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*Research Article*

# A Dual of the Compression-Expansion Fixed Point Theorems

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This paper presents a dual of the fixed point theorems of compression and expansion of functional type as well as the original Leggett-Williams fixed point theorem. The multi-valued situation is also discussed.



## 1. Introduction

In this paper, we present a dual of the fixed point theorems of expansion and compression using an axiomatic index theory as well as the original Leggett-Williams fixed point which is itself a generalization of the fixed point theorems of expansion and compression. In [1] Leggett and Williams presented criteria which guaranteed the existence of a fixed point for single-valued, continuous, compact maps that did not require the operator to be invariant on the underlying sets utilizing a concave functional and the norm. In that sense, the Leggett-Williams fixed point theorem generalized the compression-expansion fixed point theorem of norm type by Guo [2]. In [3] Anderson and Avery generalized the fixed point theorem of Guo [2] by replacing the norm in places by convex functionals and in [4] Zhang and Sun extended this result by showing that a certain set was a retract thus completely removing the norm from the argument. In this paper, we provide, in a sense, a generalization of all of the compression-expansion arguments that have utilized the norm and/or functionals (including [2–6]) which does not require sets to be invariant under our operator and yet maintains the freedom gained by using concave and convex functionals. The main result changes the roles of the concave and convex functionals from the techniques of [1] that have been employed in numerous multiple fixed point theorems ([7–10] to mention a few) which yields an additional technique for researchers interested in finding multiple fixed point theorems. It is in the sense of this exchange in the roles of concave and convex, yet resulting in somewhat analogous fixed point results, that we think of the main result of this paper as being *dual* to aforementioned fixed point results.

We conclude by applying the techniques of Agarwal and O'Regan [11] to generalize the fixed point theorem to maps which obey an axiomatic index theory, so in particular the results apply to all multivalued maps in the literature which have a well-defined fixed point index (see [11–13] and the references therein).

## 2. Preliminaries

In this section, we will state the definitions that are used in the remainder of the paper.

*Definition 2.1.* Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$$x \leq y \quad \text{iff } y - x \in P. \tag{2.1}$$

*Definition 2.2.* An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

*Definition 2.3.* A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if

$$\alpha : P \longrightarrow [0, \infty) \tag{2.2}$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y) \tag{2.3}$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if

$$\beta : P \longrightarrow [0, \infty) \tag{2.4}$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y) \tag{2.5}$$

for all $x, y \in P$ and $t \in [0,1]$.

Let $\alpha$ and $\psi$ be nonnegative continuous concave functionals on $P$ and let $\beta$ be a nonnegative continuous convex functional on $P$; then, for positive real numbers $r$, $\tau$, and $R$, we define the following sets:

$$
\begin{aligned}
Q(\alpha,r) &= \{x \in P : r \le \alpha(x)\},\\
Q(\alpha,\beta,r,R) &= \{x \in P : r \le \alpha(x),\ \beta(x) \le R\},\\
Q(\alpha,\psi,\beta,r,\tau,R) &= \{x \in P : r \le \alpha(x),\ \tau \le \psi(x),\ \beta(x) \le R\}.
\end{aligned}
\tag{2.6}
$$

*Definition 2.4.* Let $D$ be a subset of a real Banach space $E$. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then $D$ is a *retract* of $E$, and the map $r$ is a *retraction*. The *convex hull* of a subset $D$ of a real Banach space $X$ is given by

$$
\operatorname{conv}(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1,\ n \in \mathbb{N} \right\}. \tag{2.7}
$$

The following theorem is due to Dugundji and a proof can be found in [14, page 44].

Theorem 2.5. *For Banach spaces $X$ and $Y$, let $D \subset X$ be closed and let*

$$
F : D \longrightarrow Y \tag{2.8}
$$

*be continuous. Then $F$ has a continuous extension*

$$
\widetilde{F} : X \longrightarrow Y \tag{2.9}
$$

*such that*

$$
\widetilde{F}(X) \subset \overline{\operatorname{conv}(F(D))}. \tag{2.10}
$$

Corollary 2.6. *Every closed convex set of a Banach space is a retract of the Banach space.*

Note that for any positive real number $r$ and nonnegative continuous concave functional $\alpha$, $Q(\alpha,r)$ is a retract of $E$ by Corollary 2.6. Note also, if $r$ is a positive number and if $\alpha : P \to [0,\infty)$ is a uniformly continuous convex functional with $\alpha(0) = 0$ and $\alpha(x) > 0$ for $x \neq 0$, then [4, Theorem 2.1] guarantees that $Q(\alpha,r)$ is a retract of $E$.

