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Research Article

Some Extensions of Banach's Contraction Principle in Complete Cone Metric Spaces

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In this paper we consider complete cone metric spaces. We generalize some definitions such as c-nonexpansive and (c,λ)-uniformly locally contractive functions f-closure, c-isometric in cone metric spaces, and certain fixed point theorems will be proved in those spaces. Among other results, we prove some interesting applications for the fixed point theorems in cone metric spaces.

1. Introduction

The study of fixed points of functions satisfying certain contractive conditions has been at the center of vigorous research activity, for example see [1–5] and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, see [6–10]. Recently, Huang and Zhang generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions [11]. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [12–15]. The aim of this paper is to generalize some definitions such as c-nonexpansive and (c,λ)-uniformly locally contractive functions in these spaces and by using these definitions, certain fixed point theorems will be proved.

Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geqslant 0$, and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

There are non-normal cones.

Example 1.1. Let $E = C^2_{\mathbb{R}}([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone $P = \{f \in E : f \geqslant 0\}$. For each $K \geqslant 1$, put $f(x) = x$ and $g(x) = x^{2K}$. Then, $0 \leqslant g \leqslant f$, $\|f\| = 2$, and $\|g\| = 2K + 1$. Since $K\|f\| < \|g\|$, K is not normal constant of P [16].

In the following, we always suppose E is a real Banach space, P is a cone in E with $\operatorname{int} P \neq \varnothing$, and $\leqslant$ is partial ordering with respect to P.

Let X be a nonempty set. As it has been defined in [11], a function $d : X \times X \to E$ is called a cone metric on X if it satisfies the following conditions:

(i) $d(x, y) \geqslant 0$, for every $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, for every $x, y \in X$,

(iii) $d(x, y) \leqslant d(x, z) + d(y, z)$, for every $x, y, z \in X$.

Then (X, d) is called a cone metric space.

Example 1.2. Let $E = l^1$, $P = \{\{x_n\}_{n \geqslant 1} \in E : x_n \geqslant 0, \text{ for all } n\}$, (X, ρ) a metric space, and $d : X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n \geqslant 1}$. Then (X, d) is a cone metric space and the normal constant of P is equal to 1 [16].

The sequence $\{x_n\}$ in X is called to be convergent to $x \in X$ if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$, for every $n \geqslant n_0$, and is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$, for every $m, n \geqslant n_0$. A cone metric space (X, d) is said to be a complete cone metric space if every Cauchy sequence in X is convergent to a point of X. A self-map$T$ on X is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$ in X. The following lemmas are useful for us to prove our main results.

Lemma 1.3 (see [11, Lemma 1]). *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$.

Lemma 1.4 (see [11, Lemma 3]). *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent, then it is a Cauchy sequence, too.*

Lemma 1.5 (see [11, Lemma 4]). *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is a Cauchy sequence if and only if* $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

The following example is a cone metric space, see [11].

Example 1.6. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geqslant 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geqslant 0$ is a constant. Then (X, d) is a cone metric space.

2. Certain nonexpansive mappings

Definition 2.1. Let (X, d) be a cone metric space, where P is a cone and $f : X \to X$ is a function. Then f is said to be c-nonexpansive, for $0 \ll c$, if

$$d\big(f(x), f(y)\big) \leqslant d(x, y), \tag{2.1}$$

for every $x, y \in X$ with $d(x, y) \ll c$. If we have

$$d\big(f(x), f(y)\big) < d(x, y), \tag{2.2}$$

for every $x, y \in X$ with $x \neq y$ and $d(x, y) \ll c$, then f is called c-contractive.

Definition 2.2. Let (X, d) be a cone metric space, where P is a cone. A point $y \in Y \subseteq X$ is said to belong to the f-closure of Y and is denoted by $y \in Y^f$, if $f(Y) \subseteq Y$ and there are a point $x \in Y$ and an increasing sequence $\{n_i\} \subseteq \mathbb{N}$ such that $\lim_{i\to\infty} f^{n_i}(x) = y$.

