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Research Article

Some Sufficient Conditions for Fixed Points of Multivalued Nonexpansive Mappings

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We show some sufficient conditions on a Banach space X concerning the generalized James constant, the generalized Jordan-von Neumann constant, the generalized Zbagănu constant, the coefficient $\widetilde{\varepsilon}_0(X)$, the weakly convergent sequence coefficient $WCS(X)$, and the coefficient of weak orthogonality, which imply the existence of fixed points for multivalued nonexpansive mappings. These fixed point theorems improve some previous results in the recent papers.

1. Introduction

In 1969, Nadler [1] established the multivalued version of Banach contraction principle. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. However, many questions remain open, for instance, the possibility of extending the well-known Kirk's theorem [2], that is, "Do Banach spaces with weak normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings?"

Since weak normal structure is implied by different geometric properties of Banach spaces, it is natural to study whether those properties imply the FPP for multivalued mappings. Dhompongsa et al. [3, 4] introduced the DL condition and property (D) which imply the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the DL condition or property (D). In this setting the following results have been obtained.

(i) Kaewkhao [5] proved that a Banach space X with

$$J(X) < 1 + \frac{1}{\mu(X)} \tag{1.1}$$

satisfies the DL condition. He also showed that the condition

$$C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2} \tag{1.2}$$

implies the DL condition [6].

(ii) Saejung [7] showed that a Banach space X has property (D) whenever $\varepsilon_0(X) <$ WCS(X).

In this paper, we show some sufficient conditions on a Banach space X concerning the generalized James constant, the generalized Jordan-von Neumann constant, the generalized Zbăganu constant, the coefficient $\widetilde{\varepsilon_0}(X)$, the weakly convergent sequence coefficient WCS(X), and the coefficient of weak orthogonality, which imply the existence of fixed points for multivalued nonexpansive mappings. These theorems improve the above results.

2. Preliminaries

Before going to the result, let us recall some concepts and results which will be used in the following sections. Let X be a Banach space with the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The two constants of a Banach space

$$\begin{gathered} C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} : x, y \in X \text{ and } \|x\| + \|y\| > 0 \right\}, \\ J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\} \end{gathered} \tag{2.1}$$

are called the von Neumann-Jordan [8] and James constants [9], respectively, and are widely studied by many authors [10–20]. Recently, both constants are generalized in the following ways for $o \leq a \leq 2$ (see [12, 13]):

$$\begin{gathered} C_{\mathrm{NJ}}(a, X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X,\ \|x\| + \|y\| + \|z\| > 0, \text{ and } \|y-z\| \leq a\|x\| \right\}, \\ J(a, X) = \sup\{\min\{\|x+y\|, \|x-z\|\} : x, y, z \in S_X, \text{ and } \|y-z\| \leq a\|x\|\}. \end{gathered} \tag{2.2}$$

It is clear that $C_{\mathrm{NJ}}(0, X) = C_{\mathrm{NJ}}(X)$ and $J(0, X) = J(X)$.

Recently, Gao and Saejung in [6] define a new constant for $a \geq 0$:

$$C_Z(a, X) = \sup\left\{ \frac{2\|x+y\|\|x-z\|}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X,\ \|x\| + \|y\| + \|z\| > 0,\ \text{and}\ \|y-z\| \leq a\|x\| \right\}, \tag{2.3}$$

which is inspired by Zbăganu paper [21]. It is clear that

$$C_Z(0, X) = C_Z(X) = \sup\left\{ \frac{\|x+y\|\|x-z\|}{\|x\|^2 + \|y\|^2} : x, y \in X,\ \|x\| + \|y\| > 0 \right\}. \tag{2.4}$$

The modulus of convexity of X (see [22]) is a function $\delta_X(\epsilon) : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{ 1 - \frac{\|x+y\|}{2} : x, y \in S_X,\ \|x-y\| \geq \epsilon \right\}. \tag{2.5}$$

The function $\delta_X(\epsilon)$ is strictly increasing on $[\epsilon_0(X), 2]$. Here $\epsilon_0(X) = \sup\{\epsilon : \delta_X(\epsilon) = 0\}$ is the characteristic of convexity of X, and the space is called uniformly nonsquare if $\epsilon_0(X) < 2$.

