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Research Article

On Uniqueness of Conjugacy of Continuous and Piecewise Monotone Functions

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We investigate the existence and uniqueness of solutions $\varphi : I \to J$ of the functional equation $\varphi(f(x)) = F(\varphi(x))$, $x \in I$, where I, J are closed intervals, and $f : I \to I$, $F : J \to J$ are some continuous piecewise monotone functions. A fixed point principle plays a crucial role in the proof of our main result.

1. Introduction

Let $I = [a, b]$, $J = [c, d]$ be closed, bounded, and nondegenerate (i.e., neither of them consists of a single point) real intervals, and let $f : I \to I$, $F : J \to J$ be continuous functions. The aim of this paper is to discuss, under some additional assumptions on the maps f and F, the problem of (topological) conjugacy of f and F. More precisely, we investigate the existence and uniqueness of solutions $\varphi : I \to J$ of the following functional equation:

$$\varphi(f(x)) = F(\varphi(x)), \quad x \in I. \tag{1.1}$$

Let us recall that a homeomorphism $\varphi : I \to J$ satisfying (1.1) is said to be a *topological conjugacy* between f and F (f and F are then called *topologically conjugate*), whereas an arbitrary function $\varphi : I \to J$ fulfilling (1.1) is called a *conjugacy* between them (so the conjugacy needs not to be continuous, surjective, or injective).

A continuous function $f : I \to I$ is said to be a *horseshoe map* (see [1]) if there exist an integer $n > 1$ and a sequence $(t_i)_{i=0}^{n}$ of reals such that

$$a = t_0 < t_1 < \cdots < t_n = b, \tag{1.2}$$

and for every $i \in \{0, \ldots, n-1\}$, $f_{|[t_i, t_{i+1}]}$ is a homeomorphism of the interval $[t_i, t_{i+1}]$ (which is called a *lap* of f) onto I. We say that horseshoe maps $f : I \to I$ and $F : J \to J$ having that the same number of laps are of the *same type* if f and F are of the same type of monotonicity on their leftmost laps.

It is known (see [1, 2]) that two horseshoe maps of the same type and without homtervals (i.e., intervals on which all their iterates are monotone) are topologically conjugate. So are also transitive horseshoe maps having two laps each, and, in this case, topological conjugacy is only one (see [3]). Moreover, if f is transitive and $F : [0,1] \to [0,1]$ is an arbitrary function, then every increasing, continuous, and surjective solution $\varphi : I \to [0,1]$ of (1.1) is homeomorphic (see [4]). We will show (Example 3.7) that if we omit the assumption of the transitivity of f, then such a solution needs not to be injective even if F is continuous and transitive and f is continuous and piecewise monotone.

However, the main purpose of this paper is to find some regularity conditions on f and F ensuring the uniqueness of conjugacy between them as well as implying that conjugacy is topological. The following fixed point principle plays a crucial role in the proof of our main result (Theorem 3.1).

Theorem 1.1 (see [5, Theorem 1.2, page 8] and also [6, Theorem 3.2, page 12]). *Let (X, d) be a complete metric space and $T : X \to X$. If*

$$d(T(x), T(y)) \leq \gamma(d(x, y)), \quad x, y \in X, \tag{1.3}$$

for a nondecreasing function $\gamma : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{n \to \infty} \gamma^n(t) = 0, \quad t > 0, \tag{1.4}$$

then T has a unique fixed point.

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper E stands for the integer part function.

Let (X, d) be a metric space. A function $T : X \to X$ is called *strictly contractive* if

$$d(T(x), T(y)) < d(x, y), \quad x, y \in X,\ x \neq y. \tag{2.1}$$

Given a nondecreasing function $\gamma : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{n \to \infty} \gamma^n(t) = 0, \tag{2.2}$$

for $t > 0$ and a selfmap T of X, we say that T is γ-*contractive* if it satisfies condition (1.3).

