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Research Article

An Implicit Iterative Scheme for an Infinite Countable Family of Asymptotically Nonexpansive Mappings in Banach Spaces

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Recommended by William Kirk

Let K be a nonempty closed convex subset of a reflexive Banach space E with a weakly continuous dual mapping, and let $\{T_i\}_{i=1}^{\infty}$ be an infinite countable family of asymptotically nonexpansive mappings with the sequence $\{k_{in}\}$ satisfying $k_{in} \geq 1$ for each $i = 1,2,\ldots$, $n = 1,2,\ldots$, and $\lim_{n\to\infty} k_{in} = 1$ for each $i = 1,2,\ldots$. In this paper, we introduce a new implicit iterative scheme generated by $\{T_i\}_{i=1}^{\infty}$ and prove that the scheme converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$, which solves some certain variational inequality.

1. Introduction and preliminaries

Let E be a Banach space and let K be a nonempty closed convex subset of E. Let $T : K\to K$ be a mapping. Then T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$. T is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ that converges to 1 as $n\to\infty$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and all $n \geq 1$. Obviously, a nonexpansive mapping is asymptotically nonexpansive. In [1], Goebel and Kirk originally introduced the concept of asymptotically nonexpansive mappings and proved that if E is a uniformly convex Banach space and K is a nonempty closed convex bounded subset of E, then every asymptotically nonexpansive self-mapping on K has a fixed point. After that, many authors began to study the convergence of the iterative scheme generated by asymptotically nonexpansive mappings [2–12].

In [8], the authors introduced an iterative scheme generated by a finite family of asymptotically nonexpansive mappings:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{r_n}^{l_n+1} x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{T_i\}_{i=1}^N : K \to K$ are N asymptotically nonexpansive mappings, where K is a nonempty closed convex subset of a uniformly convex Banach space satisfying Opial's condition [13], and where $n = l_n N + r_n$ for some integers $l_n \geq 0$ and $1 \leq r_n \leq N$. Then the authors proved that if $\cap_{i=1}^N F(T_i) \neq \phi$, then $\{x_n\}$ generated by (1.3) strongly converges to a common fixed point of $\{T_i\}_{i=1}^N$.

Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S : K \to K$ be a nonexpansive mapping and let $T : K \to K$ be an asymptotically nonexpansive mapping. In [10], the authors introduced the following modified Ishikawa iteration sequence with errors with respect to S and T:

$$\begin{aligned} y_n &= a_n' S x_n + b_n' T^n x_n + c_n' v_n, \\ x_{n+1} &= a_n S x_n + b_n T^n y_n + c_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$ are three real numbers sequences in $(0,1)$ satisfying $a_n' + b_n' + c_n' = 1$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are also three real numbers sequences in $(0,1)$ satisfying $a_n + b_n + c_n = 1$, and $\{u_n\}$ and $\{v_n\}$ are given bounded sequences in K. Then the authors proved that the sequence $\{x_n\}$ generated by (1.4) strongly converges to a common fixed point of S and T if some certain conditions are satisfied.

Let K be a nonempty closed convex subset of a Banach space E and let $f : K \to K$ be a contraction with efficient λ $(0 < \lambda < 1)$ such that

$$\|f(x) - f(y)\| \leq \lambda \|x - y\| \tag{1.5}$$

for all $x, y \in K$. Shahzad and Udomene [9] studied the following implicit and explicit iterative schemes for an asymptotically nonexpansive mapping T with the sequence $\{k_n\}$ in a uniformly smooth Banach space:

$$\begin{aligned} x_n &= \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n, \\ x_{n+1} &= \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n, \end{aligned} \tag{1.6}$$

where $\{t_n\}$ is a sequence in $(0,1)$. They proved that the sequence $\{x_n\}$ converges strongly to the unique solution of some variational inequality if the sequence $\{t_n\}$ satisfies some certain conditions and the mapping T satisfies $\|Tx_n - x_n\| \to 0$ as $n \to \infty$.

