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Research Article

A Strong Convergence Theorem for a Family of Quasi-ϕ-Nonexpansive Mappings in a Banach Space

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The purpose of this paper is to propose a modified hybrid projection algorithm and prove a strong convergence theorem for a family of quasi-ϕ-nonexpansive mappings. The strong convergence theorem is proven in the more general reflexive, strictly convex, and smooth Banach spaces with the property (K). The results of this paper improve and extend the results of S. Matsushita and W. Takahashi (2005), X. L. Qin and Y. F. Su (2007), and others.

1. Introduction

It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, the so-called hybrid projection iteration method is such a modification.

The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [1] in 1968. For 40 years, HPIA has received rapid developments. For details, the readers are referred to papers [2–7] and the references therein.

In 2005, Matsushita and Takahashi [5] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily},\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n),\\
&C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n}(x_0).
\end{aligned} \tag{1.1}$$

They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.1), where J is the normalized duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

In 2007, Qin and Su [2] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily,}\\
&z_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) JTx_n\right),\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTz_n),\\
&C_n = \left\{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\right\},\\
&Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n}(x_0).
\end{aligned} \tag{1.2}$$

They proved the following convergence theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself such that* $\mathrm{Fix}(T) \neq \emptyset$. *Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$ and $\beta_n \to 1$. Suppose that $\{x_n\}$ is given by (1.2). If T is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{Fix}(T)}x_0$.*

Question 1. Can both Theorems MT and QS be extended to more general reflexive, strictly convex, and smooth Banach spaces with the property (K)?

Question 2. Can both Theorems MT and QS be extended to more general class of quasi-ϕ-nonexpansive mappings?

The purpose of this paper is to give some affirmative answers to the questions mentioned previously, by introducing a modified hybrid projection iteration algorithm and by proving a strong convergence theorem for a family of closed and quasi-ϕ-nonexpansive mappings by using new analysis techniques in the setting of reflexive, strictly convex, and smooth Banach spaces with the property (K). The results of this paper improve and extend the results of Matsushita and Takahashi [5], Qin and Su [2], and others.

2. Preliminaries

In this paper, we denote by X and X^* a Banach space and the dual space of X, respectively. Let C be a nonempty closed convex subset of X. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \left\{ j \in X^* : \langle x, j \rangle = \|x\|^2 = \|j\|^2 \right\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between X and X^*. It is well known that if X is reflexive, strictly convex, and smooth, then $J : X \to X^*$ is single-valued, demicontinuous and strictly monotone (see, e.g., [8, 9]).

It is also very well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [10] recently introduced a generalized projection operator Π_C in a Banach space X which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that X is a real reflexive, strictly convex, and smooth Banach space. Let us consider the functional defined as in [4, 5] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.2}$$

Observe that, in a Hilbert space H, (2.2) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

The generalized projection $\Pi_C : X \to C$ is a map that assigns to an arbitrary point $x \in X$ the unique minimum point of the functional $\phi(\cdot, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the unique solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.3}$$

Remark 2.1. The existence and uniqueness of the element $\overline{x} \in C$ follow from the reflexivity of X, the properties of the functional $\phi(\cdot, x)$, and strict monotonicity of the mapping J (see, e.g., [8–12]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \quad \forall x, y \in X. \tag{2.4}$$

Remark 2.2. If X is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.4), we have $\|x\| = \|y\|$. This in turn implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the smoothness of X, we know that J is single valued, and hence we have $Jx = Jy$. Since X is strictly convex, J is strictly monotone, in particular, J is one to one, which implies that $x = y$; one may consult [8, 9] for the details.

Let C be a closed convex subset of X, and T a mapping from C into itself. A point p in C is said to be asymptotic fixed point of T [13] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive [5, 14–16] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [14–16].

T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 2.3. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [5, 14–16] which requires the strong restriction: $\widetilde{F(T)} = F(T)$.

We present two examples which are closed and quasi-ϕ-nonexpansive.

Example 2.4. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space X onto a nonempty closed convex subset C of X. Then, Π_C is a closed and quasi-ϕ-nonexpansive mapping from X onto C with $F(\Pi_C) = C$.

Example 2.5. Let X be a reflexive, strictly convex, and smooth Banach space, and $A \subset X \times X^*$ is a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed and quasi-ϕ-nonexpansive mapping from X onto $D(A)$ and $F(J_r) = A^{-1}0$.

Recall that a Banach space X has the property (K) if for any sequence $\{x_n\} \subset X$ and $x \in X$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning property (K) the reader is referred to [17] and references cited therein.

In order to prove our main result of this paper, we need to the following facts.