In [23] the author introduces a modulus that scales the 3-dimensional convexity of the unit ball: he considers the number

$$\widetilde{\delta}(X) = \sup\{\varepsilon \in [0,2] : \exists x, y, z \in B_X \text{ such that } \min\{\|x-y\|, \|x-z\|, \|y-z\|\} \geq \varepsilon\} \tag{2.6}$$

and defines the function $\widetilde{\delta}_X : [0, \widetilde{\delta}(X)) \to [0,1]$ by

$$\widetilde{\delta}_X(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y+z}{3} \right\| : x, y, z \in B_X,\ \min\{\|x-y\|, \|x-z\|, \|y-z\|\} \geq \varepsilon \right\}. \tag{2.7}$$

He also considers the coefficient corresponding to this modulus:

$$\widetilde{\varepsilon}_0(X) := \sup\left\{ \varepsilon \in \left[0, \widetilde{\delta}(X)\right) : \widetilde{\delta}(X) = 0 \right\}. \tag{2.8}$$

It is evident that $\widetilde{\delta}_X(\varepsilon) \geq \delta_X(\varepsilon)$ for all $\varepsilon \in [0, \widetilde{\delta}(X))$ and in consequence $\widetilde{\varepsilon}_0(X) \leq \varepsilon_0(X)$. Moreover this last inequality can be strict, since it was shown in [23] the existence of Banach spaces with $\widetilde{\varepsilon}_0(X) < 2$ which are not uniformly nonsquare.

The weakly convergent sequence coefficient WCS(X) of X is defined as follows: $\mathrm{WCS}(X) = \inf\{\lim_{n \neq m}\|x_n - x_m\|\}$ where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n\to\infty}\|x_n\| = 1$ and $\lim_{n,m\to\infty, n\neq m}\|x_n - x_m\|$ exist.

The WORTH property was introduced by Sims in [24] as follows. A Banach space X has the WORTH property if

$$\lim_{n\to\infty} |\|x_n + x\| - \|x_n - x\|| = 0, \tag{2.9}$$

for all $x \in X$ and all weakly null sequences (x_n). In [25], Jiménez-Melado and Llorens-Fuster defined the coefficient of weak orthogonality, which measures the degree of WORTH wholeness , by

$$\mu(X) = \inf\left\{ \lambda : \limsup_{n\to\infty} \|x_n + x\| \le \lambda \limsup_{n\to\infty} \|x_n - x\| \right\}, \tag{2.10}$$

where the infimum is taken over all $x \in X$ and all weakly null sequence (x_n). It is known that X has the WORTH property if and only if $\mu(X) = 1$.

Let C be a nonempty subset of a Banach space X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in C, \tag{2.11}$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\left\{ \sup_{x\in A}\inf_{y\in B} \|x - y\|, \ \sup_{y\in B}\inf_{x\in A} \|x - y\| \right\}, \quad A, B \in CB(X). \tag{2.12}$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C, \{x_n\})$ and the asymptotic center $A(C, \{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C, \{x_n\}) = \inf\left\{ \limsup_n \|x_n - x\| : x \in C \right\}, \tag{2.13}$$

$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}, \tag{2.14}$$

respectively. It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is.

The sequence $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences x_{n_i} of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.1. (i) *(See Goebel [26] and Lim [27]) There always exists a subsequence of $\{x_n\}$ which is regular with respect to C.*

(ii) *(See Kirk [28]) If C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C.*

If D is a bounded subset of X, then the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x \in C} \sup_{y \in D} \|x - y\|. \tag{2.15}$$

Dhompongsa et al. [4] introduced the property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset C of X, any sequence $\{x_n\} \subset C$ which is regular asymptotically uniform relative to C, and any sequence $\{y_n\} \subset A(C,\{x_n\})$ which is regular asymptotically uniform relative to X we have

$$r(C,\{y_n\}) \leq \lambda r(C,\{x_n\}). \tag{2.16}$$

The Domínguez-Lorenzo condition, DL condition in short form, introduced in [3] is defined as follows: if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C we have,

$$r_C(A(C,\{x_n\})) \leq \lambda r(C,\{x_n\}). \tag{2.17}$$

It is clear from the definition that property (D) is weaker than the DL condition. The next results show that property (D) is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings [4].