Definition 2.3. Let (X, d) be a cone metric space, where P is a cone. A sequence $\{x_i\} \subseteq X$ is said to be a c-isometric sequence if

$$d\big(x_m, x_n\big) = d\big(x_{m+k}, x_{n+k}\big), \tag{2.3}$$

for all $k, m \in \mathbb{N}$ with $d(x_m, x_n) < c$. A point $x \in X$ is said to generate a c-isometric sequence under the function $f : X \to X$, if $\{f^n(x)\}$ is a c-isometric sequence.

Theorem 2.4. *Let (X, d) be a cone metric space, where P is a normal cone with normal constant K. If $f : X \to X$ is c-nonexpansive, for some $0 \ll c$, and $x \in X^f$, then there is an increasing sequence $\{m_j\} \subseteq \mathbb{N}$ such that $\lim_{j\to\infty} f^{m_j}(x) = x$.*

Proof. Since $x \in X^f$, there are $y \in X$ and a sequence $\{n_i\}$ such that $\lim_{i\to\infty} f^{n_i}(y) = x$. If $f^m(y) = x$, for some $m \in \mathbb{N}$. Put $m_j = n_j - m (n_j > m)$, is a sequence as desired, then $\{m_j\}$ is a sequence with desired property. Otherwise, for $\epsilon > 0$, fix δ, $0 < \delta < \epsilon$. Choose $c \in E$ with $0 \ll c$ and $K\|c\| < \delta$. Then there is $i = i(c)$ such that

$$d\big(x, f^{n_i+j}(y)\big) \ll \frac{c}{4}, \tag{2.4}$$

for every $j \in \mathbb{N} \cup \{0\}$. So by c-nonexpansivity of f and putting $j = 0$, we have

$$d\big(f^{n_{i+k}-n_i}(x), f^{n_i+k}(y)\big) < \frac{c}{4}, \tag{2.5}$$

for every $k \in \mathbb{N}$. Therefore,

$$d\big(f^{n_i}(y), f^{n_{i+k}}(y)\big) \leqslant d\big(x, f^{n_i}(y)\big) + d\big(x, f^{n_{i+k}}(y)\big) \ll \frac{c}{2}, \tag{2.6}$$

for every $k \in \mathbb{N}$. Hence,

$$d\big(x, f^{n_{i+1}-n_i}(x)\big) \leqslant d\big(x, f^{n_i}(y)\big) + d\big(f^{n_i}(y), f^{n_{i+1}}(y)\big) + d\big(f^{n_{i+1}}(y), f^{n_{i+1}-n_i}(x)\big) < \frac{c}{4} + \frac{c}{2} + \frac{c}{4} = c, \tag{2.7}$$

which implies

$$\left\|d\left(x, f^{n_{i+1}-n_i}(x)\right)\right\| \leqslant K\|c\| < \delta. \tag{2.8}$$

Put $m_1 = n_{i+1} - n_i$ and suppose that $m_1 < m_2 < \cdots < m_{j-1}$ chosen such that

$$\left\|d\left(x, f^{m_i}(y)\right)\right\| \leqslant \frac{1}{2} \min_{m=1,\ldots,m_{i-1}} \left\|d\left(x, f^{m}(y)\right)\right\|, \tag{2.9}$$

for $i = 2, 3, \ldots, j-1$. We put $m_j = n_{l+1} - n_l$, where l is chosen so as to satisfy $d(x, f^{l+j}(y)) \ll c/4$, with δ replaced by

$$\min\left\{\delta, \frac{1}{2} \min_{m=1,\ldots,m_{i-1}} \left\|d\left(x, f^{m}(y)\right)\right\|\right\}. \tag{2.10}$$

It is easily seen that the sequence $\{m_j\}$ that is defined in the above satisfies the requirements of the theorem. The proof is complete. □

Theorem 2.5. *Let (X, d) be a cone metric space, where P is a normal cone with normal constant K. If $f : X \to X$ is a c-nonexpansive function, then every $x \in X^f$ generates a c-isometric sequence.*

