2.3), and (2.15) that

$$\begin{aligned}
&\left\|H\left(\frac{x+y}{2}+z\right)+H\left(\frac{x-y}{2}+z\right)-H(x)-2H(z)\right\|_B \\
&\quad = \lim_{n\to\infty}\frac{1}{2^n}\left\|f\left(2^{n-1}(x+y)+2^n z\right)+f\left(2^{n-1}(x-y)+2^n z\right)-f(2^n x)-2f(2^n z)\right\|_B \\
&\quad \le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n y,2^n z)=0
\end{aligned} \tag{2.17}$$

for all $x,y,z \in A$. So

$$H\left(\frac{x+y}{2}+z\right)+H\left(\frac{x-y}{2}+z\right)=H(x)+2H(z) \tag{2.18}$$

for all $x,y,z \in A$. By [1, Lemma 2.1], the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem of [4], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.4) that

$$\begin{aligned}
\|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty}\frac{1}{4^n}\|f(4^n xy)-f(2^n x)f(2^n y)\|_B \\
&\le \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^n x,2^n y,0) \le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n y,0)=0
\end{aligned} \tag{2.19}$$

for all $x,y \in A$. So

$$H(xy)=H(x)H(y) \tag{2.20}$$

for all $x,y \in A$. Thus, $H : A \to B$ is a homomorphism satisfying (2.5), as desired. □

Corollary 2.2. *Let $r<1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\begin{aligned}
\|Cf(x,y,z)\|_B &\le \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right), \\
\|f(xy)-f(x)f(y)\|_B &\le \theta\left(\|x\|_A^r+\|y\|_A^r\right)
\end{aligned} \tag{2.21}$$

for all $x,y,z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{3\theta}{2-2^r}\|x\|_A^r \tag{2.22}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y,z) := \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) \tag{2.23}$$

for all $x, y, z \in A$. Then, $L = 2^{r-1}$ and we get the desired result. □

THEOREM 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ satisfying (2.3) and (2.4) such that*

$$\sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{2.24}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x,x,x) \le (1/2)L\varphi(2x,2x,2x)$ for all $x \in A$ and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{L}{2-2L}\varphi(x,x,x) \tag{2.25}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{2.26}$$

for all $x \in A$.

It follows from (2.10) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \le \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\varphi(x,x,x) \tag{2.27}$$

for all $x \in A$. Hence $d(f, Jf) \le L/2$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that the following hold.

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.28}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}. \tag{2.29}$$

This implies that H is a unique mapping satisfying (2.28) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x,x,x) \tag{2.30}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x) \tag{2.31}$$

for all $x \in A$.

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{L}{2-2L}, \tag{2.32}$$

which implies that the inequality (2.25) holds.

It follows from (2.3), (2.24), and (2.31) that

$$\begin{aligned} &\left\|H\left(\frac{x+y}{2}+z\right)+H\left(\frac{x-y}{2}+z\right)-H(x)-2H(z)\right\|_B \\ &\quad = \lim_{n\to\infty} 2^n\left\|f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)+f\left(\frac{x-y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x}{2^n}\right)-2f\left(\frac{z}{2^n}\right)\right\|_B \\ &\quad \le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \le \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0 \end{aligned} \tag{2.33}$$

for all $x, y, z \in A$. So

$$H\left(\frac{x+y}{2}+z\right)+H\left(\frac{x-y}{2}+z\right) = H(x)+2H(z) \tag{2.34}$$

for all $x, y, z \in A$. By [1, Lemma 2.1], the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem of [4], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.4) that

$$\begin{aligned} \|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty} 4^n\left\|f\left(\frac{xy}{4^n}\right)-f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},0\right) = 0 \end{aligned} \tag{2.35}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y) \tag{2.36}$$

for all $x, y \in A$. Thus, $H : A \to B$ is a homomorphism satisfying (2.25), as desired. □

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.21). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x)-H(x)\|_B \le \frac{3\theta}{2^r-2}\|x\|_A^r \tag{2.37}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y,z) := \theta\big(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\big) \tag{2.38}$$

for all $x,y,z \in A$. Then, $L = 2^{1-r}$ and we get the desired result. □

3. Stability of generalized derivations on real Banach algebras

Throughout this section, assume that A is a real Banach algebra with norm $\|\cdot\|_A$.

For a given mapping $f : A \to A$, we define

$$Df(x,y,z) := 2f\left(\frac{x+y}{2}+z\right) - f(x) - f(y) - 2f(z) \tag{3.1}$$

for all $x,y,z \in A$.