Theorem 2.2. *Let X be a Banach space satisfying property (D). Then X has weak normal structure.*

Theorem 2.3. *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies the property (D). Let $T : C \to KC(C)$ be a nonexpansive mapping, then T has a fixed point.*

3. Main Results

Theorem 3.1. *Let C be a weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in C regular with respect to C. Then for every $a \in [0,2]$,*

$$r_C(A(C,\{x_n\})) \leq \frac{J(a,X)}{1 + |1-a|/\mu(X)} r(C,\{x_n\}). \tag{3.1}$$

Proof. Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\})$. We can assume that $r > 0$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, then passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence$\{x_n\}$. Let $z \in A$, then we have that

$$\limsup_n \|x_n - z\| = r. \tag{3.2}$$

Denote $\mu = \mu(X)$. By the definition of μ, we have that

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \|(x_n - x) + (z - x)\| \\ &\leq \mu \limsup_n \|(x_n - x) - (z - x)\| = \mu r.\end{aligned} \tag{3.3}$$

Convexity of C implies that $(2/(1+\mu))x + ((\mu - 1)/(1+\mu))z \in C$, and thus, we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{1+\mu} x + \frac{\mu - 1}{1+\mu} z \right) \right\| \geq r. \tag{3.4}$$

On the other hand, by the weak lower semicontinuity of the norm, we have that

$$\liminf_n \left\| \left(1 - \frac{1-a}{\mu}\right)(x_n - x) - \left(1 + \frac{1-a}{\mu}\right)(z - x) \right\| \geq \left(1 + \frac{|1-a|}{\mu}\right)\|z - x\|. \tag{3.5}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leq r + \varepsilon$,

(2) $\|x_N - 2x + z\| \leq \mu(r + \varepsilon)$,

(3) $\|x_N - ((2/(1+\mu))x + ((\mu - 1)/(1+\mu))z)\| \geq r - \varepsilon$,

(4) $\|(1 - (1-a)/\mu)(x_N - x) - (1 + (1-a)/\mu)(z - x)\| \geq (1 + |1-a|/\mu)\|z - x\|((r - \varepsilon)/r)$.

Now, put $u = (1/(r+\varepsilon))(x_N - z)$, $v = (1/\mu(r+\varepsilon))(x_N - 2x + z)$, and $\omega = ((1-a)/\mu(r+\varepsilon))(x_N - 2x + z)$. Using the above estimates, we obtain $u, v, \omega \in B_X$, $\|v - \omega\| \leq a\|u\|$, and

$$\begin{aligned}\|u + v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z - x}{\mu(r+\varepsilon)} \right\| \\ &= \left\| \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)} \right)(x_N - x) - \left(\frac{1}{r+\varepsilon} - \frac{1}{\mu(r+\varepsilon)} \right)(z - x) \right\| \\ &= \frac{1}{r+\varepsilon}\left(1 + \frac{1}{\mu}\right) \left\| (x_N - x) - \left(\frac{1 - 1/\mu}{1 + 1/\mu} \right)(z - x) \right\| \\ &= \frac{1}{r+\varepsilon}\left(1 + \frac{1}{\mu}\right) \left\| x_N - \left(\frac{2}{1+\mu} x + \frac{\mu - 1}{1+\mu} z \right) \right\| \\ &\geq \left(1 + \frac{1}{\mu}\right)\left(\frac{r - \varepsilon}{r + \varepsilon} \right), \\ \|u - \omega\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} - \frac{(1-a)(x_N - x)}{\mu(r+\varepsilon)} - \frac{(1-a)(z - x)}{\mu(r+\varepsilon)} \right\| \\ &= \frac{1}{r+\varepsilon} \left\| \left(1 - \frac{1-a}{\mu}\right)(x_N - x) - \left(1 + \frac{1-a}{\mu}\right)(z - x) \right\| \\ &\geq \left(1 + \frac{|1-a|}{\mu}\right) \frac{\|z - x\|}{r} \left(\frac{r - \varepsilon}{r + \varepsilon} \right).\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned} J(a,X) &\geq \|u+v\| \wedge \|u-\omega\| \\ &\geq \left(1+\frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right) \wedge \left(1+\frac{|1-a|}{\mu}\right)\frac{\|z-x\|}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right). \end{aligned} \tag{3.7}$$

By the weak lower semicontinuity of the norm again, we conclude that $\|z-x\| \leq r$, and hence,

$$\left(1+\frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right) \wedge \left(1+\frac{|1-a|}{\mu}\right)\frac{\|z-x\|}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right) = \left(1+\frac{|1-a|}{\mu}\right)\frac{\|z-x\|}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right). \tag{3.8}$$

Therefore $J(a,X) \geq (1+|1-a|/\mu)(\|z-x\|/r)(r-\varepsilon/r+\varepsilon)$. Since the above inequality is true for every $\varepsilon > 0$ and every $z \in A$, we obtain

$$\sup_{z\in A}\|x-z\| \leq \left(\frac{J(a,X)}{1+|1-a|/\mu}\right)r, \tag{3.9}$$

and therefore,

$$r_C(A) \leq \left(\frac{J(a,X)}{1+|1-a|/\mu}\right)r. \tag{3.10}$$

□

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $J(a,X) < 1+|1-a|/\mu(X)$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof. When $J(a,X) < 1+|1-a|/\mu(X)$, then X satisfies the DL condition by Theorem 3.1. So T has a fixed point by Theorem 2.3. □

Remark 3.3. In particular, when $a = 0$, we get the result of Kaewkhao; a Banach space X with

$$J(X) < 1 + \frac{1}{\mu(X)} \tag{3.11}$$

satisfies the DL condition.