## 3. Fixed point index

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [15, pages 82–86]; an elementary proof can be found in [14, pages 58–238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 3.1. *Let $X$ be a retract of a real Banach space $E$. Then, for every bounded relatively open subset $U$ of $X$ and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on $\partial U$ (relative to $X$), there exists an integer $i(A,U,X)$ satisfying the following conditions:*

(G1) *normality: $i(A,U,X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*

(G2) *additivity: $i(A,U,X) = i(A,U_1,X) + i(A,U_2,X)$ whenever $U_1$ and $U_2$ are disjoint open subsets of $U$ such that $A$ has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*

(G3) *homotopy invariance:* $i(H(t,\cdot),U,X)$ *is independent of* $t \in [0,1]$ *whenever* $H : [0,1] \times \overline{U} \to X$ *is completely continuous and* $H(t,x) \neq x$ *for any* $(t,x) \in [0,1] \times \partial U$;

(G4) *permanence:* $i(A,U,X) = i(A,U \cap Y,Y)$ *if* $Y$ *is a retract of* $X$ *and* $A(\overline{U}) \subset Y$;

(G5) *excision:* $i(A,U,X) = i(A,U_0,X)$ *whenever* $U_0$ *is an open subset of* $U$ *such that* $A$ *has no fixed points in* $\overline{U} - U_0$;

(G6) *solution: if* $i(A,U,X) \neq 0$, *then* $A$ *has at least one fixed point in* $U$.

*Moreover,* $i(A,U,X)$ *is uniquely defined.*

## 4. Main result

Theorem 4.1. *Suppose that* $P$ *is a cone in a real Banach space* $E$, $\alpha$, *and* $\psi$ *are nonnegative continuous concave functionals on* $P$, $\beta$ *is a nonnegative continuous convex functional on* $P$, *and there exist nonnegative numbers* $r$, $\tau$, *and* $R$ *such that*

$$A : Q(\alpha,\beta,r,R) \longrightarrow P \tag{4.1}$$

*is a completely continuous operator and* $Q(\alpha,\beta,r,R)$ *is a bounded set. If*

(1) $\{x \in Q(\alpha,\psi,\beta,r,\tau,R) : \beta(x) < R\} \neq \emptyset$ *and* $\beta(Ax) < R$ *for all* $x \in Q(\alpha,\psi,\beta,r,\tau,R)$;

(2) $\alpha(Ax) \geq r$ *for all* $x \in Q(\alpha,\beta,r,R)$;

(3) $\beta(Ax) < R$ *for all* $x \in Q(\alpha,\beta,r,R)$ *with* $\psi(Ax) < \tau$,

*then* $A$ *has a fixed point* $x$ *in* $Q(\alpha,\beta,r,R)$.

*Proof.* Let

$$U = \{x \in Q(\alpha,\beta,r,R) : \beta(x) < R\}, \tag{4.2}$$

then $U$ is the interior of $Q(\alpha,\beta,r,R)$ in $Q(\alpha,r)$ and we have assumed that $U$ is a bounded set.

*Claim 1.* $Ax \neq x$ for all $x \in \partial U$.

Suppose the opposite, that is, there is an $x_0 \in \partial U$ such that $Ax_0 = x_0$. Since $x_0 \in \partial U$, we have that $\beta(x_0) = R$. Either $\psi(x_0) < \tau$ or $\psi(x_0) \geq \tau$. If $\psi(x_0) < \tau$, then $\psi(Ax_0) = \psi(x_0) < \tau$ which implies by condition (3) that $\beta(x_0) = \beta(Ax_0) < R$ which is a contradiction. If $\psi(x_0) \geq \tau$, then $x_0 \in Q(\alpha,\psi,\beta,r,\tau,R)$ and by condition (1) we have that $\beta(x_0) = \beta(Ax_0) < R$ which is a contradiction. Therefore, $Ax \neq x$ for all $x \in \partial U$.

Let $x^* \in Q(\alpha,\psi,\beta,r,\tau,R)$ with $\beta(x^*) < R$ (see condition (1)) and let (see condition (2))

$$H : [0,1] \times \overline{U} \longrightarrow Q(\alpha,r) \tag{4.3}$$

be defined by

$$H(t,x) = (1-t)Ax + tx^*. \tag{4.4}$$

Clearly, $H$ is continuous and the image of $[0,1]\times\overline{U}$ is relatively compact.