Definition 3.1 [23]. A *generalized derivation* $\delta : A \to A$ is $\mathbb{R}$-linear and fulfills the generalized Leibniz rule

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{3.2}$$

for all $x,y,z \in A$.

We prove the Hyers-Ulam-Rassias stability of generalized derivations on real Banach algebras for the functional equation $Df(x,y,z) = 0$.

Theorem 3.2. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ satisfying (2.2) such that*

$$||Df(x,y,z)||_A \le \varphi(x,y,z), \tag{3.3}$$

$$||f(xyz) - f(xy)z + xf(y)z - xf(yz)||_A \le \varphi(x,y,z) \tag{3.4}$$

for all $x,y,z \in A$. If there exists an $L < 1$ such that $\varphi(x,x,x) \le 2L\varphi(x/2,x/2,x/2)$ for all $x \in A$ and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that

$$||f(x) - \delta(x)||_A \le \frac{1}{4-4L}\varphi(x,x,x) \tag{3.5}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to A\} \tag{3.6}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : ||g(x) - h(x)||_A \le C\varphi(x,x,x),\ \forall x \in A\}. \tag{3.7}$$

It is easy to show that (X,d) is complete.

We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{3.8}$$

for all $x \in A$.

By [21, Theorem 3.1],

$$d(Jg,Jh) \le Ld(g,h) \tag{3.9}$$

for all $g,h \in X$.

Letting $y = z = x$ in (3.3), we get

$$||2f(2x) - 4f(x)||_A \le \varphi(x,x,x) \tag{3.10}$$

for all $x \in A$. So

$$\left|\left|f(x) - \frac{1}{2}f(2x)\right|\right|_A \le \frac{1}{4}\varphi(x,x,x) \tag{3.11}$$

for all $x \in A$. Hence $d(f,Jf) \le 1/4$.

By Theorem 1.4, there exists a mapping $\delta : A \to A$ such that the following hold.

(1) δ is a fixed point of J, that is,

$$\delta(2x) = 2\delta(x) \tag{3.12}$$

for all $x \in A$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}. \tag{3.13}$$

This implies that δ is a unique mapping satisfying (3.12) such that there exists $C \in (0,\infty)$ satisfying

$$||\delta(x) - f(x)||_A \le C\varphi(x,x,x) \tag{3.14}$$

for all $x \in A$.

(2) $d(J^n f,\delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \delta(x) \tag{3.15}$$

for all $x \in A$.

(3) $d(f,\delta) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,\delta) \le \frac{1}{4-4L}. \tag{3.16}$$

This implies that the inequality (3.5) holds.

It follows from (2.2), (3.3), and (3.15) that

$$\begin{aligned}\left\|2\delta\left(\frac{x+y}{2}+z\right)-\delta(x)-\delta(y)-2\delta(z)\right\|_A\\ &=\lim_{n\to\infty}\frac{1}{2^n}\left\|2f\left(2^{n-1}(x+y)+2^nz\right)-f(2^nx)-f(2^ny)-2f(2^nz)\right\|_A\\ &\leq\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny,2^nz)=0\end{aligned}\tag{3.17}$$

for all $x,y,z\in A$. So

$$2\delta\left(\frac{x+y}{2}+z\right)=\delta(x)+\delta(y)+2\delta(z)\tag{3.18}$$

for all $x,y,z\in A$. By [1, Lemma 2.1], the mapping $\delta:A\to A$ is Cauchy additive.

By the same reasoning as in the proof of Theorem of [4], the mapping $\delta:A\to A$ is $\mathbb{R}$-linear.

It follows from (3.4) that

$$\begin{aligned}\|\delta(xyz)-\delta(xy)z+x\delta(y)z-x\delta(yz)\|_A\\ &=\lim_{n\to\infty}\frac{1}{8^n}\left\|f(8^nxyz)-f(4^nxy)\cdot 2^nz+2^nxf(2^ny)\cdot 2^nz-2^nxf(4^nyz)\right\|_A\\ &\leq\lim_{n\to\infty}\frac{1}{8^n}\varphi(2^nx,2^ny,2^nz)\leq\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny,2^nz)=0\end{aligned}\tag{3.19}$$

for all $x,y,z\in A$. So

$$\delta(xyz)=\delta(xy)z-x\delta(y)z+x\delta(yz)\tag{3.20}$$

for all $x,y,z\in A$. Thus, $\delta:A\to A$ is a generalized derivation satisfying (3.5). □