Quite recently, Ceng et al. [12] introduced the following two implicit and explicit iterative schemes generated by a finite family of asymptotically nonexpansive mappings

$\{T_i\}_{i=1}^N$ with the same sequence $\{k_n\}$ in a reflexive Banach space with a weakly continuous duality map:

$$x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{1-t_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n,$$
$$x_{n+1} = \left(1 - \frac{1}{k_n}\right)x_n + \frac{1-t_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n, \tag{1.7}$$

where $r_n = n \bmod N$ and $\{t_n\}$ is a sequence in $[0,1]$. Then they proved that if the control sequence $\{t_n\}$ satisfies some certain condition and $T_i x_n - x_n \to 0$ as $n \to \infty$ for each $i = 1, 2, \ldots, N$, then both schemes (1.7) strongly converge a common fixed point x^* of $\{T_i\}_{i=1}^N$ which solves the variational inequality

$$\langle (I - f)x^*, J(p - x^*)\rangle \geq 0, \quad p \in \bigcap_{i=1}^N F(T_i), \tag{1.8}$$

where $F(T_i)$ denotes the set of fixed points of the mapping T_i for each $i = 1, 2, \ldots, N$.

Let E be a Banach space and let E^* be the dual space of E. Given a continuous strictly increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$, we associate a (possibly multivalued) generalized duality map $J_\varphi : E \to 2^{E^*}$, defined as

$$J_\varphi(x) = \{x^* \in E^* : x^*(x) = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\} \tag{1.9}$$

for every $x \in E$. We call the function φ a gauge. If $\varphi(t) = t$ for all $t \geq 0$, then we call J_φ a normalized duality mapping and write it as J.

A Banach space E is said to have a weakly continuous generalized duality map if there exists a continuous strictly increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$, and J_φ is single valued and sequentially continuous from E with the weak topology to E^* with the weak* topology. For instance, every l^p-space $(1 < p < \infty)$ has a weakly continuous generalized duality map for $\varphi(t) = t^{p-1}$.

For each $t \geq 0$, let $\Phi(t) = \int_0^t \varphi(x)\mathrm{d}x$. The following property may be seen in many literatures.

Property 1.1. Let E be a real Banach space and let J_φ be the duality map associated with the gauge φ. Then for all $x, y \in E$ and $j(x+y) \in J_\varphi(x+y)$ one holds

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j(x+y)\rangle. \tag{1.10}$$

One also holds

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{1.11}$$

for all $x, y \in E$ and $j(x+y) \in J(x+y)$.

Lemma 1.2 (see [14]). *Let E be a Banach space satisfying a weakly continuous duality map and let K be a nonempty closed convex subset of E. Let $T : K \to K$ be an asymptotically nonexpansive mapping with fixed point. Then $I - T$ is demiclosed at zero.*

2. Strong convergence results

In this section, let E be a reflexive Banach space with a weakly continuous duality map J_φ, where φ is a gauge and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty} : K \to K$ be an infinite countable family of asymptotically nonexpansive mappings such that

$$\left\| T_i^n x - T_i^n y \right\| \le k_{in} \|x - y\| \tag{2.1}$$

for all $x, y \in K$, where the sequence $\{k_{in}\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_{in} = 1$ for each $i = 1, 2, \ldots$.

For each $n = 1, 2, \ldots$, let $b'_n = \sup\{k_{in} \mid i = 1, 2, \ldots\}$ and assume

$$\begin{gathered} \sup \left\{ b'_n \mid n = 1, 2, \ldots \right\} < \infty, \\ \lim_{n\to\infty} b'_n = b < \infty. \end{gathered} \tag{2.2}$$

Taking $b_n = \max\{b'_n, b\}$ for each $n = 1, 2, \ldots$, obviously, we have

$$\begin{gathered} \lim_{n\to\infty} b_n = b \ge 1, \\ b' = \sup \left\{ b_n \mid n = 1, 2, \ldots \right\} < \infty. \end{gathered} \tag{2.3}$$

Moreover, the following inequality

$$\left\| T_i^n x - T_i^n y \right\| \le b_n \|x - y\| \tag{2.4}$$

holds for all $x, y \in K$ and each $i = 1, 2 \ldots$.

Take an integer $r > 1$ arbitrarily. For each $n \ge 1$, define the mapping $S_{ni} : K \to K$ by

$$S_{ni} = T_{(n-1)r+i} \tag{2.5}$$

for each $i = 1, 2, \ldots, r$, that is,

$$S_{11} = T_1, \ldots, S_{1r} = T_r, S_{21} = T_{r+1}, \ldots, S_{2r} = T_{2r}, \ldots. \tag{2.6}$$

