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Research Article

Some Results for a Finite Family of Uniformly L-Lipschitzian Mappings in Banach Spaces

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The purpose of this paper is to prove a strong convergence theorem for a finite family of uniformly L-Lipschitzian mappings in Banach spaces. The results presented in the paper not only correct some mistakes appeared in the paper by Ofoedu (2006) but also improve and extend some recent results by Chang (2001), Cho et al. (2005), Ofoedu (2006), Schu (1991), and Zeng (2003, 2005).

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\|\}, \quad x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. Let $T : K \to K$ be a mapping.

(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ (without loss of generality, assume that $L \geq 1$) such that for any $x, y \in K$,

$$\left\|T^n x - T^n y\right\| \leq L\|x - y\| \quad \forall n \geq 1; \tag{1.2}$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that for any given $x, y \in K$,

$$||T^n x - T^n y|| \le k_n \|x - y\| \quad \forall n \ge 1; \tag{1.3}$$

(3) T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that, for any $x, y \in K$, there exists $j(x-y) \in J(x-y)$:

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2 \quad \forall n \ge 1. \tag{1.4}$$

Remark 1.2. (1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\ge1} k_n$, and every asymptotically nonexpansive mapping is asymptotically pseudocontractive, but the inverse is not true, in general.

(2) The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1], while the concept of asymptotically pseudocontractive mappings was introduced by Schu [2] who proved the following theorem.

Theorem 1.3 (Schu [2]). *Let H be a Hilbert space, let K be a nonempty bounded closed convex subset of H, and let $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian, and asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ satisfying the following conditions:*

(i) $k_n \to 1$ *as* $n \to \infty$*;*

(ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ such that $\varepsilon < \alpha_n < b$, for all $n \ge 1$, where $\varepsilon > 0$ and $b \in (0, L^{-2}[(1+L^2)^{1/2} - 1])$ are some positive numbers. For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \quad \forall n \ge 1. \tag{1.5}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T in K.

In [3], the first author extended Theorem 1.3 to a real uniformly smooth Banach space and proved the following theorem.

Theorem 1.4 (Chang [3]). *Let E be a uniformly smooth Banach space, let K be a nonempty bounded closed convex subset of E, and let $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$, and $F(T) \ne \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$*;*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \quad \forall n \ge 0. \tag{1.6}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \le k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|) \quad \forall n \ge 0, \tag{1.7}$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$.

Very recently, in [4] Ofoedu proved the following theorem.

Theorem 1.5 (Ofoedu [4]). *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$, such that $x^* \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \quad \forall n \ge 0. \tag{1.8}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|) \quad \forall x \in K, \tag{1.9}$$

then $\{x_n\}$ converges strongly to x^*.

Remark 1.6. It should be pointed out that although Theorem 1.5 extends Theorem 1.4 from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K, but the proof of [4, Theorem 3.1] has some problems.

The purpose of this paper is, by using a simple and quite different method, to prove some strong convergence theorems for a finite family of L-Lipschitzian mappings in stead of the assumption that T is a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in a Banach space. Our results not only correct some mistakes appeared in [4] but also extend and improve some recent results in [2–7].

For this purpose, we first give the following lemmas.

Lemma 1.7 (Change [8]). *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall j(x+y) \in J(x+y). \tag{1.10}$$

Lemma 1.8 (Moore and Nnoli [9]). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and let $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \le \lambda_n \le 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty. \tag{1.11}$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ *such that*

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n \quad \forall n \geq n_0, \tag{1.12}$$

where n_0 *is some nonnegative integer and* $\{\sigma_n\}$ *is a sequence of nonnegative number such that* $\sigma_n = \circ(\lambda_n)$, *then* $\theta_n \to 0$ *as* $n \to \infty$.

2. Main results

Definition 2.1. Let E be real Banach space, let K be a nonempty closed convex subset, and let $T_i : K \to K$, $i = 1,2,\dots,N$ be a finite family of mappings. $\{T_i,\ i = 1,2,\dots,N\}$ is called a finite family of uniformly L-Lipschitzian mappings if there exists a positive constant L (without loss of generality, assume that $L \geq 1$) such that for all $x, y \in K$,

$$||T_i^n x - T_i^n y|| \leq L\|x - y\| \quad \forall n \geq 1,\ i = 1,2,\dots,N. \tag{2.1}$$

The following theorem is the main result in this paper.