Lemma 2.6 (see, e.g., [10–12]). *Let C be a convex subset of a real smooth Banach space X, $x \in X$, and $x_0 \in C$. Then,*

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\} \tag{2.5}$$

if and only if

$$\langle z - x_0, Jx_0 - Jx \rangle \geq 0, \quad \forall z \in C. \tag{2.6}$$

Lemma 2.7 (see, e.g., [10–12]). *Let C be a convex subset of a real reflexive, strictly convex, and smooth Banach space X. Then the following inequality holds:*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \tag{2.7}$$

for all $x \in X$ and $y \in C$.

Now we are in a proposition to prove the main results of this paper.

3. Main Results

Theorem 3.1. *Let X be a reflexive, strictly convex, smooth Banach space such that X and X^* have the property (K). Assume that C is a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinitely countable family of closed and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$.*

Assume that $\{\alpha_{n,i}\}$ *are real sequences in* $[0,1]$ *such that* $b_{0,i} = \liminf_{n\to\infty}\alpha_{n,i} < 1$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{aligned}
&x_0 \in X \text{ chosen arbitrarily},\\
&C_{1,i} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \quad x_1 = \Pi_{C_1}(x_0),\\
&y_{n,i} = J^{-1}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})J(T_i x_n)), \quad n \ge 1,\\
&C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \le \phi(z, x_n)\},\\
&C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i},\\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0.
\end{aligned} \tag{3.1}$$

Then $\{x_n\}$ *converges strongly to* $p_0 = \Pi_F x_0$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. We split the proof into six steps.

Step 1. Show that $\Pi_F x_0$ is well defined for every $x_0 \in X$.

To this end, we prove first that $F(T_i)$ is closed and convex for any $i \in N$. Let $\{p_n\}$ be a sequence in $F(T_i)$ with $p_n \to p$ as $n \to \infty$, we prove that $p \in F(T_i)$. From the definition of quasi-ϕ-nonexpansive mappings, one has $\phi(p_n, T_i p) \le \phi(p_n, p)$, which implies that $\phi(p_n, T_i p) \to 0$ as $n \to \infty$. Noticing that

$$\phi(p_n, T_i p) = \|p_n\|^2 - 2\langle p_n, J(T_i p)\rangle + \|T_i p\|^2. \tag{3.2}$$

By taking limit in (3.2), we have

$$\lim_{n\to\infty} \phi(p_n, T_i p) = \|p\|^2 - 2\langle p, J(T_i p)\rangle + \|T_i p\|^2 = \phi(p, T_i p). \tag{3.3}$$

Hence $\phi(p, T_i p) = 0$. It implies that $p = T_i p$ for all $i \in N$. We next show that $F(T_i)$ is convex. To this end, for arbitrary $p_1, p_2 \in F(T_i)$, $t \in (0,1)$, putting $p_3 = tp_1 + (1-t)p_2$, we prove that $T_i p_3 = p_3$. Indeed, by using the definition of $\phi(x,y)$, we have

$$\begin{aligned}
\phi(p_3, T_i p_3) &= \|p_3\|^2 - 2\langle p_3, J(T_i p_3)\rangle + \|T_i p_3\|^2\\
&= \|p_3\|^2 - 2\langle tp_1 + (1-t)p_2, J(T_i p_3)\rangle + \|T_i p_3\|^2\\
&= \|p_3\|^2 - 2t\langle p_1, J(T_i p_3)\rangle - 2(1-t)\langle p_2, J(T_i p_3)\rangle + \|T_i p_3\|^2\\
&= \|p_3\|^2 + t\phi(p_1, T_i p_3) + (1-t)\phi(p_2, T_i p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2\\
&\le \|p_3\|^2 + t\phi(p_1, p_3) + (1-t)\phi(p_2, p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2\\
&= \|p_3\|^2 - 2\langle p_3, Jp_3\rangle + \|p_3\|^2\\
&= 0.
\end{aligned} \tag{3.4}$$

This implies that $T_i p_3 = p_3$. Hence $F(T_i)$ is closed and convex for all $i \in N$ and consequently $F = \bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex. By our assumption that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, we have $\Pi_F x_0$ is well defined for every $x_0 \in X$.

Step 2. Show that C_n is closed and convex for each $n \geq 1$.