For each $i = 1, 2, \ldots, r$, let $\{\alpha_{ni}\} \subset (0,1)$ be a sequence real numbers. For each $n \geq 1$, define the mapping W_n of K into itself by

$$W_n = U_{nr} = \alpha_{nr} S_{nr}^n U_{nr-1} + (1 - \alpha_{nr}) I, \tag{2.7}$$

where

$$\begin{aligned} U_{n1} &= \alpha_{n1} S_{n1}^n + (1 - \alpha_{n1}) I, \\ U_{n2} &= \alpha_{n2} S_{n2}^n U_{n1} + (1 - \alpha_{n2}) I, \\ &\vdots \\ U_{nr-1} &= \alpha_{nr-1} S_{nr-1}^n U_{nr-2} + (1 - \alpha_{nr-1}) I. \end{aligned} \tag{2.8}$$

We call W_n a W-mapping generated by $S_{n1}, S_{n2}, \ldots, S_{nr}$ and $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nr.}$

Let $f : K \to K$ be a λ-contraction with $0 < \lambda < 1/b'^{r}$. Take a sequence of real numbers$\{t_n\} \subset [0, b]$ such that

$$\lim_{n\to\infty} t_n = 0, \quad t_n < \frac{b(1 - b_n^r \lambda)}{(1 - \lambda) b_n^r}, \quad n \geq 1. \tag{2.9}$$

Note that since $\lambda < 1/b'^{r}$, one has $0 < b(1 - b_n^r \lambda)/(1 - \lambda) b_n^r \leq b$. Therefore, the sequence $\{t_n\}$ can be taken easily to satisfy the condition (2.9), for example, $t_n = (1/n)(b(1 - b_n^r \lambda)/(1 - \lambda) b_n^r)$.

Then, we introduce an implicit iterative scheme

$$x_n = \left(1 - \frac{b}{b_n^{r+1}}\right) x_n + \frac{b - t_n}{b_n^{r+1}} f(W_n x_n) + \frac{t_n}{b_n^{r+1}} W_n x_n, \quad n \geq 1. \tag{2.10}$$

By using the following lemmas, we will prove that the implicit scheme (2.10) is well defined.

Lemma 2.1. *Let $\{T_i\}_{i=1}^{\infty} : K \to K$ be an infinite countable family of asymptotically nonexpansive mappings with the sequences $\{k_{in}\}$ and let W_n be a W-mapping generated by* (2.7) *for each $n = 1, 2, \ldots$. If $\cap_{i=1}^{\infty} F(T_i) \neq \phi$, then $\cap_{i=1}^{\infty} F(T_i) \subset F(W_n)$ for each $n = 1, 2, \ldots$.*

Proof. The conclusion is obtained directly from the definition of W_n. □

Lemma 2.2. *Let $\{T_i\}_{i=1}^{\infty} : K \to K$ with the sequences $\{k_{in}\}$ and let W_n be the W-mapping generated by* (2.7) *for each $n = 1, 2, \ldots$. Then one holds*

$$\|W_n x - W_n y\| \leq b_n^r \|x - y\| \tag{2.11}$$

for all $n \geq 1$ and all $x, y \in K$.

Proof. For any $x, y \in K$ all $n \geq 1$, we first see (noting that $b_n \geq 1$)

$$
\begin{aligned}
\|U_{n1}x - U_{n1}y\| &= \left\|(\alpha_{n1}S_{n1}^n + (1-\alpha_{n1})I)x - (\alpha_{n1}S_{n1}^n + (1-\alpha_{n1})I)y\right\| \\
&\leq \alpha_{n1}\left\|S_{n1}^n x - S_{n1}^n y\right\| + (1-\alpha_{n1})\|x-y\| \\
&= \alpha_{n1}\left\|T_{(n-1)r+1}^n x - T_{(n-1)r+1}^n y\right\| + (1-\alpha_{n1})\|x-y\| \\
&\leq \alpha_{n1}k_{(n-1)r+1n}\|x-y\| + (1-\alpha_{n1})\|x-y\| \\
&\leq \alpha_{n1}b_n\|x-y\| + (1-\alpha_{n1})\|x-y\| \\
&\leq \alpha_{n1}b_n\|x-y\| + (1-\alpha_{n1})b_n\|x-y\| \\
&= b_n\|x-y\|, \\
\|U_{n2}x - U_{n2}y\| &= \left\|(\alpha_{n2}S_{n2}^n U_{n1} + (1-\alpha_{n2})I)x - (\alpha_{n2}S_{n2}^n U_{n1} + (1-\alpha_{n2})I)y\right\| \\
&\leq \alpha_{n2}\left\|S_{n2}^n U_{n1}x - S_{n2}^n U_{n1}y\right\| + (1-\alpha_{n2})\|x-y\| \\
&= \alpha_{n2}\left\|T_{(n-1)r+2}^n U_{n1}x - T_{(n-1)r+2}^n U_{n1}y\right\| + (1-\alpha_{n2})\|x-y\| \\
&\leq \alpha_{n2}k_{(n-1)r+2n}\left\|U_{n1}x - U_{n1}y\right\| + (1-\alpha_{n2})\|x-y\| \\
&\leq \alpha_{n2}b_n\left\|U_{n1}x - U_{n1}y\right\| + (1-\alpha_{n2})\|x-y\| \\
&\leq \alpha_{n2}b_n^2\|x-y\| + (1-\alpha_{n1})b_n^2\|x-y\| \\
&= b_n^2\|x-y\|.
\end{aligned}
\tag{2.12}
$$