*Claim 2.* $H(t,x) \neq x$ for all $(t,x) \in [0,1]\times\partial U$.

Suppose the opposite, that is, there exists $(t_1,x_1) \in [0,1]\times\partial U$ such that $H(t_1,x_1) = x_1$. Since $x_1 \in \partial U$, we have that $\beta(x_1) = R$. Either $\psi(Ax_1) < \tau$ or $\psi(Ax_1) \geq \tau$.

*Case 1.* $\psi(Ax_1) < \tau$. By condition (3), we have

$$\beta(x_1) = \beta((1-t_1)Ax_1 + t_1x^*) \leq (1-t_1)\beta(Ax_1) + t_1\beta(x^*) < R, \tag{4.5}$$

which is a contradiction.

*Case 2.* $\psi(Ax_1) \geq \tau$. We have that $x_1 \in Q(\alpha,\psi,\beta,r,\tau,R)$ since

$$\psi(x_1) = \psi((1-t_1)Ax_1 + t_1x^*) \geq (1-t_1)\psi(Ax_1) + t_1\psi(x^*) \geq \tau, \tag{4.6}$$

and thus by condition (1), we have

$$\beta(x_1) = \beta((1-t_1)Ax_1 + t_1x^*) \leq (1-t_1)\beta(Ax_1) + t_1\beta(x^*) < R, \tag{4.7}$$

which is a contradiction.

Therefore, we have shown that $H(t,x) \neq x$ for all $(t,x) \in [0,1]\times\partial U$ and thus by the homotopy invariance property (G3) of the fixed point index

$$i(A,U,Q(\alpha,r)) = i(x^*,U,Q(\alpha,r)), \tag{4.8}$$

and by the normality property (G1) of the fixed point index

$$i(A,U,Q(\alpha,r)) = i(x^*,U,Q(\alpha,r)) = 1, \tag{4.9}$$

therefore by the solution property (G6) of the fixed point index, the operator $A$ has a fixed point $x \in U$. □

The argument in the proof of Theorem 4.1 immediately guarantees the following generalization.

Theorem 4.2. *Suppose that $P$ is a cone in a real Banach space $E$, $\alpha$ is a nonnegative continuous functional on $P$, $\psi$ is a nonnegative continuous concave functionals on $P$, $\beta$ is a nonnegative continuous convex functional on $P$, and there exist nonnegative numbers $r$, $\tau$, and $R$ such that*

$$A : Q(\alpha,\beta,r,R) \longrightarrow P \tag{4.10}$$

*is a completely continuous operator and $Q(\alpha,\beta,r,R)$ is a bounded set. Also assume $Q(\alpha,r)$ is a retract of $E$ and suppose (1), (2), and (3) in Theorem 4.1 hold. In addition, assume the following is satisfied:*

(4) *there exists* $x^* \in Q(\alpha,\psi,\beta,r,\tau,R)$ *with* $\beta(x^*) < R$ *such that the map* $H$ *given by* $H(t,x) = (1-t)Ax + tx^*$ *maps* $[0,1] \times \{x \in Q(\alpha,\beta,r,R) : \beta(x) \le R\}$ *into* $Q(\alpha,r)$.

*Then* $A$ *has a fixed point* $x$ *in* $Q(\alpha,\beta,r,R)$.

## 5. Multivalued generalization

In this section, we provide some background material from fixed point theory related to multivalued maps.

Let $X$ be a closed, convex subset of some Banach space $E = (E,\|\cdot\|)$. Suppose for every open subset $U$ of $X$ and every upper semicontinuous map $A : \overline{U^X} \to 2^X$ (here $2^X$ denotes the family of nonempty subsets of $X$) which satisfies property (B) (to be specified later) with $x \notin Ax$ for $x \in \partial_X U$ (here $\overline{U^X}$ and $\partial_X U$ denote the closure and boundary of $U$ in $X$, resp.), there exists an integer, denoted by $i_X(A,U)$, satisfying the following properties.

(P1) If $x_0 \in U$, then $i_X(\hat{x}_0,U) = 1$ (here $\hat{x}_0$ denotes the map whose constant value is $x_0$).