It suffices to show that for any $i \in N$, $C_{n,i}$ is closed and convex for every $n \geq 1$. This can be proved by induction on n. In fact, for $n = 1$, $C_{1,i} = C$ is closed and convex. Assume that $C_{n,i}$ is closed and convex for some $n \geq 1$. For $z \in C_{n+1,i}$, one obtains that

$$\phi(z, y_{n,i}) \leq \phi(z, x_n) \tag{3.5}$$

is equivalent to

$$2\langle z, Jx_n - Jy_{n,i}\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2. \tag{3.6}$$

It is easy to see that $C_{n+1,i}$ is closed and convex. Then, for all $n \geq 1$, $C_{n,i}$ is closed and convex. Consequently, $C_n = \bigcap_{i=1}^{\infty} C_{n,i}$ is closed and convex for all $n \geq 1$.

Step 3. Show that $F = \bigcap_{i=1}^{\infty} F(T_i) \subset \bigcap_{n=1}^{\infty} C_n = D$.

It suffices to show that for any $i \in N$, $F \subset C_{n,i}$ for every $n \geq 1$. For any $c_0 \in F$, from the definition of quasi-ϕ-nonexpansive mappings, we have $\phi(c_0, T_i x) \leq \phi(c_0, x)$, for all $x \in C$ and $i \in N$. Noting that for any $x \in C$ and $\alpha \in [0,1]$, we have

$$\begin{aligned}
&\phi\Big(c_0, J^{-1}(\alpha Jx + (1-\alpha)J(T_i x))\Big) \\
&\quad = \|c_0\|^2 - 2\langle c_0, \alpha Jx + (1-\alpha)J(T_i x)\rangle + \Big\|J^{-1}(\alpha Jx + (1-\alpha)J(T_i x))\Big\|^2 \\
&\quad \leq \|c_0\|^2 - 2\langle c_0, \alpha Jx + (1-\alpha)J(T_i x)\rangle + \alpha\|x\|^2 + (1-\alpha)\|T_i x\|^2 \\
&\quad = \alpha\phi(c_0, x) + (1-\alpha)\phi(c_0, T_i x) \\
&\quad \leq \alpha\phi(c_0, x) + (1-\alpha)\phi(c_0, x) \\
&\quad = \phi(c_0, x),
\end{aligned} \tag{3.7}$$

which implies that $c_0 \in C_{n,i}$ and consequently $F \subset C_{n,i}$. So $F \subset \bigcap_{n=1}^{\infty} C_n$. Hence $x_{n+1} = \Pi_{C_{n+1}} x_0$ is well defined for each $n \geq 0$. Therefore, the iterative algorithm (3.1) is well defined.

Step 4. Show that $\|x_n - p_0\| \to 0$, where $p_0 = \Pi_D x_0$.

From Steps 2 and 3, we obtain that D is a nonempty, closed, and convex subset of C. Hence $\Pi_D x_0$ is well defined for every $x_0 \in C$. From the construction of C_n, we know that

$$C \supset C_1 \supset C_2 \supset \cdots. \tag{3.8}$$

Let $p_0 = \Pi_D x_0$, where $p_0 \in C$ is the unique element that satisfies $\inf_{x\in D}\phi(x, x_0) = \phi(p_0, x_0)$. Since $x_n = \Pi_{C_n} x_0$, by Lemma 2.7, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \le \cdots \le \phi(p_0, x_0). \tag{3.9}$$

By the reflexivity of X, we can assume that $x_n \to g_1 \in X$ weakly. Since $C_j \subset C_n$, for $j \ge n$, we have $x_j \in C_n$ for $j \ge n$. Since C_n is closed and convex, by the Marzur theorem, $g_1 \in C_n$ for any $n \in N$. Hence $g_1 \in D$. Moreover, by using the weakly lower semicontinuity of the norm on X and (3.9), we obtain

$$\begin{aligned}\phi(p_0, x_0) &\le \phi(g_1, x_0) \le \liminf_{n\to\infty}\phi(x_n, x_0)\\ &\le \limsup_{n\to\infty}\phi(x_n, x_0) \le \inf_{x\in D}\phi(x, x_0) = \phi(p_0, x_0),\end{aligned} \tag{3.10}$$

which implies that $\lim_{n\to\infty}\phi(x_n, x_0) = \phi(p_0, x_0) = \phi(g_1, x_0) = \inf_{x\in D}\phi(x, x_0)$. By using Lemma 2.6, we have

$$\langle p_0 - g_1, Jp_0 - Jg_1\rangle = 0, \tag{3.11}$$

and hence $p_0 = g_1$, since J is strictly monotone.

Further, by the definition of ϕ, we have

$$\lim_{n\to\infty}\left(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\right) = \left(\|p_0\|^2 - 2\langle p_0, Jx_0\rangle + \|x_0\|^2\right), \tag{3.12}$$

which shows that $\lim_{n\to\infty}\|x_n\| = \|p_0\|$. By the property (K) of X, we have $\|x_n - p_0\| \to 0$, where $p_0 = \Pi_D x_0$.