Proof. By contradiction, suppose that there are $k, m, n \in \mathbb{N}$ such that $d(f^m(x), f^n(x)) < c$ and $p = d(f^m(x), f^n(x)) - d(f^{m+k}(x), f^{n+k}(x)) \neq 0$. By the assumption, $p \in P$ and

$$0 < p \leqslant d\left(f^m(x), f^n(x)\right) - d\left(f^{m+l}(x), f^{n+l}(x)\right), \tag{2.11}$$

for $l \geqslant k,\ l \in \mathbb{N}$. It means that

$$\|p\| \leqslant K\left\|d\left(f^m(x), f^n(x)\right) - d\left(f^{m+k}(x), f^{n+k}(x)\right)\right\|, \tag{2.12}$$

for $l \geqslant k,\ l \in \mathbb{N}$. Also by the assumption and Theorem 2.4,

$$\lim_{j\to\infty} f^{n_j}\left(f^l(x)\right) = \lim_{j\to\infty} f^{n_j+l}(x) = f^l(x). \tag{2.13}$$

Put $\delta = \|p\|$ and choose $c \in E$ such that $0 \ll c$ and $\|c\| < (1/K^2)\delta$. By Lemma 1.4, there is $i \in \mathbb{N}$ such that

$$d\left(f^{m+n_j}(x), f^m(x)\right) \ll \frac{c}{2}, \qquad d\left(f^{n+n_j}(x), f^n(x)\right) \ll \frac{c}{2}, \tag{2.14}$$

for every $j \geqslant i$. However,

$$\begin{aligned} d\left(f^m(x), f^n(x)\right) &\leqslant d\left(f^m(x), f^{m+n_j}(x)\right) + d\left(f^{m+n_j}(x), f^{n+n_j}(x)\right) \\ &\quad + d\left(f^{n+n_j}(x), f^n(x)\right) \ll c + d\left(f^{m+n_j}(x), f^{n+n_j}(x)\right) + \frac{c}{2}. \end{aligned} \tag{2.15}$$

So

$$d\left(f^m(x), f^n(x)\right) - d\left(f^{m+n_j}(x), f^{n+n_j}(x)\right) \ll c. \tag{2.16}$$

It means that

$$\left\|d\left(f^m(x), f^n(x)\right) - d\left(f^{m+n_j}(x), f^{n+n_j}(x)\right)\right\| \leqslant K\|c\| < \frac{1}{K}\delta, \tag{2.17}$$

that is a contradiction by (2.12), for $n_j \geqslant \max\{n_i, k\}$. Therefore, $p = 0$ and the proof is complete. □

The following corollary implies immediately.

Corollary 2.6. *Let (X, d) be a cone metric space, where P is a normal cone with normal constant K. If $f : X \to X$ is a nonexpansive function and $x \in X^f$ generates an isometric sequence.*

3. Extended contraction principle

We have the following generalized form of Banach's contraction for cone metric spaces.

Theorem 3.1 (see [11, Theorem 1]). *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(T(x), T(y)) \leqslant \beta d(x, y), \tag{3.1}$$

for every $x, y \in X$, where $\beta \in [0, 1)$ is a constant. Then T has a unique fixed point in X, and for any $x \in X$, the sequence $\{T^n(x)\}$ converges to the fixed point.

It is natural to ask whether the mentioned theorem could be modified if (3.1) holds for just sufficiently close points. To be more specific, we introduce the following definitions.

Definition 3.2. Let (X, d) be a cone metric space. A function $f : X \to X$ is said to be locally contractive, if for every $x \in X$ there is $c \in X$ with $0 \ll c$ and $0 \leqslant \lambda < 1$ such that

$$d(f(p), f(q)) \leqslant \lambda d(p, q), \tag{3.2}$$

for every $p, q \in \{y \in X : d(x, y) \ll c\}$. A function $f : X \to X$ is said to be (c, λ)-uniformly locally contractive if it is locally contractive and both c and λ do not depend on x.

It is easy to find cone metric spaces which admit locally contractive which are not globally contractive.