Theorem 2.2. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K$, $i = 1,2,\dots,N$ be a finite family of uniformly L-Lipschitzian mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $L \geq 1$ is a constant and $F(T_i)$ is the set of fixed points of T_i in K. Let x^* be a given point in $\bigcap_{i=1}^N F(T_i)$ and let $\{k_n\} \subset [1,\infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *(as $n \to \infty$);*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_n^n y_n \quad \forall n \geq 1, \\ y_n &= (1 - \beta_n)x_n + \beta_n T_n^n x_n \quad \forall n \geq 1, \end{aligned} \tag{2.2}$$

where $T_n^n = T_{n(\mathrm{mod}N)}^n$. If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for any $x \in K$,

$$\langle T_n^n x - x^*, j(x - x^*)\rangle \leq k_n||x - x^*||^2 - \phi(||x - x^*||) \quad \forall n \geq 1, \tag{2.3}$$

then $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^N F(T_i)$, if and only if $\{y_n\}$ is bounded.

Proof

Necessity. If the sequence $\{x_n\}$ defined by (2.2) converges strongly to $x^* \in \bigcap_{i=1}^N F(T_i)$, from (2.2) we have

$$\begin{aligned} ||y_n - x^*|| &= ||(1 - \beta_n)(x_n - x^*) + \beta_n(T_n^n x_n - x^*)|| \\ &\leq (1 - \beta_n)||x_n - x^*|| + \beta_n||T_n^n x_n - x^*|| \\ &\leq (1 - \beta_n)||x_n - x^*|| + \beta_n L||x_n - x^*|| \\ &\leq L||x_n - x^*|| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned} \tag{2.4}$$

This implies that $y_n \to x^*$, as $n \to \infty$, and so $\{y_n\}$ is bounded.

Sufficiency. Let $\{y_n\}$ be a bounded sequence. Denote $M = \sup_{n\geq 1}\|y_n - x^*\|$. It follows from (2.2) that

$$\begin{aligned}||x_{n+1}-x^*|| &= ||(1-\alpha_n)(x_n-x^*)+\alpha_n(T_n^n y_n - x^*)||\\ &\leq (1-\alpha_n)||x_n-x^*||+\alpha_n L||y_n-x^*||\\ &\leq (1-\alpha_n)||x_n-x^*||+\alpha_n ML\\ &\leq \max\{||x_n-x^*||, ML\}.\end{aligned} \tag{2.5}$$

By induction, we can prove that

$$||x_{n+1}-x^*|| \leq \max\{||x_1-x^*||, ML\} \quad \forall n \geq 1. \tag{2.6}$$

This implies that $\{x_n\}$ is bounded, and so $\{\|T_n^n x_n\|\}$ and $\{\|T_n^n y_n\|\}$ both are bounded. Denote

$$M_1 = \sup_{n\geq 1}\{||x_n-x^*||+||T_n^n x_n - x_n||+||T_n^n y_n - x_n||\} < \infty. \tag{2.7}$$

Again from (2.2) and Lemma 1.7, we have

$$\begin{aligned}||x_{n+1}-x^*||^2 &= ||(x_n-x^*)+\alpha_n(T_n^n y_n - x_n)||^2\\ &\leq ||(x_n-x^*)||^2+2\alpha_n\langle T_n^n y_n - x_n, j(x_{n+1}-x^*)\rangle.\end{aligned} \tag{2.8}$$

Now we consider the second term on the right side of (2.8) It follows from (2.2) and (2.3) that

$$\begin{aligned}\langle T_n^n y_n - x_n, j(x_{n+1}-x^*)\rangle &= \langle T_n^n x_{n+1}-x^*, j(x_{n+1}-x^*)\rangle\\ &\quad + \langle T_n^n y_n - T_n^n x_{n+1}, j(x_{n+1}-x^*)\rangle\\ &\quad + \langle x^*-x_n, j(x_{n+1}-x^*)\rangle\\ &\leq k_n||x_{n+1}-x^*||^2-\phi(||x_{n+1}-x^*||)\\ &\quad + L||y_n-x_{n+1}||\cdot||x_{n+1}-x^*||\\ &\quad + \langle x_{n+1}-x_n, j(x_{n+1}-x^*)\rangle\\ &\quad - \langle x_{n+1}-x^*, j(x_{n+1}-x^*)\rangle\\ &\leq k_n||x_{n+1}-x^*||^2-\phi(||x_{n+1}-x^*||)\\ &\quad + L||y_n-x_{n+1}||\cdot||x_{n+1}-x^*||\\ &\quad + \alpha_n\langle T_n^n y_n - x_n, j(x_{n+1}-x^*)\rangle - ||x_{n+1}-x^*||^2\\ &\leq (k_n-1)||x_{n+1}-x^*||^2-\phi(||x_{n+1}-x^*||)\\ &\quad + L||y_n-x_{n+1}||\cdot||x_{n+1}-x^*||+\alpha_n||T_n^n y_n - x_n||\cdot||x_{n+1}-x^*||\\ &\leq (k_n-1)M_1^2-\phi(||x_{n+1}-x^*||)+LM_1||y_n-x_{n+1}||+\alpha_n M_1^2.\end{aligned} \tag{2.9}$$