Theorem 3.4. *Let C be a weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in C regular with respect to C. Then for every $a \in [0,2]$,*

$$r_C(A(C,\{x_n\})) \leq \left(\sqrt{\frac{C_{\mathrm{NJ}}(a,X)\left[2\mu^4+\mu^2+(1-a)^2\mu^2\right]-\left(\mu^2+1\right)^2}{\left(\mu^2+1-a\right)^2}}\right) r(C,\{x_n\}). \tag{3.12}$$

Proof. Let $r, A, \{x_n\}, x, z$, and μ be as in the proof of the previous theorem. Thus,

$$\limsup_n \|x_n - z\| = r,$$
$$\limsup_n \|x_n - 2x + z\| \le \mu r. \tag{3.13}$$

Since $(2/(\mu^2+1))x + (\mu^2-1)/(\mu^2+1)z \in C$ and by the definition of r, we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \right) \right\| \ge r. \tag{3.14}$$

On the other hand, by the weak lower semicontinuity of the norm, we have that

$$\liminf_n \left\| \left(\mu^2 - 1 + a\right)(x_n - x) - \left(\mu^2 + 1 - a\right)(z - x) \right\| \ge \left(\mu^2 + 1 - a\right)\|z - x\|. \tag{3.15}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \le r + \varepsilon$,

(2) $\|x_N - 2x + z\| \le \mu(r + \varepsilon)$,

(3) $\|x_N - ((2/(\mu^2+1))x + ((\mu^2-1)/(\mu^2+1))z)\| \ge r - \varepsilon$,

(4) $\|(\mu^2 - 1 + a)(x_N - x) - (\mu^2 + 1 - a)(z - x)\| \ge (\mu^2 + 1 - a)\|z - x\|((r - \varepsilon)/r)$.

Now, put $u = \mu^2(x_N - z)$, $v = (x_N - 2x + z)$, and $\omega = (1 - a)(x_N - 2x + z)$ and use the above estimates to obtain $\|u\| \le \mu^2(r + \varepsilon)$, $\|v\| \le \mu(r + \varepsilon)$, $\|\omega\| \le (|1 - a|)\mu(r + \varepsilon)$, and $\|v - \omega\| \le a\|u\|$, so that

$$\begin{aligned}
\|u + v\| &= \left\| \mu^2((x_N - x) - (z - x)) + (x_N - x) + (z - x) \right\| \\
&= \left(\mu^2 + 1\right) \left\| (x_N - x) - \frac{\mu^2 - 1}{\mu^2 + 1}(z - x) \right\| \\
&= \left(\mu^2 + 1\right) \left\| x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \right) \right\| \\
&\ge \left(\mu^2 + 1\right)(r - \varepsilon), \\
\|u - \omega\| &= \left\| \mu^2((x_N - x) - (z - x)) - (1 - a)((x_N - x) + (z - x)) \right\| \\
&= \left\| \left(\mu^2 - 1 + a\right)(x_N - x) - \left(\mu^2 + 1 - a\right)(z - x) \right\| \\
&\ge \left(\mu^2 + 1 - a\right)\|z - x\| \left(\frac{r - \varepsilon}{r} \right).
\end{aligned} \tag{3.16}$$

By the definition of $C_{NJ}(a,X)$, we get

$$C_{NJ}(a,X) \geq \left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \right\} \geq \left(\frac{r-\varepsilon}{r+\varepsilon}\right)^2 \frac{(\mu^2+1)^2 + (\mu^2+1-a)^2(\|z-x\|/r)^2}{2\mu^4 + \mu^2 + (1-a)^2\mu^2}. \tag{3.17}$$

Let $\varepsilon \to 0^+$; we obtain that $C_{NJ}(a,X) \geq ((\mu^2+1)^2 + (\mu^2+1-a)^2(\|z-x\|/r)^2)/(2\mu^4+\mu^2+(1-a)^2\mu^2)$. Then we have

$$\|z-x\| \leq \left(\sqrt{\frac{C_{NJ}(a,X)\left[2\mu^4+\mu^2+(1-a)^2\mu^2\right] - (\mu^2+1)^2}{(\mu^2+1-a)^2}}\right) r. \tag{3.18}$$