Example 3.3. Let $E = \mathbb{R}^2$,

$$X = \left\{ (x, y) \mid x = \cos t,\ y = \sin t;\ 0 \leqslant t \leqslant \frac{3\pi}{2} \right\} \subseteq \mathbb{R}^2, \tag{3.3}$$

and $P = \{(x, y) \in E \mid x, y \geqslant 0\}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geqslant 0$ is a constant. It is easily checked that (X, d) is a cone metric space. Suppose that $f((\cos t, \sin t)) = (\cos(t/2), \sin(t/2))$. It is not hard to see that f is locally contractive but not globally contractive.

Note that every locally contractive function is c-nonexpansive for some $c \gg 0$.

Definition 3.4. A cone metric space (X, d) is called c-chainable, for $0 \ll c$, if for every $a, b \in X$, there is a finite set of points $a = x_0, x_1, \ldots, x_n = b$, n depends on both a and b, such that $d(x_{i-1}, x_i) < c$, for i, $1 \leqslant i \leqslant n$.

Example 3.5. It is easily seen that the cone metric space that is defined in Example 1.6 is c-chainable.

Theorem 3.6. *Let (X, d) be a complete c-chainable cone metric space, P be a normal cone with normal constant K. If $f : X \to X$ is (c, β)-uniformly locally contractive, then there is a unique point $z \in X$ such that $f(z) = z$.*

Proof. Let $x \in X$ be arbitrary. Consider the c-chain $x = x_0, x_1, \ldots, x_n = f(x)$. We have

$$d(x, f(x)) \leqslant \sum_{i=1}^{n} d(x_{i-1}, x_i) < nc. \tag{3.4}$$

We have

$$d(f(x_{i-1}), f(x_i)) \leqslant \beta d(x_{i-1}, x_i) < \beta c, \tag{3.5}$$

for every $1 \leqslant i \leqslant n$, and by induction

$$d(f^m(x_{i-1}), f^m(x_i)) < \beta d(f^{m-1}(x_{i-1}), f^{m-1}(x_i)) < \cdots < \beta^m c, \tag{3.1}$$

for every $m \in \mathbb{N}$. Hence

$$d(f^m(x), f^{m+1}(x)) \leqslant \sum_{i=1}^{n} d(f^m(x_{i-1}), f^m(x_i)) < \beta^m nc, \tag{3.2}$$

for every $m \in \mathbb{N}$. Now, for $m, p \in \mathbb{N}$ with $m < p$, we have

$$d(f^m(x), f^p(x)) \leqslant \sum_{i=m}^{p-1} d(f^i(x), f^{i+1}(x)) < nc(\beta^m + \cdots + \beta^{p-1}) < nc\frac{\beta^m}{1-\beta}. \tag{3.3}$$

It means that

$$\|d(f^m(x), f^p(x))\| \leqslant n\|c\|\frac{\beta^m}{1-\beta}, \tag{3.4}$$

for $m, p \in \mathbb{N}$ with $m < p$. Since $k \in [0,1)$, then $\lim_{m,p\to\infty}\|d(f^m(x), f^p(x))\| = 0$. So $\lim_{m,p\to\infty} d(f^m(x), f^p(x)) = 0$, and by Lemma 1.5, $\{f^m(x)\}$ is a Cauchy sequence. Since X is complete, then $\lim_{m\to\infty} f^m(x) = z$, for some $z \in X$. From the continuity of f it follows that $f(z) = z$. To complete the proof it is enough to show that z is the unique point with this property. To do this, suppose that there is $z' \in X$ such that $f(z') = z'$. Let $z' = x_0, x_1, \ldots, x_t = z$ be a c-chain. By (3.1), we obtain

$$d(f(z), f(z')) = d(f^l(z), f^l(z')) \leqslant \sum_{i=1}^{t} d(f^l(x_{i-1}), f^l(x_i)) < \beta^l tc. \tag{3.5}$$