(P2) For every pair of disjoint open subsets $U_1$, $U_2$ of $U$ such that $A$ has no fixed points on $\overline{U^X} \backslash (U_1 \cup U_2)$,

$$i_X(A,U) = i_X(A,U_1) + i_X(A,U_2). \tag{5.1}$$

(P3) For every upper semicontinuous map $H : [0,1] \times \overline{U^X} \to 2^X$ which satisfies property (B) and $x \notin H(t,x)$ for $(t,x) \in [0,1] \times \partial_X U$,

$$i_X(H(1,\cdot),U) = i_X(H(0,\cdot),U). \tag{5.2}$$

(P4) If $Y$ is a closed convex subset of $X$ and $A(\overline{U^X}) \subseteq Y$, then

$$i_X(A,U) = i_Y(A,U \cap Y). \tag{5.3}$$

Also assume the family

$$\begin{aligned} i_X(A,U) : & X \text{ a closed, convex subset of a Banach space } E, \quad U \text{ open in } X, \\ & \text{and } A : \overline{U^X} \longrightarrow 2^X \text{ is an upper semicontinuous map} \\ & \text{that satisfies property (B) with } x \notin Ax \text{ on } \partial_X U \end{aligned} \tag{5.4}$$

is uniquely determined by the properties (P1)–(P4).

We note that property (B) is any property on the map so that the fixed point index is well defined. Usually in application property, (B) will mean that the map is compact with convex compact values. Other examples of maps with a well-defined fixed point index (e.g., property (B) could mean that the map is countably condensing with convex compact values) can be found in the literature.

If the above holds, notice also that

(P5) for every open subset $V$ of $U$ such that $A$ has no fixed points on $\overline{U^X} \backslash V$,

$$i_X(A,U) = i_X(A,V); \tag{5.5}$$

(P6) if $i_X(A,U) \neq 0$, then $A$ has at least one fixed point in $U$.

The proof of the following generalization of Theorem 4.1 to multivalued maps is essentially the same as the proof of Theorem 4.1 following the techniques applied in [7] and is therefore omitted.

Theorem 5.1. *Let $E = (E, \|\cdot\|)$ be a Banach space and $X$ a closed, convex subset of $E$. Suppose for every open subset $U$ of $X$ and every upper semicontinuous map $A : \overline{U^X} \to 2^X$ which satisfies property (B) with $x \notin Ax$ for $x \in \partial_X U$, there exists an integer $i_X(A,U)$ satisfying (P1)–(P4). In addition, assume the family*

$$\begin{aligned} i_X(A,U) :& X \text{ a closed, convex subset of a Banach space } E, \quad U \text{ open in } X, \\ & \text{and } A : \overline{U^X} \longrightarrow 2^X \text{ is an upper semicontinuous map} \\ & \text{that satisfies property (B) with } x \notin Ax \text{ on } \partial_X U \end{aligned} \tag{5.6}$$

*is uniquely determined by the properties (P1)–(P4). Let $P \subset E$ be a cone in $E$ and suppose there exist nonnegative, continuous, concave functionals $\alpha$ and $\psi$ on $P$, and a nonnegative, continuous, convex functional $\beta$ on $P$ and there exist nonnegative numbers $r$, $\tau$, and $R$ such that $Q(\alpha,\beta,r,R)$ is a bounded set. Furthermore, suppose*

$$F : Q(\alpha,\beta,r,R) \longrightarrow 2^P \tag{5.7}$$

*is an upper semicontinuous map which satisfies property (B) such that the following properties are satisfied:*

(H1) *$\{x \in Q(\alpha,\psi,\beta,r,\tau,R) : \beta(x) < R\} \neq \emptyset$ and if $x \in Q(\alpha,\psi,\beta,r,\tau,R)$, then $\beta(y) < R$ for all $y \in Fx$;*

(H2) *if $x \in Q(\alpha,\beta,r,R)$ with $\psi(y) < \tau$ for some $y \in Fx$, then $\beta(y) < R$;*

(H3) *if $x \in Q(\alpha,\beta,r,R)$, then $\alpha(y) \geq r$ for all $y \in Fx$;*

(H4) *there exists $x^* \in \{x \in Q(\alpha,\psi,\beta,r,\tau,R) : \beta(x) < R\}$ such that the mapping $H : [0,1] \times \{x \in Q(\alpha,\beta,r,R) : \beta(x) \leq R\} \to 2^{Q(\alpha,r)}$, given by $H(t,x) = (1-t)Fx + tx^*$, satisfies property (B).*