Similarly, for each $i = 3, \ldots, r-1$, we have

$$
\|U_{ni}x - U_{ni}y\| \leq b_n^i\|x-y\|. \tag{2.13}
$$

Hence,

$$
\begin{aligned}
\|W_n x - W_n y\| &= \left\|(\alpha_{nr}S_{nr}^n U_{n\,r-1} + (1-\alpha_{nr})I)x - (\alpha_{nr}S_{nr}^n U_{n\,r-1} + (1-\alpha_{nr})I)y\right\| \\
&\leq \alpha_{nr}\left\|S_{nr}^n U_{n\,r-1}x - S_{nr}^n U_{n\,r-1}y\right\| + (1-\alpha_{nr})\|x-y\| \\
&\leq b_n^r\|x-y\|.
\end{aligned}
\tag{2.14}
$$

This completes the proof. □

Now we prove that the implicit scheme (2.10) is well defined. Since $0 < t_n < b(1 - b_n^r\lambda)/(1-\lambda)b_n^r$, we obtain

$$
0 < 1 - \frac{b}{b_n^{r+1}} + \frac{b - t_n}{b_n}\lambda + \frac{t_n}{b_n} < 1. \tag{2.15}
$$

Hence, the mapping

$$
x \mapsto Tx : \left(1 - \frac{b}{b_n^{r+1}}\right)x + \frac{b - t_n}{b_n^{r+1}} f(W_n x) + \frac{t_n}{b_n^{r+1}} W_n x \tag{2.16}
$$

is a contraction on K. In fact, to see this, taking any $x, y \in K$, by Lemma 2.2 we have

$$\begin{aligned}
\|Tx - Ty\| &= \left\| \left(1 - \frac{b}{b_n^{r+1}}\right)(x-y) + \frac{b-t_n}{b_n^{r+1}}\left(f(W_n x) - f(W_n y)\right) + \frac{t_n}{b_n^{r+1}}\left(W_n x - W_n y\right) \right\| \\
&\le \left(1 - \frac{b}{b_n^{r+1}}\right)\|x-y\| + \frac{(b-t_n)\lambda b_n^r}{b_n^{r+1}}\|x-y\| + \frac{t_n}{b_n^{r+1}} b_n^r \|x-y\| \\
&= \left(1 - \frac{b}{b_n^{r+1}} + \frac{b-t_n}{b_n}\lambda + \frac{t_n}{b_n}\right)\|x-y\| \\
&\le \|x-y\|,
\end{aligned} \tag{2.17}$$

which implies that the implicit scheme (2.10) is well defined.

For the implicit scheme (2.10), we have strong convergence as follows.

Theorem 2.3. *Assume* (2.9), $F(T) = \cap_{i=1}^{\infty} F(T_i) \ne \phi$ *and* $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ *for each* $i = 1, 2, \ldots$. *Then* $\{x_n\}$ *converges strongly to a common fixed point* $x \in F(T)$, *where* x *solves the variational inequality*

$$\left\langle (I-f)x, J(p-x) \right\rangle \ge 0, \quad p \in F(T). \tag{2.18}$$