It means that

$$\|d(z, z')\| = \|d(f(z), f(z'))\| \leqslant \beta^l t\|c\|. \tag{3.6}$$

Since $\beta \in [0,1)$, then $\|d(z, z')\| = 0$ and $z = z'$. This completes the proof. □

Corollary 3.7. *Let (X, d) be a complete c-chainable cone metric space, P be a normal cone with normal constant K. If f is a one to one, (c, λ)-uniformly locally expansive function of Y onto X, where $Y \subseteq X$, then f has a unique fixed point.*

Proof. It is an immediate consequence of the fact that for the inverse function all assumptions of the Theorem 3.6 are satisfied. □

In the following theorem we investigate a kind of functions which are not necessarily contractions but have a unique fixed point. First, we will prove the following lemma which will be used later.

Lemma 3.8. *Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K, $f : X \to X$be a continuous function, and $\beta \in [0,1)$ such that for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\big(f^{n(x)}(x), f^{n(x)}(y)\big) \leqslant \beta d(x,y), \tag{3.7}$$

for every $y \in X$. Then for every $x \in X$, $r(x) = \sup_n \|d(f^{n(x)}(x), x)\|$ is finite.

Proof. Let $x \in X$ and $l(x) = \max\{\|d(f^j(x), x)\| : j = 1, 2, \ldots, n(x)\}$. If $n \in \mathbb{N}$ and $n > n(x)$, then there is $s \in \mathbb{N} \cup \{0\}$ such that $sn(x) < n \leqslant (s+1)n(x)$ and we have

$$\begin{aligned} d\big(f^n(x), x\big) &\leqslant d\big(f^{n(x)}\big(f^{n-n(x)}(x)\big), f^{n(x)}(x)\big) + d\big(f^{n(x)}(x), x\big) \\ &\leqslant \beta d\big(f^{n-n(x)}(x), x\big) + d\big(f^{n(x)}(x), x\big) \\ &\leqslant d\big(f^{n(x)}(x), x\big) + \beta\big(d\big(f^{n-n(x)}(x), f^{n(x)}(x)\big) + d\big(f^{n(x)}(x), x\big)\big) \\ &\leqslant d\big(f^{n(x)}(x), x\big) + \beta\big(\beta d\big(f^{n-2n(x)}(x), x\big) + d\big(f^{n(x)}(x), x\big)\big) \\ &\leqslant \cdots \leqslant d\big(f^{n(x)}(x), x\big)\big(1 + \beta + \beta^2 + \cdots + \beta^s\big). \end{aligned} \tag{3.8}$$

It means that

$$\big\|d\big(f^n(x), x\big)\big\| \leqslant K\frac{1}{1-\beta}\big\|d\big(f^{n(x)}(x), x\big)\big\| \leqslant K\frac{1}{1-\beta}l(x). \tag{3.9}$$

Hence $r(x)$ is finite and the proof is complete. □

Theorem 3.9. *Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K, $\beta \in [0,1)$, and $f : X \to X$ be a continuous function such that for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\big(f^{n(x)}(x), f^{n(x)}(y)\big) \leqslant \beta d(x,y), \tag{3.10}$$

for every $y \in X$. Then f has a unique fixed point $u \in X$ and $\lim_{n\to\infty} f^n(x_0) = u$, for every $x_0 \in X$.

Proof. Let $x_0 \in X$ be arbitrary, and $m_0 = n(x_0)$. Define the sequence $x_1 = f^{m_0}(x_0)$, $x_{i+1} = f^{m_i}(x_i)$, where $m_i = n(x_i)$. We show that $\{x_n\}$ is a Cauchy sequence. We have

$$\begin{aligned} d\big(x_{n+1}, x_n\big) &= d\big(f^{m_{n-1}}\big(f^{m_n}\big(x_{n-1}\big), f^{m_{n-1}}\big(x_{n-1}\big)\big)\big) \\ &\leqslant \beta d\big(f^{m_n}\big(x_{n-1}\big), x_{n-1}\big) \\ &\leqslant \cdots \leqslant \beta^n d\big(f^{m_n}\big(x_0\big), x_0\big), \end{aligned} \tag{3.11}$$

for every $n \in \mathbb{N}$. So by Lemma 3.8, $\|d(x_{n+1}, x_n)\| \leqslant K\beta^n r(x_0)$, for every $n \in \mathbb{N}$. Now, suppose that $m, n \in \mathbb{N}$ with $m < n$, we have

$$\big\|d\big(x_n, x_m\big)\big\| \leqslant K\sum_{i=m}^{n-1}\big\|d\big(x_{i+1}, x_i\big)\big\| \leqslant K\frac{\beta^n}{1-\beta}r\big(x_0\big). \tag{3.12}$$