*Then $F$ has at least one fixed point $x$ in $Q(\alpha,\beta,r,R)$.*

## 6. Application

The use of functionals provides researchers flexibility when establishing the existence of solutions to boundary value problems. A standard technique is to assume the nonlinearity is bounded by a constant (or some appropriate function) on intervals in order to verify certain inequalities, in which case, choosing the minimum of a function over an interval (concave functional) and the maximum of a function over an interval (convex functional) often simplify the arguments. An alternative inversion technique can be employed to simplify such arguments which benefits from the choice of alternative functionals.

Consider the second-order nonlinear focal boundary-value problem

$$\begin{gathered} y''(t) + f\big(y(t)\big) = 0, \quad t \in (0,1), \\ y(0) = 0 = y'(1), \end{gathered} \tag{6.1}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous, increasing, and concave. If $x$ is a fixed point of the operator $A$ defined by

$$Ax(t) := f\left(\int_0^1 G(t,s)x(s)ds\right), \tag{6.2}$$

where

$$G(t,s) = \begin{cases} t, & t \leq s, \\ s, & s \leq t, \end{cases} \tag{6.3}$$

is the Green's function for the operator $L$ defined by

$$Lx(t) := -x'', \tag{6.4}$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1), \tag{6.5}$$

then

$$y(t) = \int_0^1 G(t,s)x(s)ds \tag{6.6}$$

is a solution of (6.1). See [16] for a thorough treatment of this alternative inversion technique. Throughout this section of the paper, we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in $t$.

Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E : x \text{ is concave, nonnegative, and nondecreasing}\}; \tag{6.7}$$

then clearly $A : P \to P$ by the properties of Green's function and the properties of $f$. Define the functionals $\alpha$ and $\beta$ by

$$\begin{aligned} \alpha(x) : &= \min_{t \in [1/4,1]} \int_0^1 G(t,s)x(s)ds = \int_0^1 G\left(\frac{1}{4}, s\right)x(s)ds, \\ \beta(x) : &= \max_{t \in [0,1]} \int_{1/4}^1 G(t,s)x(s)ds = \int_{1/4}^1 G(1,s)x(s)ds. \end{aligned} \tag{6.8}$$

In the following theorem, using the standard technique of bounding the nonlinearity by constants, we show how to employ the alternative inversion technique.

Theorem 6.1. *Suppose there exist positive real numbers $r$ and $R$, with $0 < 103r/25 < R$, and a continuous, increasing, concave function $f : [r, 4R/3] \to [0, \infty)$, such that*

$$\frac{16r}{3} \leq f(x) < \frac{32R}{15} \quad \text{for } x \in \left[r, \frac{4R}{3}\right]. \tag{6.9}$$

*Then, the operator A has at least one positive solution $x^*$ such that*

$$r \le \alpha(x^*), \qquad \beta(x^*) \le R. \tag{6.10}$$

*Moreover, this implies that the boundary value problem (6.1) has at least one positive solution $y^*$ such that*

$$y^*(t) = \int_0^1 G(t,s)x^*(s)ds \tag{6.11}$$

*with*

$$r \le y^*\left(\frac{1}{4}\right), \qquad y^*(1) \le \frac{4R}{3}. \tag{6.12}$$

*Proof.* Let $\psi = \alpha$ and $\tau = r$. Thus condition (3) of Theorem 4.1 will be satisfied once we have verified condition (2) of Theorem 4.1. The set $Q(\alpha,\beta,r,R)$ is bounded. To see this, let $x \in Q(\alpha,\beta,r,R)$. Then

$$\beta(x) = \int_{1/4}^1 G(1,s)x(s)ds \ge \left(\frac{1}{4}\right)\int_{1/4}^1 x(s)ds, \tag{6.13}$$

and by the concavity of $x$ with a standard calculus area argument, we have

$$\int_{1/4}^1 x(s)ds \ge \frac{3\big(x(1)+x(1/4)\big)}{8} \ge \frac{3x(1)}{8}, \tag{6.14}$$

and hence

$$\frac{32\beta(x)}{3} \ge x(1), \tag{6.15}$$

or

$$\|x\| \le \frac{32R}{3}. \tag{6.16}$$

Also, it can easily be shown that $r+R \in \{x \in Q(\alpha,\psi,\beta,r,\tau,R) : \beta(x) < R\}$, since we have $0 < 103r/25 < R$, and hence the set is nonempty.