Step 5. Show that $p_0 = T_i p_0$, for any $i \in N$.

Since $x_{n+1} \in C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}$ for all $n \ge 0$ and $i \in N$, we have

$$0 \le \phi(x_{n+1}, y_{n,i}) \le \phi(x_{n+1}, x_n). \tag{3.13}$$

Since $\|x_n - p_0\| \to 0, \phi(x_{n+1}, x_n) \to 0$ and consequently

$$\phi(x_{n+1}, y_{n,i}) \longrightarrow 0. \tag{3.14}$$

Note that $0 \le (\|x_{n+1}\| - \|y_{n,i}\|)^2 \le \phi(x_{n+1}, y_{n,i})$. Hence $\|y_{n,i}\| \to \|p_0\|$ and consequently $\|J(y_{n,i})\| \to \|Jp_0\|$. This implies that $\{J(y_{n,i})\}$ is bounded. Since X is reflexive, X^* is also reflexive. So we can assume that

$$J(y_{n,i}) \longrightarrow f_0 \in X^* \tag{3.15}$$

weakly. On the other hand, in view of the reflexivity of X, one has $J(X) = X^*$, which means that for $f_0 \in X^*$, there exists $x \in X$, such that $J(x) = f_0$. It follows that

$$\begin{aligned}
\lim_{n\to\infty} \phi(x_{n+1}, y_{n,i}) &= \lim_{n\to\infty}\left[\|x_{n+1}\|^2 - 2\langle x_{n+1}, J(y_{n,i})\rangle + \|y_{n,i}\|^2\right] \\
&= \lim_{n\to\infty}\left[\|x_{n+1}\|^2 - 2\langle x_{n+1}, J(y_{n,i})\rangle + \|J(y_{n,i})\|^2\right] \\
&\geq \|p_0\|^2 - 2\langle p_0, f_0\rangle + \|f_0\|^2 \\
&= \|p_0\|^2 - 2\langle p_0, Jx\rangle + \|Jx\|^2 \\
&= \phi(p_0, x),
\end{aligned} \tag{3.16}$$

where we used the weakly lower semicontinuity of the norm on X^*. From (3.14), we have $\phi(p_0, x) = 0$ and consequently $p_0 = x$, which implies that $f_0 = Jp_0$. Hence

$$J(y_{n,i}) \longrightarrow Jp_0 \in X^* \tag{3.17}$$

weakly. Since $\|J(y_{n,i})\| \to \|Jp_0\|$ and X^* has the property (K), we have

$$\|J(y_{n,i}) - Jp_0\| \longrightarrow 0. \tag{3.18}$$

Since $\|x_n - p_0\| \to 0$, noting that $J : X \to X^*$ is demi-continuous, we have

$$Jx_n \longrightarrow Jp_0 \in X^* \tag{3.19}$$

weakly. Noticing that

$$\big|\|Jx_n\| - \|Jp_0\|\big| = \big|\|x_n\| - \|p_0\|\big| \leq \|x_n - p_0\| \longrightarrow 0, \tag{3.20}$$

which implies that $\|Jx_n\| \to \|Jp_0\|$. By using the property (K) of X^*, we have

$$\|Jx_n - Jp_0\| \longrightarrow 0. \tag{3.21}$$

From (3.1), (3.18), (3.21), and $b_{0,i} = \liminf_{n\to\infty} \alpha_{n,i} < 1$, we have

$$\|J(T_i x_n) - Jp_0\| \longrightarrow 0. \tag{3.22}$$

Since $J^{-1} : X^* \to X$ is demi-continuous, we have

$$T_i x_n \longrightarrow p_0 \tag{3.23}$$

weakly in X. Moreover,

$$\big|\|T_i x_n\| - \|p_0\|\big| = \big|\|J(T_i x_n)\| - \|Jp_0\|\big| \leq \|J(T_i x_n) - Jp_0\| \longrightarrow 0, \tag{3.24}$$

which implies that $\|T_i x_n\| \to \|p_0\|$. By the property (K) of X, we have

$$T_i x_n \longrightarrow p_0. \tag{3.25}$$

From $\|x_n - p_0\| \to 0$ and the closeness property of T_i, we have

$$T_i p_0 = p_0, \tag{3.26}$$

which implies that $p_0 \in F = \bigcap_{i=1}^{\infty} F(T_i)$.

Step 6. Show that $p_0 = \Pi_F x_0$.