Since $\lim_{n\to\infty}(\beta^n/(1-\beta)) = 0$, then $\lim_{m,n\to\infty}\|d(x_n, x_m)\| = 0$, and by Lemma 1.5, $\{x_n\}$ is a Cauchy sequence. Completeness of X implies that $\lim_{n\to\infty} x_n = u$, for some $u \in X$. Now, we show that $f(u) = u$. By contradiction, suppose that $f(u) \neq u$. We claim that there are $c, d \in E$ such that $0 \ll c,\ 0 \ll d$ and $B_c(u)$ and $B_d(f(u))$have no intersection, where $B_e(x) = \{y \in X : d(x,y) \ll e\}$, for every $x \in X$ and $0 \ll e$. If not, then suppose that $\epsilon > 0$, and choose $c \in E$ with $0 \ll c$ and $K\|c\| < \epsilon$. Then clearly, $0 \ll c/2$ and for $z \in B_{c/2}(u) \cap B_{c/2}(f(u))$, we have

$$d(u, f(u)) \leqslant d(u, z) + d(z, f(u)) \ll c. \tag{3.13}$$

It means that $\|d(u, f(u))\| \leq K\|c\| < \epsilon$. Since $\epsilon > 0$ is arbitrary, then $\|d(u, f(u))\| = 0$ and so $f(u) = u$, a contradiction. Therefore, assume that $c, d \in E$ with $0 \ll c,\ 0 \ll d$ are such that $B_c(u) \cap B_d(f(u)) = \varnothing$. Since f is continuous, then there is $n_0 \in \mathbb{N}$ such that $x_n \in B_c(u)$ and $f(x_n) \in B_d(f(u))$, for every $n \in \mathbb{N}$ and $n \geqslant n_0$. Then

$$d(f(x_n), x_n) = d(f^{m_{n-1}}(f(x_{n-1}), f^{m_{n-1}}(x_{n-1})) \leq \beta d(f(x_{n-1}), x_{n-1}) \leqslant \cdots \leqslant \beta^n d(f(x_0), x_0), \tag{3.14}$$

for every $n \in \mathbb{N}$. It means that $\|d(f(x_n), x_n)\| \leqslant K\beta^n \|d(f(x_0), x_0)\|$, for every $n \in \mathbb{N}$. So $\lim_{n\to\infty} d(f(x_n), x_n) = 0$, a contradiction. Thus $f(u) = u$. The uniqueness of the fixed point follows immediately from the hypothesis.

Now, suppose that $x_0 \in X$ is arbitrary. To show that $\lim_{n\to\infty} f^n(x_0) = u$, set

$$r_0 = \max\left\{\left\|d(f^m(x_0), u)\right\| : m = 0, 1, \ldots, n(u) - 1\right\}. \tag{3.15}$$

If n is sufficiently large, then $n = rn(u) + q$, for $r > 0$ and $0 \leqslant q < n(u)$, and we have

$$d(f^n(x_0), u) = d(f^{rn(u)+q}(x_0), f^{n(u)}(u)) \leqslant \beta d(f^{(r-1)n(u)+q}(x_0), u)) \leq \cdots \leqslant \beta^r d(f^q(x_0), u). \tag{3.16}$$

It means that

$$\left\|d(f^n(x_0), u)\right\| \leqslant K\beta^r \left\|d(f^q(x_0), u)\right\| \leqslant K\beta^r r_0. \tag{3.17}$$

Therefore, $\lim_{n\to\infty} \|d(f^n(x_0), u)\| = 0$ and hence $\lim_{n\to\infty} f^n(x_0) = u$. This completes the proof. □

Definition 3.10. Let X be an ordered space. A function $\varphi : X \to X$ is said to be a comparison function if for every $x, y \in X$, $x \leqslant y$, implies that $\varphi(x) \leqslant \varphi(y)$, $\varphi(x) \leqslant x$, and $\lim_{n\to\infty} \|\varphi^n(x)\| = 0$, for every $x \in X$.