Given horseshoe maps $f : I \to I$, $F : J \to J$ having laps

$$[t_i, t_{i+1}], \quad [x_i, x_{i+1}], \quad i \in \{0, \ldots, n-1\}, \tag{2.3}$$

respectively, for every $i \in \{0, \dots, n-1\}$ put

$$I_i := [t_i, t_{i+1}], \qquad J_i := [x_i, x_{i+1}], \qquad f_i := f_{|I_i}, \qquad F_i := F_{|J_i}. \tag{2.4}$$

A horseshoe map $f : I \to I$ having n laps $[t_i, t_{i+1}]$ is said to be *piecewise expansive* (resp., *piecewise γ-expansive*) if for every $i \in \{0, \dots, n-1\}$, f_i^{-1} is strictly contractive (resp., γ-contractive for a $\gamma : [0, +\infty) \to [0, +\infty)$).

We start with the following.

Proposition 2.1. *Let $f : I \to I$ and $F : J \to J$ be horseshoe maps having n laps $[t_i, t_{i+1}]$ and $[x_i, x_{i+1}]$, respectively. Assume also that $\varphi : I \to J$ is a monotone and surjective solution of* (1.1).

If φ is increasing, then

$$\varphi(t_i) = x_i, \quad i \in \{0, \dots, n\}, \tag{2.5}$$

and $\varphi[I_i] = J_i$ for $i \in \{0, \dots, n-1\}$.

If φ is decreasing, then

$$\varphi(t_i) = x_{n-i}, \quad i \in \{0, \dots, n\}, \tag{2.6}$$

and $\varphi[I_i] = J_{n-i-1}$ for $i \in \{0, \dots, n-1\}$.

Proof. Let us first note that the fact that $\varphi : I \to J$ is monotone and surjective together with (1.1) gives $F(\varphi(t_i)) \in \{c, d\}$ for $i \in \{0, \dots, n\}$, and consequently

$$\varphi(t_i) \in \{x_0, \dots, x_n\}, \quad i \in \{0, \dots, n\}. \tag{2.7}$$

Suppose that $\varphi(t_i) = \varphi(t_{i+1})$ for an $i \in \{0, \dots, n-1\}$. Then, using the fact that φ is monotone, we see that $\varphi[I_i]$ consists of a single point, and therefore so does $F[\varphi[I_i]]$. But from (1.1) and the surjectivity of φ it follows that

$$F\big[\varphi[I_i]\big] = \varphi\big[f[I_i]\big] = \varphi[I] = J, \tag{2.8}$$

a contradiction. We have thus shown that

$$\varphi(t_i) \neq \varphi(t_{i+1}), \quad i \in \{0, \dots, n-1\}. \tag{2.9}$$

Assume that φ is increasing. Then, by (2.9), we obtain

$$\varphi(t_0) < \varphi(t_1) < \cdots < \varphi(t_n), \tag{2.10}$$

which together with (2.7) gives (2.5). Hence we immediately see that for any $i \in \{0, \dots, n-1\}$, $\varphi[I_i] = J_i$.

The rest of the proof runs as before. $\square$

Proposition 2.2. *Assume that*

(H) $f : I \to I$ *and* $F : J \to J$ *are horseshoe maps of the same type and having* n *laps* $[t_i, t_{i+1}]$ *and* $[x_i, x_{i+1}]$*, respectively,*

and F *is piecewise expansive. If* $\varphi : I \to J$ *is a continuous and nonconstant solution of* (1.1)*, then* φ *is surjective. If, moreover,* n *is even and* φ *is injective, then* φ *is strictly increasing.*

Proof. Let α, β with $\alpha < \beta$ be such that $[\alpha, \beta] = \varphi[I]$. Then, by (1.1), we have

$$F\big[[\alpha, \beta]\big] = \big[\alpha, \beta\big]. \tag{2.11}$$

This and the fact that for every $i \in \{0, \dots, n-1\}$, F_i^{-1} is strictly contractive shows that α and β are not in the same interval J_i. Therefore, $c \in F[[\alpha, \beta]]$ or $d \in F[[\alpha, \beta]]$, and consequently, by (2.11), $\alpha = c$ or $\beta = d$.

Assume that $\alpha = c \in J_0$. Since $\beta \notin J_0$, $J_0 \subset [\alpha, \beta]$, (2.11) now gives

$$[c, d] = F[J_0] \subset F\big[[\alpha, \beta]\big] = \big[\alpha, \beta\big]. \tag{2.12}$$

Therefore, $\beta = d$. Similarly, $\beta = d$ implies $\alpha = c$. We have thus shown that φ is a surjection.