*Claim 3.* $\beta(Ax) < R$ for all $x \in Q(\alpha,\psi,\beta,r,\tau,R)$.

For $s \in [1/4,1]$ and $x \in Q(\alpha,\psi,\beta,r,\tau,R)$, we have

$$\begin{gathered} r \le \alpha(x) = \int_0^1 G\left(\frac{1}{4},w\right)x(w)dw \le \int_0^1 G(s,w)x(w)dw, \\ \int_0^1 G(s,w)x(w)dw \le \int_0^1 G(1,w)x(w)dw \le \left(\frac{4}{3}\right)\int_{1/4}^1 G(1,w)x(w)dw \le \frac{4R}{3}, \end{gathered} \tag{6.17}$$

thus if $x \in Q(\alpha,\psi,\beta,r,\tau,R)$, then

$$\beta(Ax) = \int_{1/4}^1 G(1,s)f\left(\int_0^1 G(s,w)x(w)dw\right)ds < \int_{1/4}^1 G(1,s)\left(\frac{32R}{15}\right)ds = R. \tag{6.18}$$

*Claim 4.* $\alpha(Ax) \geq r$ for all $x \in Q(\alpha,\beta,r,R)$.

If $x \in Q(\alpha,\beta,r,R)$, then

$$\begin{aligned}\alpha(Ax) &= \int_0^1 G\left(\frac{1}{4},s\right) f\left(\int_0^1 G(s,w)x(w)dw\right) ds \\ &\geq \int_{1/4}^1 G\left(\frac{1}{4},s\right) f\left(\int_0^1 G(s,w)x(w)dw\right) ds \\ &\geq \int_{1/4}^1 G\left(\frac{1}{4},s\right)\left(\frac{16r}{3}\right) ds = r\end{aligned} \tag{6.19}$$

for the same reasons in Claim 3.

Therefore, the hypotheses of Theorem 4.1 have been satisfied; thus the operator $A$ has at least one positive solution $x^*$ such that

$$r \leq \alpha(x^*), \qquad \beta(x^*) \leq R. \tag{6.20}$$

□

## References


[1] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.

[2] D. J. Guo, "A new fixed-point theorem," *Acta Mathematica Sinica*, vol. 24, no. 3, pp. 444–450, 1981.

[3] D. R. Anderson and R. I. Avery, "Fixed point theorem of cone expansion and compression of functional type," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1073–1083, 2002.

[4] G. Zhang and J. Sun, "A generalization of the cone expansion and compression fixed point theorem and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 2, pp. 579–586, 2007.

[5] D. J. Guo, "Some fixed point theorems on cone maps," *Kexue Tongbao*, vol. 29, no. 5, pp. 575–578, 1984.

[6] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

[7] R. P. Agarwal, R. I. Avery, J. Henderson, and D. O'Regan, "The five functionals fixed point theorem generalized to multivalued maps," *Journal of Nonlinear and Convex Analysis*, vol. 4, no. 3, pp. 455–462, 2003.

[8] R. I. Avery, "A generalization of the Leggett-Williams fixed point theorem," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 7, pp. 9–14, 1999.

[9] R. I. Avery and J. Henderson, "An extension of the five functionals fixed point theorem," *International Journal of Differential Equations and Applications*, vol. 1, no. 3, pp. 275–290, 2000.

[10] R. I. Avery and J. Henderson, "Two positive fixed points of nonlinear operators on ordered Banach spaces," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 27–36, 2001.

[11] R. P. Agarwal and D. O'Regan, "A generalization of the Petryshyn-Leggett-Williams fixed point theorem with applications to integral inclusions," *Applied Mathematics and Computation*, vol. 123, no. 2, pp. 263–274, 2001.

[12] D. O'Regan, "Integral inclusions of upper semi-continuous or lower semi-continuous type," *Proceedings of the American Mathematical Society*, vol. 124, no. 8, pp. 2391–2399, 1996.

[13] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, vol. 41 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2002.

[14] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.
[15] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.
[16] R. I. Avery and A. C. Peterson, "Multiple positive solutions of a discrete second order conjugate problem," *PanAmerican Mathematical Journal*, vol. 8, no. 3, pp. 1–12, 1998.

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