Corollary 3.3. *Let $r<1$ and θ be nonnegative real numbers, and let $f:A\to A$ be a mapping such that*

$$\begin{gathered}\|Df(x,y,z)\|_A\leq\theta\cdot\|x\|_A^{r/3}\cdot\|y\|_A^{r/3}\cdot\|z\|_A^{r/3},\\ \|f(xyz)-f(xy)z+xf(y)z-xf(yz)\|_A\leq\theta\cdot\|x\|_A^{r/3}\cdot\|y\|_A^{r/3}\cdot\|z\|_A^{r/3}\end{gathered}\tag{3.21}$$

for all $x,y,z\in A$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique generalized derivation $\delta:A\to A$ such that

$$\|f(x)-\delta(x)\|_A\leq\frac{\theta}{4-2^{r+1}}\|x\|_A^r\tag{3.22}$$

for all $x\in A$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x,y,z):=\theta\cdot\|x\|_A^{r/3}\cdot\|y\|_A^{r/3}\cdot\|z\|_A^{r/3}\tag{3.23}$$

for all $x,y,z\in A$. Then, $L=2^{r-1}$ and we get the desired result. □

Theorem 3.4. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ satisfying (3.3) and (3.4) such that*

$$\sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{3.24}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x,x,x) \le (1/2)L\varphi(2x,2x,2x)$ for all $x \in A$ and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{L}{4-4L}\varphi(x,x,x) \tag{3.25}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{3.26}$$

for all $x \in A$.

It follows from (3.10) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_A \le \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x,x,x) \tag{3.27}$$

for all $x \in A$. Hence $d(f, Jf) \le L/4$.

By Theorem 1.4, there exists a mapping $\delta : A \to A$ such that the following hold.

(1) δ is a fixed point of J, that is,

$$\delta(2x) = 2\delta(x) \tag{3.28}$$

for all $x \in A$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}. \tag{3.29}$$

This implies that δ is a unique mapping satisfying (3.28) such that there exists $C \in (0,\infty)$ satisfying

$$\|\delta(x) - f(x)\|_A \le C\varphi(x,x,x) \tag{3.30}$$

for all $x \in A$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \delta(x) \tag{3.31}$$

for all $x \in A$.

(3) $d(f,\delta) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,\delta) \le \frac{L}{4-4L}, \tag{3.32}$$

which implies that the inequality (3.25) holds.

It follows from (3.3), (3.24), and (3.31) that

$$\begin{aligned} &\left\|2\delta\left(\frac{x+y}{2}+z\right)-\delta(x)-\delta(y)-2\delta(z)\right\|_A \\ &\quad = \lim_{n\to\infty} 2^n\left\|2f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)-2f\left(\frac{z}{2^n}\right)\right\|_A \\ &\quad \le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \le \lim_{n\to\infty} 8^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0 \end{aligned} \tag{3.33}$$

for all $x,y,z \in A$. So

$$2\delta\left(\frac{x+y}{2}+z\right) = \delta(x)+\delta(y)+2\delta(z) \tag{3.34}$$

for all $x,y,z \in A$. By [1, Lemma 2.1], the mapping $\delta : A \to A$ is Cauchy additive.

By the same reasoning as in the proof of Theorem of [4], the mapping $\delta : A \to A$ is $\mathbb{R}$-linear.

It follows from (3.4) that

$$\begin{aligned} &\|\delta(xyz)-\delta(xy)z+x\delta(y)z-x\delta(yz)\|_A \\ &\quad = \lim_{n\to\infty} 8^n\left\|f\left(\frac{xyz}{8^n}\right)-f\left(\frac{xy}{4^n}\right)\cdot\frac{z}{2^n}+\frac{x}{2^n}f\left(\frac{y}{2^n}\right)\cdot\frac{z}{2^n}-\frac{x}{2^n}f\left(\frac{yz}{4^n}\right)\right\|_A \\ &\quad \le \lim_{n\to\infty} 8^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0 \end{aligned} \tag{3.35}$$

for all $x,y,z \in A$. So

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{3.36}$$

for all $x,y,z \in A$. Thus, $\delta : A \to A$ is a generalized derivation satisfying (3.28). □

Corollary 3.5. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.21). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that*

$$\|f(x)-\delta(x)\|_A \le \frac{\theta}{2^{r+1}-4}\|x\|_A^r \tag{3.37}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x,y,z) := \theta\cdot\|x\|_A^{r/3}\cdot\|y\|_A^{r/3}\cdot\|z\|_A^{r/3} \tag{3.38}$$

for all $x,y,z \in A$. Then, $L = 2^{1-r}$ and we get the desired result. □

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