Now, assume that n is even (which obviously yields $f(a) = f(b)$). Suppose also, contrary to our claim, that φ is decreasing, and let us consider the case when $f(a) = a$. Then $\varphi(b) = c$ and $\varphi(a) = d$, and (1.1) now gives

$$d = \varphi(a) = \varphi\big(f(a)\big) = \varphi\big(f(b)\big) = F\big(\varphi(b)\big) = F(c), \tag{2.13}$$

which contradicts the fact that f and F are of the same type. Similar considerations apply to the case when $f(a) = b$. □

Lemma 2.3. *Let* n *be odd, and let assumption (H) hold. If* $\varphi : I \to J$ *is a solution of* (1.1) *such that*

$$\varphi[I_i] \subset J_{n-i-1}, \quad i \in \{0, \dots, n-1\}, \tag{2.14}$$

then

$$\varphi(x) = F_{n-i-1}^{-1}\big(\varphi\big(f_i(x)\big)\big), \quad x \in I_i,\ i \in \{0, \dots, n-1\}. \tag{2.15}$$

If, moreover, F *is piecewise expansive, then* (2.6) *holds true.*

Conversely, every function $\varphi : I \to J$ *satisfying* (2.15) *is a solution of* (1.1) *fulfilling* (2.14).

Proof. It is obvious that if $\varphi : I \to J$ fulfils (1.1) and (2.14), then (2.15) holds.

Assume that F is piecewise expansive. If $f(a) = a$, then from (1.1) and (2.14) it follows that $\varphi(a) \in J_{n-1}$ is a fixed point of F_{n-1}, and the strict contractivity of F_{n-1}^{-1} gives $\varphi(a) = d$.

Next, assume that $f(a) = b$, which clearly forces $f(b) = a$, $F_0^{-1}(d) = c$, and $F_{n-1}^{-1}(c) = d$. Since by (2.14) we obtain $\varphi(b) \in J_0$ and $\varphi(a) \in J_{n-1}$, (1.1) implies $F_{n-1}^{-1}(\varphi(b)) = \varphi(a)$ and

$F_0^{-1}(\varphi(a)) = \varphi(b)$. If it were true that $\varphi(a) \neq d$, we would conclude from the strict contractivity of F_{n-1}^{-1} and F_0^{-1} that

$$\begin{aligned} \left|\varphi(b) - c\right| &\geq \left|F_{n-1}^{-1}(\varphi(b)) - F_{n-1}^{-1}(c)\right| \\ &= \left|\varphi(a) - d\right| \\ &> \left|F_0^{-1}(\varphi(a)) - F_0^{-1}(d)\right| \\ &= \left|\varphi(b) - c\right|, \end{aligned} \tag{2.16}$$

a contradiction.

We have thus shown that $\varphi(a) = d$. In the same manner we can see that $\varphi(b) = c$.

Now, fix an $i \in \{1, \ldots, n-1\}$. Suppose that i is even and note that from (1.1), the equalities $\varphi(a) = d$, $\varphi(b) = c$, and the fact that $n - i$ is odd, we get

$$\begin{aligned} F(\varphi(t_i)) &= \varphi(f(t_i)) \\ &= \begin{cases} \varphi(a), & f(a) = a, \\ \varphi(b), & f(a) = b, \end{cases} \\ &= \begin{cases} d, & f(a) = a, \\ c, & f(a) = b, \end{cases} \\ &= F(x_{n-i}). \end{aligned} \tag{2.17}$$

Since, by (2.14), $\varphi(t_i), x_{n-i} \in J_{n-i-1}$, the injectivity of F_{n-i-1} gives now $\varphi(t_i) = x_{n-i}$. As the same conclusion can be drawn for odd i, the proof of (2.6) is complete.

The rest of the proof is immediate. □

Analysis similar to that in the proof of Lemma 2.3 (due to condition (2.18) it also applies to the case when n is even) gives the following.