Example 3.11. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geqslant 0\}$. It is easy to check that $\varphi : E \to E$, with $\varphi(x, y) = (ax, ay)$, for some $a \in (0, 1)$ is a comparison function. Also if $\varphi_1,\ \varphi_2$ are two comparison functions over $\mathbb{R}$, then $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ is also a comparison function over E.

Recall that for a cone metric space (X, d), where P is a cone with normal constant K, since for every $x \in X,\ x \leqslant x$, and therefore $\|x\| \leqslant K\|x\|$, then $K \geqslant 1$.

Theorem 3.12. *Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\varphi : P \to P$ such that*

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)), \tag{3.18}$$

for every $x, y \in X$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We have

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leqslant \varphi(d(f^{n-1}(x_0), f^n(x_0))) \\ &\leqslant \varphi^2(d(f^{n-2}(x_0), f^{n-1}(x_0))) \\ &\leqslant \cdots \leqslant \varphi^n(d(x_0, f(x_0)), \end{aligned} \tag{3.19}$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\|\varphi^n(d(x_0, f(x_0)))\| = 0$, for an arbitrary $\epsilon > 0$, we can choose $n \in \mathbb{N}$ such that

$$\|d(f^n(x_0), f^{n+1}(x_0))\| < \frac{(\epsilon - K\|\varphi(c)\|)}{K}, \tag{3.20}$$

for every $n \geq n_0$ and $c \in P$ with

$$\|c\| < \frac{\epsilon}{K^2}, \qquad \frac{1}{K}\|\varphi(c)\| \geqslant \|\varphi(d(f^n(x_0), f^{n+1}(x_0)))\|. \tag{3.21}$$

For $n \geq n_0$, we have

$$d(f^n(x_0), f^{n+2}(x_0)) \leqslant d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0)). \tag{3.22}$$

So

$$\begin{aligned} \|d(f^n(x_0), f^{n+2}(x_0))\| &\leqslant K\|d(f^n(x_0), f^{n+1}(x_0))\| + K\|d(f^{n+1}(x_0), f^{n+2}(x_0))\| \\ &< K\left(\frac{\epsilon - K\|\varphi(c)\|}{K}\right) + K^2\|\varphi(d(f^n(x_0), f^{n+1}(x_0)))\| \\ &\leqslant \epsilon. \end{aligned} \tag{3.23}$$

Now, for every $n \geqslant n_0$, we have

$$d(f^n(x_0), f^{n+3}(x_0)) \leqslant d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+3}(x_0)). \tag{3.24}$$

Since $K \geq 1$, then we have

$$\begin{aligned} \|d(f^n(x_0), f^{n+3}(x_0))\| &\leqslant K\|d(f^n(x_0), f^{n+1}(x_0))\| + K\|d(f^{n+1}(x_0), f^{n+3}(x_0))\| \\ &< K\left(\frac{\epsilon - K\|\varphi(c)\|}{K}\right) + K^2\|\varphi(d(f^n(x_0), f^{n+2}(x_0)))\| \\ &\leqslant \epsilon. \end{aligned} \tag{3.25}$$