Proof. First, we prove that $\{x_n\}$ is bounded. By using Property 1.1, Lemmas 2.1, 2.2, for any $z \in F(T)$, we have (noting $0 < 1 - b/b_n^{r+1} + ((b-t_n)/b_n)\lambda + t_n/b_n < 1$)

$$\begin{aligned}
\|x_n - z\|^2 &= \left\| \left(1 - \frac{b}{b_n^{r+1}}\right)(x_n - z) + \frac{b-t_n}{b_n^{r+1}}\left(f(W_n x_n) - f(z)\right) + \frac{t_n}{b_n^{r+1}}\left(W_n x_n - z\right) \right. \\
&\qquad \left. + \frac{b-t_n}{b_n^{r+1}}\left(f(z) - z\right) \right\|^2 \\
&\le \left\| \left(1 - \frac{b}{b_n^{r+1}}\right)(x_n - z) + \frac{b-t_n}{b_n^{r+1}}\left(f(W_n x_n) - f(z)\right) + \frac{t_n}{b_n^{r+1}}\left(W_n x_n - z\right) \right\|^2 \\
&\quad + \frac{2(b-t_n)}{b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle \\
&\le \left[\left(1 - \frac{b}{b_n^{r+1}}\right)\|x_n - z\| + \frac{b-t_n}{b_n^{r+1}}\left\|f(W_n x_n) - f(W_n z)\right\| + \frac{t_n}{b_n^{r+1}}\left\|W_n x_n - W_n z\right\| \right]^2 \\
&\quad + \frac{2(b-t_n)}{b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle \\
&\le \left(1 - \frac{b}{b_n^{r+1}} + \frac{(b-t_n)\lambda}{b_n} + \frac{t_n}{b_n}\right)^2 \left\|x_n - z\right\|^2 + \frac{2(b-t_n)}{b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle \\
&\le \left(1 - \frac{b}{b_n^{r+1}} + \frac{(b-t_n)\lambda}{b_n} + \frac{t_n}{b_n}\right) \left\|x_n - z\right\|^2 + \frac{2(b-t_n)}{b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle \\
&= (1-\eta_n)\left\|x_n - z\right\|^2 + \frac{2(b-t_n)}{b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle,
\end{aligned} \tag{2.19}$$

where

$$\eta_n = \frac{b}{b_n^{r+1}} - \frac{b-t_n}{b_n}\lambda - \frac{t_n}{b_n} > 0. \tag{2.20}$$

It follows from (2.19) that

$$\|x_n - z\|^2 \leq \frac{2(b-t_n)}{\eta_n b_n^{r+1}}\langle f(z) - z, j(x_n - z)\rangle. \tag{2.21}$$

Since $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} t_n = 0$, we have

$$\lim_{n\to\infty}\frac{b-t_n}{\eta_n b_n^{r+1}} = \frac{1}{1-\lambda b^r}. \tag{2.22}$$

Hence, $\{x_n\}$ is bounded.

Now we prove that $\{x_n\}$ strongly converges to a common fixed point $x \in F(T)$. To see this, we assume that x is a weak limit point of $\{x_n\}$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to x. Then by the assumption of the theorem and Lemma 1.2, we have $x \in F(T_i)$ for every $i = 1, 2, \ldots$. In (2.21), replacing x_n with x_{n_j} and z with x, respectively, and then taking the limit as $j\to\infty$, we obtain by the weak continuity of the duality map J

$$\lim_{j\to\infty}\|x_{n_j} - x\| = 0. \tag{2.23}$$

Therefore, $x_{n_j}\to x$. We further show that x solves the variational inequality

$$\langle (I-f)x, J(p-x)\rangle \geq 0, \quad p \in F(T). \tag{2.24}$$

To see this result, taking any $p \in F(T)$, then by using Property 1.1, Lemmas 2.1 and 2.2 we compute

$$\begin{aligned}
&\Phi(\|x_n - p\|)\\
&\quad= \Phi\left(\left\|\left(1-\frac{b}{b_n^{r+1}}\right)(x_n-p) + \frac{b-t_n}{b_n^{r+1}}(x_n-p) + \frac{t_n}{b_n^{r+1}}(W_nx_n - p) + \frac{b-t_n}{b_n^{r+1}}(f(W_nx_n) - x_n)\right\|\right)\\
&\quad\leq \Phi\left(\left\|\left(1-\frac{t_n}{b_n^{r+1}}\right)(x_n-p) + \frac{t_n}{b_n^{r+1}}(W_nx_n-p)\right\|\right) + \frac{b-t_n}{b_n^{r+1}}\langle f(W_nx_n) - x_n, J_\varphi(x_n-p)\rangle\\
&\quad\leq \left(1-\frac{t_n}{b_n^{r+1}} + t_n\right)\Phi(\|x_n-p\|) + \frac{b-t_n}{b_n^{r+1}}\langle f(W_nx_n) - x_n, J_\varphi(x_n-p)\rangle,
\end{aligned} \tag{2.25}$$

which implies that

$$\langle x_n - f(W_n x_n), J_\varphi(x_n - p)\rangle \le \frac{(b_n^{r+1} - 1)t_n}{b - t_n}\Phi(\|x_n - p\|). \tag{2.26}$$