Lemma 2.4. *Let assumption (H) hold. If $\varphi : I \longrightarrow J$ is a solution of* (1.1) *for which*

$$\varphi[I_i] \subset J_i, \quad i \in \{0, \ldots, n-1\}, \tag{2.18}$$

then

$$\varphi(x) = F_i^{-1}(\varphi(f_i(x))), \quad x \in I_i,\ i \in \{0, \ldots, n-1\}. \tag{2.19}$$

If, moreover, F is piecewise expansive, then (2.5) *holds true.*

Conversely, every function $\varphi : I \longrightarrow J$ satisfying (2.19) *is a solution of* (1.1) *fulfilling* (2.18).

3. Main Results

We can now formulate our main results.

Theorem 3.1. *If assumption (H) holds and F is piecewise γ-expansive, then there exists a unique function $\varphi : I \to J$ satisfying* (1.1) *and condition* (2.18). *This function is continuous, surjective and increasing. If, moreover, f is piecewise expansive, then φ is also strictly increasing.*

Proof. Put

$$\begin{aligned}
\mathcal{B} &:= \{\varphi : I \to J : \varphi(a) = c,\ \varphi(b) = d\},\\
\mathcal{C} &:= \{\varphi \in \mathcal{B} : \varphi \text{ is continuous}\},\\
\mathcal{M} &:= \{\varphi \in \mathcal{B} : \varphi \text{ is increasing}\}.
\end{aligned} \tag{3.1}$$

It is easily seen that all these spaces with the metric $d(\varphi,\psi) := \sup_{x\in I}|\varphi(x) - \psi(x)|$ are complete.

Fix a $\varphi \in \mathcal{B}$, and set

$$T(\varphi)(x) := F_i^{-1}(\varphi(f_i(x))), \quad x \in I_i,\ i \in \{0,\dots,n-1\}. \tag{3.2}$$

We will show that the above formula correctly defines a selfmap of $\mathcal{B}$. In order to do this let us first fix an $i \in \{0,\dots,n-1\}$ and observe that we have

$$\begin{aligned}
F_i^{-1}(\varphi(f_i(t_i))) &= \begin{cases} F_i^{-1}(\varphi(b)), & (i \text{ is odd and } f(a) = a) \text{ or } (i \text{ is even and } f(a) = b)\\ F_i^{-1}(\varphi(a)), & (i \text{ is odd and } f(a) = b) \text{ or } (i \text{ is even and } f(a) = a)\end{cases}\\
&= \begin{cases} F_i^{-1}(d), & (i \text{ is odd and } f(a) = a) \text{ or } (i \text{ is even and } f(a) = b)\\ F_i^{-1}(c), & (i \text{ is odd and } f(a) = b) \text{ or } (i \text{ is even and } f(a) = a)\end{cases}\\
&= x_i.
\end{aligned} \tag{3.3}$$

In the same manner we can see that $F_{i-1}^{-1}(\varphi(f_{i-1}(t_i))) = x_i$ for $i \in \{1,\dots,n\}$. Thus, $T : \mathcal{B} \to \mathcal{B}$ and, by Lemma 2.4, $\varphi : I \to J$ is a solution of (1.1) fulfilling (2.18) if and only if it is a fixed point of T. Moreover, it is easily seen that $T_{|\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $T_{|\mathcal{M}} : \mathcal{M} \to \mathcal{M}$. Now, fix $\varphi, \psi \in \mathcal{B}$, and note that by the facts that for every $i \in \{0,\dots,n-1\}$, F_i^{-1} is γ-contractive and γ is increasing, we get

$$\begin{aligned}
d(T(\varphi), T(\psi)) &= \sup_{x\in I}\left|T(\varphi)(x) - T(\psi)(x)\right|\\
&= \max_{i\in\{0,\dots,n-1\}} \sup_{x\in I_i}\left|T(\varphi)(x) - T(\psi)(x)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \max_{i\in\{0,\dots,n-1\}} \sup_{x\in I_i} \left| F_i^{-1}(\varphi(f_i(x))) - F_i^{-1}(\psi(f_i(x))) \right| \\
&= \max_{i\in\{0,\dots,n-1\}} \sup_{x\in I} \left| F_i^{-1}(\varphi(x)) - F_i^{-1}(\psi(x)) \right| \\
&\leq \gamma\left(\sup_{x\in I} \left| \varphi(x) - \psi(x) \right| \right) \\
&= \gamma(d(\varphi,\psi)).
\end{aligned}
\tag{3.4}
$$