By induction, we have $\|d(f^n(x_0), f^{n+r}(x_0))\| < \epsilon$, for every $r \in \mathbb{N}$ and $n \geqslant n_0$. Hence by Lemma 1.5, we have $\{f^n(x_0)\}$ is a Cauchy sequence in (X, d). So $\lim_{n\to\infty} f^n(x_0) = x^*$, for some $x^* \in X$. Now, we will prove $f(x^*) = x^*$. Since $\lim_{n\to\infty} f^n(x_0) = x^*$, for every $c \gg 0$, there exists $n_c \in \mathbb{N}$ such that for every $n \geqslant n_c$, we have $d(f^n(x_0), x^*) < c$. Therefore,

$$\begin{aligned} d(x^*, f(x^*)) &\leqslant d(x^*, f^{n+1}(x_0)) + d(f(f^n(x_0)), f(x^*)) \\ &\leqslant d(x^*, f^{n+1}(x_0)) + \varphi(d(f^n(x_0), x^*)) \\ &< d(x^*, f^{n+1}(x_0)) + d(f^n(x_0), x^*) < 2c, \end{aligned} \tag{3.26}$$

for every $c \gg 0$. So $f(x^*) = x^*$. For the uniqueness of the fixed point, suppose that there exists $y^* \in X$ such that $f(y^*) = y^*$. Hence

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leqslant \varphi^n(d(y^*, x^*)). \tag{3.27}$$

So

$$\|d(x^*, y^*)\| \leqslant K\|\varphi^n(d(y^*, x^*))\|. \tag{3.28}$$

Since $\lim_{n\to\infty}\|\varphi^n(d(y^*, x^*))\| = 0$, then $x^* = y^*$ and the proof is complete. □

4. Applications

Theorem 4.1. *Consider the integral equation*

$$x(t) = \int_a^b k(t,s,x(s))ds + g(t), \quad t \in [a,b]. \tag{i}$$

Suppose that

(i) $k : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [a,b] \to \mathbb{R}^n$;

(ii) $k(t,s,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for every* $t,s \in [a,b]$;

(iii) *there exists a continuous function* $p : [a,b] \times [a,b] \to \mathbb{R}^+$ *and a comparison function* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$(|k(t,s,u) - k(t,s,v)|, \alpha|k(t,s,u) - k(t,s,v)|) \leqslant (p(t,s), \alpha p(t,s))\varphi(d(u,v)), \tag{4.1}$$

for every $t,s \in [a,b]$, $u,v \in \mathbb{R}^n$;

(iv) $\sup_{t\in[a,b]} \int_a^b (p(t,s), \alpha p(t,s))ds = 1$.

Then the integral equation (i) *has a unique solution* x^* *in* $C([a,b],\mathbb{R}^n)$.

Proof. Let $X = C([a,b],\mathbb{R}^n)$, $P = \{(x,y) : x,y \geqslant 0\} \subseteq \mathbb{R}^2$, and define $d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$, for every $f,g \in X$. Then it is easily seen that (X,d) is a cone metric space. Define $A : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$, by

$$Ax(t) := \int_a^b k(t,s,x(s))ds + g(t), \quad t \in [a,b]. \tag{4.2}$$

For every $x,y \in X$, we have

$$\begin{aligned}
&(|Ax(t) - Ay(t)|, \alpha|Ax(t) - Ay(t)|) \\
&\quad = \left(\left|\int_a^b [k(t,s,x(s)) - k(t,s,y(s))]ds\right|, \alpha\left|\int_a^b [k(t,s,x(s)) - k(t,s,y(s))]ds\right|\right) \\
&\quad \leqslant \left(\int_a^b |k(t,s,x(s)) - k(t,s,y(s))|ds, \int_a^b \alpha|k(t,s,x(s)) - k(t,s,y(s))|ds\right) \\
&\quad \leqslant \int_a^b (p(t,s), \alpha p(t,s))\varphi(|x(s) - y(s)|, \alpha|x(s) - y(s)|)ds \\
&\quad \leqslant \varphi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)\int_a^b (p(t,s), \alpha p(t,s))ds \\
&\quad = \varphi(\|x-y\|_\infty, \alpha\|x-y\|_\infty).
\end{aligned} \tag{4.3}$$

Hence $d(Ax,Ay) \leqslant \varphi(d(x,y))$, for every $x,y \in X$. The conclusion follows now from Theorem 3.12. □

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