It follows from Steps 3, 4, and 5 that

$$\phi(p_0, x_0) \leq \phi(\Pi_F x_0, x_0) \leq \phi(p_0, x_0), \tag{3.27}$$

which implies that $\phi(\Pi_F x_0, x_0) = \phi(p_0, x_0)$. Hence, $p_0 = \Pi_F x_0$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_F x_0$. This completes the proof. □

From Theorem 3.1, we can obtain the following corollary.

Corollary 3.2. *Let X be a reflexive, strictly convex and smooth Banach space such that both X and X^* have the property (K). Assume that C is a nonempty closed convex subset of X. Let $T : C \to C$ be a closed and quasi-ϕ-nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $b_0 = \liminf_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily},\\
&C_1 = C, \quad x_1 = \Pi_{C_1}(x_0),\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J(T x_n)), \quad n \geq 1,\\
&C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0.
\end{aligned} \tag{3.28}$$

Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Remark 3.3. Theorem 3.1 and its corollary improve and extend Theorems MT and QS at several aspects.

(i) From uniformly convex and uniformly smooth Banach spaces extend to reflexive, strictly convex and smooth Banach spaces with the property (K). In Theorem 3.1 and its corollary the hypotheses on X are weaker than the usual assumptions of uniform convexity and uniform smoothness. For example, any strictly convex, reflexive and smooth Musielak-Orlicz space satisfies our assumptions [17] while, in general, these spaces need not to be uniformly convex or uniformly smooth.

(ii) From relatively nonexpansive mappings extend to closed and quasi-ϕ-nonexpansive mappings.

(iii) The continuity assumption on mapping T in Theorem QS is removed.

(iv) Relax the restriction on $\{\alpha_n\}$ from $\limsup_{n\to\infty}\alpha_n < 1$ to $\liminf_{n\to\infty}\alpha_n < 1$.

Remark 3.4. Corollary 3.2 presents some affirmative answers to Questions 1 and 2.

4. Applications

In this section, we present some applications of the main results in Section 3.

Theorem 4.1. *Let X be a reflexive, strict, and smooth Banach space that both X and X^* have the property (K), and let C be a nonempty closed convex subset of X. Let $\{f_i\}_{i\in\mathbb{N}} : X \to (-\infty, +\infty]$ be a family of proper, lower semicontinuous, and convex functionals. Assume that the common fixed point set $F = \bigcap_{i\in\mathbb{N}} F(J_i)$ is nonempty, where $J_i = (J + r_i\partial f_i)^{-1}J$ for $r_i > 0$ and $i \in \mathbb{N}$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{aligned}
&x_0 \in X \text{ chosen arbitrarily},\\
&C_{1,i} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \quad x_1 = \Pi_{C_1}(x_0),\\
&y_{n,i} = J^{-1}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})Jz_{n,i}), \quad n \geq 1,\\
&z_{n,i} = \operatorname*{argmin}_{z\in X}\left\{ f_i(z) + \frac{1}{2r_i}\|z\|^2 - \frac{1}{r_i}\langle z, Jx_n\rangle \right\},\\
&C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \phi(z, x_n)\},\\
&C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i},\\
&x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 0,
\end{aligned} \tag{4.1}$$

where $\{\alpha_{n,i}\}$ satisfies the restriction: $0 \leq \alpha_{n,i} < 1$ and $\liminf_{n\to\infty}\alpha_{n,i} < 1$. Then $\{x_n\}$ defined by (4.1) *converges strongly to a minimizer $\Pi_F x_0$ of the family $\{f_i\}_{i\in\mathbb{N}}$.*

Proof. By a result of Rockafellar [18], we see that $\partial f_i : X \to 2^{X^*}$ is a maximal monotone mapping for every $i \in \mathbb{N}$. It follows from Example 2.5 that $J_i : X \to X$ is a closed and quasi-ϕ-nonexpansive mapping for every $i \in \mathbb{N}$. Notice that

$$z_{n,i} = \operatorname*{argmin}_{z\in X}\left\{ f_i(z) + \frac{1}{2r_i}\|z\|^2 - \frac{1}{r_i}\langle z, Jx_n\rangle \right\} \tag{4.2}$$

is equivalent to

$$0 \in \partial f_i(z_{n,i}) + \frac{1}{r_i} J z_{n,i} - \frac{1}{r_i} J x_n, \quad n, i \in \mathbb{N}, \tag{4.3}$$

and the last inclusion relation is equivalent to

$$z_{n,i} = \left(J + r_i \partial f_i\right)^{-1} J x_n = J_i x_n. \tag{4.4}$$

Now the desired conclusion follows from Theorem 3.1. This completes the proof. □

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