Thus, the function T is γ-contractive, and Theorem 1.1 now shows that T has a unique fixed point $\varphi \in \mathcal{B}$. Similarly, $T_{|\mathcal{C}}$ has a unique fixed point $\varphi_{\mathcal{C}} \in \mathcal{C} \subset \mathcal{B}$, and $T_{|\mathcal{M}}$ has a unique fixed point $\varphi_{\mathcal{M}} \in \mathcal{M} \subset \mathcal{B}$. Therefore, $\varphi_{\mathcal{M}} = \varphi = \varphi_{\mathcal{C}}$ and, in consequence, φ is a continuous, surjective and increasing solution of (1.1) fulfilling (2.18).

Assume, additionally, that f is piecewise expansive. We will first show that φ is constant in no neighbourhood of a or b.

To do this, suppose that φ is constant on a neighbourhood of a, and denote by $[a,\alpha]$, where $\alpha \in (a,b)$, the maximal interval of constancy of φ. Since φ is surjective and satisfies (2.18), $[a,\alpha]$ is a proper subset of I_0. Therefore, $f(a) \neq f(\alpha)$ and from (1.1) it follows that φ is constant on $f[[a,\alpha]] = f_0[[a,\alpha]]$.

If $f(a) = a$, then

$$
[a, f(\alpha)] = f[[a,\alpha]] \subset [a,\alpha]. \tag{3.5}
$$

But from the strict contractivity of f_0^{-1} we also get

$$
|a-\alpha| < \left| f_0(a) - f_0(\alpha) \right| = \left| a - f(\alpha) \right|, \tag{3.6}
$$

a contradiction.

Now, assume that $f(a) = b$. Then $f[[a,\alpha]] = [f(\alpha), b]$, so φ is constant on a neighbourhood of b. Denote by $[\beta,b]$, where $\beta \in (a,b)$, the maximal interval of constancy of φ. Since φ is surjective and satisfies (2.18), $[\beta,b]$ is a proper subset of I_{n-1}. Therefore, $f(b) \neq f(\beta)$ and from (1.1) it follows that φ is constant on $f[[\beta,b]] = f_{n-1}[[\beta,b]]$.

If $f(b) = b$, then

$$
[f(\beta), b] = f[[\beta,b]] \subset [\beta,b], \tag{3.7}
$$

which contradicts the strict contractivity of f_{n-1}^{-1}.

Finally, assume that $f(b) = a$. Then $f[[\beta,b]] = [a, f(\beta)]$, so both $[f(\alpha), b]$ and $[a, f(\beta)]$ are intervals of constancy of φ, and therefore

$$
a < f(\beta) \leq \alpha < \beta \leq f(\alpha) < b. \tag{3.8}
$$

This together with the strict contractivity of f_0^{-1} and f_{n-1}^{-1} gives

$$\begin{aligned} |b-\beta| &< |f(b)-f(\beta)| \\ &= |a-f(\beta)| \\ &\le |a-\alpha| \\ &< |f(a)-f(\alpha)| \\ &= |b-f(\alpha)| \\ &\le |b-\beta|, \end{aligned} \tag{3.9}$$

a contradiction.

Analysis similar to the above shows that φ is constant in no neighbourhood of b.

Now, suppose that φ is constant on a neighbourhood of t_i for an $i \in \{1,\ldots,n-1\}$. Then from (1.1) it follows that φ is constant on a neighbourhood of a or b, which is impossible.

We have thus shown that if φ is not injective, then any interval of its constancy is contained in I_i for an $i \in \{0,\ldots,n-1\}$. Let Δ be an interval of constancy of φ having the maximal length. By (1.1), φ is constant on $f[\Delta]$. But since $\Delta \subset I_i$ for an $i \in \{0,\ldots,n-1\}$, from the strict contractivity of f_i^{-1} it follows that the interval $f[\Delta]$ is of greater length than Δ, a contradiction. □

Analysis similar to that in the proof of Theorem 3.1 with

$$\begin{gathered} \mathcal{B} := \{\varphi : I \longrightarrow J : \varphi(a) = d,\ \varphi(b) = c\}, \\ \mathcal{M} := \{\varphi \in \mathcal{B} : \varphi \text{ is decreasing}\}, \\ T(\varphi)(x) := F_{n-i-1}^{-1}(\varphi(f_i(x))), \quad x \in I_i,\ i \in \{0,\ldots,n-1\}, \end{gathered} \tag{3.10}$$

and application of Lemma 2.3 instead of Lemma 2.4 gives the following.