This holds for arbitrary $z \in A$; hence, we have that

$$r_C(A) \leq \left(\sqrt{\frac{C_{NJ}(a,X)\left[2\mu^4+\mu^2+(1-a)^2\mu^2\right] - (\mu^2+1)^2}{(\mu^2+1-a)^2}}\right) r. \tag{3.19}$$

□

Corollary 3.5. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{NJ}(a,X) < ((\mu^2+1-a)^2 + (\mu^2+1)^2)/(2\mu^4+\mu^2+(1-a)^2\mu^2)$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof. When $C_{NJ}(a,X) < ((\mu^2+1-a)^2 + (\mu^2+1)^2)/(2\mu^4+\mu^2+(1-a)^2\mu^2)$, then X satisfies the DL condition by Theorem 3.4. So T has a fixed point by Theorem 2.3. □

Remark 3.6. In particular, when $a = 0$, we get the result of Kaewkhao; a Banach space X with

$$C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2} \tag{3.20}$$

satisfies the DL condition.

Repeating the arguments in the proof of Theorem 3.4, we can easily get the following conclusion.

Theorem 3.7. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_Z(a,X) < (2(\mu^2+1)(\mu^2+1-a))/(2\mu^4+\mu^2+(1-a)^2\mu^2)$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Remark 3.8. In particular, when $a = 0$, we get that

$$C_Z(X) < 1 + \frac{1}{\mu(X)^2} \tag{3.21}$$

satisfies the DL condition which improves the result of Kaewkhao; a Banach space X with

$$C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2} \tag{3.22}$$

satisfies the DL condition.

Theorem 3.9. *A Banach space X has property (D) whenever $\widetilde{\varepsilon}_0(X) < WCS(X)$.*

Proof. Let C be a nonempty weakly compact convex subset of X. Suppose that $\{x_n\} \subset C$ and $\{y_n\} \subset A(C, \{x_n\})$ are regular asymptotically uniforms relative to C. Passing to a subsequence, we may assume that $\{y_n\}$ is weakly convergent to a point $y_0 \in C$ and $d = \lim_{n \neq m}\|y_n - y_m\|$ exists. Let $r = r(C, \{x_n\})$. Again, passing to a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, we assume in addition that

$$\begin{gathered} \|x_n - y_n\| \leq r + \frac{1}{n}, \qquad \|x_n - y_{n+1}\| \leq r + \frac{1}{n}, \qquad \|x_n - y_{n+2}\| \leq r + \frac{1}{n}, \\ \left\|x_n - \frac{1}{3}(y_n + y_{n+1} + y_{n+2})\right\| \geq r - \frac{1}{n}, \end{gathered} \tag{3.23}$$

for all $n \in \mathbb{N}$. Now, put

$$\begin{aligned} u_n &= \frac{1}{r + 1/n}(x_n - y_n), \\ v_n &= \frac{1}{r + 1/n}(x_n - y_{n+1}), \\ \omega_n &= \frac{1}{r + \dfrac{1}{n}}(x_n - y_{n+2}). \end{aligned} \tag{3.24}$$

It is easy to see that $\lim_n\|u_n - v_n\| = d/r$, $\lim_n\|u_n - \omega_n\| = d/r$, $\lim_n\|\omega_n - v_n\| = d/r$, and $\lim_n\|u_n + v_n + \omega_n\| = 3$. This implies that $\widetilde{\delta}_X(d/r) = 0$ or $\widetilde{\varepsilon}_0(X) \geq d/r$. Now we estimate d as follows:

$$\begin{aligned} d = \lim_{n \neq m}\|y_n - y_m\| &= \lim_{n \neq m}\|(y_n - y_0) - (y_m - y_0)\| \geq \mathrm{WCS}(X)\limsup_n\|y_n - y_0\| \\ &\geq \mathrm{WCS}(X) r(C, \{y_n\}). \end{aligned} \tag{3.25}$$

Hence,

$$r(C, \{y_n\}) \leq \frac{\tilde{\varepsilon}_0(X)}{\mathrm{WCS}(X)} r(C, \{x_n\}). \tag{3.26}$$

□

Remark 3.10. (1) Theorem 3.9 strengthens the result of Saejung [7] and X has property (D) whenever $\varepsilon_0(X) < \mathrm{WCS}(X)$.

(2) Theorem 3.9 also improves the result $\tilde{\varepsilon}_0(X) < 1$ implying that the Banach space X has normal structure from Theorem 2.2.

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