Now in (2.26), replacing x_n with x_{n_j} and noting $\lim_{n\to\infty} b_n = b$ and $\lim_{n\to\infty} t_n = 0$, we obtain

$$\begin{aligned}\langle x - f(x), J_\varphi(x - p)\rangle &= \lim_{j\to\infty}\langle x_{n_j} - f(W_{n_j} x_{n_j}), J_\varphi(x_{n_j} - p)\rangle \\ &\le \limsup_{j\to\infty}\frac{(b_{n_j}^{r+1} - 1)t_{n_j}}{b - t_{n_j}}\Phi(\|x_{n_j} - p\|) = 0,\end{aligned} \tag{2.27}$$

which implies that x is a solution to (2.24).

Finally, we prove that the sequence $\{x_n\}$ strongly converges to x. It suffices to prove that the variational inequality (2.24) can have only one solution. To see this, assuming that both $u \in F(T)$ and $v \in F(T)$ are solutions to (2.24), we have

$$\begin{aligned}\langle (I - f)u, J(u - v)\rangle &\le 0, \\ \langle (I - f)v, J(v - u)\rangle &\le 0.\end{aligned} \tag{2.28}$$

Adding them yields

$$\langle (I - f)u - (I - f)v, J(u - v)\rangle \le 0. \tag{2.29}$$

However, since f is a λ-contraction, we have that

$$(1 - \lambda)\|u - v\|^2 \le \langle (I - f)u - (I - f)v, J(u - v)\rangle, \tag{2.30}$$

which implies that $u = v$. This completes the proof. □

Remark 2.4. In Theorem 2.3, the condition that $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for each $i = 1, 2, \ldots$ is necessary (see [9, 12]). This theorem shows that if for each $n = 1, 2, \ldots$, the supremum of the sequence $\{k_{in}\}$, that is, $\sup\{k_{in} \mid i = 1, 2, \ldots\}$, is finite and the limit of the sequence $\sup\{k_{in} \mid i = 1, 2, \ldots\}_{n=1}^{\infty}$ exists, then by choosing the contraction constant λ and the control sequence $\{t_n\}$ we can obtain the common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Corollary 2.5. *Let $\{T_i\}_{i=1}^{N} K \to K$ be a finite family of asymptotically nonexpansive mappings with the sequences $\{k_{in}\}$ and let W_n be a W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nN}$ for each $n = 1, 2, \ldots$. Let the sequence $\{t_n\} \subset [0, 1]$ and satisfy $t_n < (1 - k_n^N \lambda)/(1 - \lambda)k_n^N$ and $t_n \to 0$, where $k_n = \max\{k_{1n}, k_{2n}, \ldots, k_{Nn}\}$ for each $n = 1, 2, \ldots$. Assume that $k = \sup\{k_n \mid n = 1, 2, \ldots\} < \infty$. Let f be a contraction with $\lambda (0 < \lambda < 1/k^N)$. Consider the implicit iterative scheme*

$$x_n = \left(1 - \frac{1}{k_n^{N+1}}\right)x_n + \frac{1 - t_n}{k_n^{N+1}} f(W_n x_n) + \frac{t_n}{k_n^{N+1}} W_n x_n. \tag{2.31}$$

If $\{T_i\}_{i=1}^N$ *satisfy the condition* $\cap_{i=1}^N F(T_i) \neq \phi$ *and* $T_i x_n - x_n \to 0$ *as* $n \to \infty$ *for each* $i = 1, 2, \ldots, N$, *then* $\{x_n\}$ *converges strongly to a common fixed point* $x \in \cap_{i=1}^N F(T_i)$, *where* x *solves the variational inequality*

$$\langle (I - f)x, J(p - x) \rangle \geq 0, \quad p \in \bigcap_{i=1}^N F(T_i). \tag{2.32}$$

Proof. In Theorem 2.3, take $b_n = k_n$, $b = \lim_{n\to\infty} k_n = 1$, $b' = k$, and $r = N$. Then, this corollary can obtained directly from Theorem 2.3. □

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