Theorem 3.2. *If n is odd, assumption (H) holds, and F is piecewise γ-expansive, then there exists a unique function $\varphi : I \to J$ satisfying* (1.1) *and condition* (2.14). *This function is continuous, surjective, and decreasing. If, moreover, f is piecewise expansive, then φ is also strictly decreasing.*

Let us next note that an immediate consequence of Theorems 3.1 and 3.2 and Proposition 2.1 is what follows.

Corollary 3.3. *If assumption (H) holds, F is piecewise γ-expansive, and n is odd, then* (1.1) *has exactly two monotone and surjective solutions. One of them is increasing, while the other is decreasing.*

The following fact follows immediately from Propositions 2.1 and 2.2.

Remark 3.4. If assumption (H) holds, F is piecewise expansive and n is even, then (1.1) has no homeomorphic solution satisfying condition (2.14).

On the other hand, we also have the following.

Theorem 3.5. *If assumption (H) holds and f and F are piecewise expansive, then there exists a unique function $\varphi : I \to J$ satisfying* (1.1) *and condition* (2.18). *This function is an increasing homeomorphism. If, moreover, n is odd, then there is also exactly one map $\varphi : I \to J$ fulfilling* (1.1) *and condition* (2.14). *This map is a decreasing homeomorphism.*

Proof. Put $L_i := [i/n, (i+1)/n]$ for $i \in \{0, \dots, n-1\}$. If $f(a) = a$, then we also set

$$H(x) := \begin{cases} nx - i, & i \text{ is even}, \\ -nx + i + 1, & i \text{ is odd}, \end{cases} \quad x \in L_i,\ i \in \{0, \dots, n-1\}, \tag{3.11}$$

while for $f(a) = b$ we put

$$H(x) = \begin{cases} -nx + i + 1, & i \text{ is even}, \\ nx - i, & i \text{ is odd}, \end{cases} \quad x \in L_i,\ i \in \{0, \dots, n-1\}. \tag{3.12}$$

By Theorem 3.1 (for every $i \in \{0, \dots, n-1\}$, $H^{-1}_{|L_i}$ is γ-contractive with $\gamma(x) := (1/2)x$ for $x \in [0, +\infty)$, so its assumptions are satisfied) there exist increasing homeomorphisms $\alpha : I \to [0,1]$, $\beta : J \to [0,1]$ such that

$$\alpha(f(x)) = H(\alpha(x)), \quad x \in I, \tag{3.13}$$

$$\beta(F(x)) = H(\beta(x)), \quad x \in J. \tag{3.14}$$

Moreover, by Proposition 2.1,

$$\alpha[I_i] = L_i = \beta[J_i], \quad i \in \{0, \dots, n-1\}. \tag{3.15}$$

Let us observe that in the case when n is odd from Theorem 3.2 and Proposition 2.1 it follows that we can also take β which is a decreasing homeomorphism such that (3.14) holds true and

$$\beta[J_i] = L_{n-i-1}, \quad i \in \{0, \dots, n-1\}. \tag{3.16}$$

Putting $\varphi := \beta^{-1} \circ \alpha$ and using (3.13) and (3.14) we obtain

$$\varphi(f(x)) = \beta^{-1}(H(\alpha(x))) = F(\varphi(x)), \quad x \in I. \tag{3.17}$$

Furthermore, from (3.15) (resp., (3.16)) it follows that condition (2.18) (resp., (2.14)) holds true.

We have thus shown that (1.1) has a homeomorphic, increasing solution fulfilling (2.18) and if, moreover, n is odd, then it also has a homeomorphic, decreasing solution satisfying (2.14).