Now we consider the third term on the right side of (2.9). From (2.2) we have

$$\begin{aligned}
||x_{n+1} - y_n|| &= ||(1-\alpha_n)(x_n - y_n) + \alpha_n(T_n^n y_n - y_n)|| \\
&\le (1-\alpha_n)||x_n - y_n|| + \alpha_n||T_n^n y_n - x^* + x^* - y_n|| \\
&\le (1-\alpha_n)||x_n - y_n|| + \alpha_n(1+L)||y_n - x^*|| \\
&\le (1-\alpha_n)||x_n - y_n|| + \alpha_n(1+L)\{||y_n - x_n|| + ||x_n - x^*||\} \\
&= (1+L\alpha_n)||x_n - y_n|| + \alpha_n(1+L)||x_n - x^*|| \\
&= (1+L\alpha_n)\{\beta_n||x_n - T_n^n x^n||\} + \alpha_n(1+L)||x_n - x^*|| \\
&\le (1+L\alpha_n)\beta_n(1+L)||x_n - x^*|| + \alpha_n(1+L)||x_n - x^*|| \le d_n M_1,
\end{aligned} \tag{2.10}$$

where

$$d_n = (1+L)\{(1+L\alpha_n)\beta_n + \alpha_n\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

Substituting (2.10) into (2.9) and then substituting the results into (2.8) and simplifying it, we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 \le{}& ||x_n - x^*||^2 - 2\alpha_n\phi(||x_{n+1} - x^*||) \\
&+ 2\alpha_n\{(k_n - 1) + Ld_n + \alpha_n\}M_1^2.
\end{aligned} \tag{2.12}$$

Taking $\theta_n = \|x_n - x^*\|$, $\lambda_n = 2\alpha_n$, and $\sigma_n = 2\alpha_n\{(k_n-1) + Ld_n + \alpha_n\}M_1^2$, then (2.12) can be written as

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n\phi(\theta_{n+1}) + \sigma_n \quad \forall n \ge n_0. \tag{2.13}$$

By the conditions (i)-(ii), we know that all the conditions in Lemma 1.8 are satisfied. Therefore, it follows that

$$||x_n - x^*|| \longrightarrow 0, \tag{2.14}$$

that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 2.3. (1) Theorem 2.2 extends and improves the corresponding results in Chang [3], Cho et al. [5], Ofoedu [4], Schu [2], and Zeng [6, 7].

(2) The method given in the proof of Theorem 2.2 is quite different from the method given in Ofoedu [4].

(3) Theorem 2.2 also corrects some mistakes appeared in the proof of [4, Theorem 3.1].

(4) Under suitable conditions, the sequence $\{x_n\}$ defined by (2.2) in Theorem 2.2 also can be generalized to the iterative sequences with errors. Because the proof is straightforward, we omit it here.

The following theorem can be obtained from Theorem 2.2 immediately.

THEOREM 2.4. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K$, $i = 1, 2, \ldots, N$ be a finite family of uniformly L-Lipschitzian mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $L \geq 1$ is a constant and $F(T_i)$ is the set of fixed points of T_i in K. Let x^* be a given point in $\bigcap_{i=1}^N F(T_i)$ and let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$ *(as $n \to \infty$);*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n^n x_n \quad \forall n \geq 1, \tag{2.15}$$

where $T_n^n = T_{n(\bmod N)}^n$. If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for any $x \in K$

$$\langle T_n^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|) \quad \forall n \geq 1, \tag{2.16}$$

then $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^N F(T_i)$ if and only if $\{x_n\}$ is bounded.

Proof. Taking $\beta_n = 0$ in Theorem 2.2, we know that $y_n = x_n$ for all $n \geq 1$. Hence the conclusion of Theorem 2.4 can be obtained from Theorem 2.2 immediately. □

Remark 2.5. Theorem 2.4 is also a generalization and improvement of Ofoedu [4, Theorem 3.2].

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