To prove the uniqueness, assume that $\psi : I \to J$ is a solution of (1.1) fulfilling condition (2.18) (resp., (2.14)). Then, by (3.14), we get

$$\beta(\psi(f(x))) = \beta(F(\psi(x))) = H(\beta(\psi(x))), \quad x \in I, \tag{3.18}$$

which means that $\beta \circ \psi$ satisfies (3.13). Moreover, from (3.15) (resp., (3.16)) it follows that $(\beta \circ \psi)[I_i] \subset L_i$ for $i \in \{0, \dots, n-1\}$. Hence, by Theorem 3.1, $\beta \circ \psi = \alpha$ and so $\psi = \varphi$. □

As an immediate consequence of Theorem 3.5, Proposition 2.1, and Remark 3.4, we get the following.

Corollary 3.6. *Let assumption (H) hold, and let f and F be piecewise expansive. If n is even, then* (1.1) *has exactly one homeomorphic solution. This solution is strictly increasing. If n is odd, then* (1.1) *has exactly two homeomorphic solutions. One of them is strictly increasing, while the other is strictly decreasing.*

Finally, we give two examples. The first of them shows that if one replaces the assumption that f is piecewise expansive by

$$\left| f_i^{-1}(x) - f_i^{-1}(y) \right| \le |x - y|, \quad x, y \in I,\ i \in \{0, \dots, n-1\}, \tag{3.19}$$

then the last assertion of Theorem 3.1 is no longer true.

Example 3.7. Let $n = 2$, $a = c = 0$, $b = d = 1$, $t_1 = x_1 = 1/2$,

$$f(x) = \begin{cases} 2x, & x \in \left[0, \dfrac{1}{2}\right], \\ -x + \dfrac{3}{2}, & x \in \left[\dfrac{1}{2}, \dfrac{5}{6}\right], \\ -4x + 4, & x \in \left[\dfrac{5}{6}, 1\right], \end{cases} \tag{3.20}$$

and let F be the standard tent map defined by

$$F(x) = 1 - |2x - 1|, \quad x \in [0, 1]. \tag{3.21}$$

It is well known (see, e.g., [7] which is also a good survey on transitive maps) that F is transitive. It is also easily seen that F fulfills the assumption of Theorem 3.1 with $\gamma(x) := (1/2)x$ for $x \in [0, +\infty)$ and

$$|f(x) - f(y)| \ge |x - y|, \quad x, y \in I_i,\ i \in \{0, 1\}. \tag{3.22}$$

By Theorem 3.1 there exists a unique (continuous, surjective, and increasing) function φ satisfying equation (1.1) and condition (2.18). Put $K := [2/3, 5/6]$, and note that $f[K] = K$. Furthermore, from (1.1) it follows that $F[\varphi[K]] = \varphi[K]$. Therefore, $\varphi[K]$ has to be one of the intervals: $\{0\}$, $\{2/3\}$, $[0,1]$. Consequently, φ is not injective, and thus (1.1) has no homeomorphic solution.

Example 3.8. Let $n = 2$, $a = c = 0$, $b = d = 1$, $t_1 = x_1 = 1/2$,

$$f(x) = 4x(1-x), \quad x \in [0,1], \tag{3.23}$$

and let F be the standard tent map. It is known (see, e.g., [7–9]) that the function $\varphi : I \to I$ given by

$$\varphi(x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad x \in [0,1] \tag{3.24}$$

is a topological conjugacy between f and F. From Theorem 3.1 it follows that φ is the unique conjugacy between these maps such that $\varphi[[0,1/2]] \subset [0,1/2]$ and $\varphi[[1/2,1]] \subset [1/2,1]$. In particular, by Proposition 2.1, φ is the only continuous, increasing, and surjective conjugacy between f and F.

Corollary 3.9. *The function φ given by* (3.24) *is the only one satisfying*

$$\varphi(4x(1-x)) = 1 - \left|2\varphi(x) - 1\right|, \quad x \in [0,1], \tag{3.25}$$

and such that $\varphi(x) \le 1/2$ for $x \le 1/2$ and $\varphi(x) \ge 1/2$ for $x \ge 1/2$.

Example 3.10. If $J = I$ and $F = f$ is a horseshoe map, then for any positive integer m the function $\varphi := f^m$ is a continuous and surjective solution of (1.1). It is obvious that this solution does not satisfy neither (2.14) nor (2.18). We thus see that in this case (1.1) can even have infinitely many solutions.

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