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Research Article

Strong Convergence of a New Iteration for a Finite Family of Accretive Operators

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The viscosity approximation methods are employed to establish strong convergence of the modified Mann iteration scheme to a common zero of a finite family of accretive operators on a strictly convex Banach space with uniformly Gâteaux differentiable norm. Our work improves and extends various results existing in the current literature.

1. Introduction

Let E be a Banach space with dual space of E^*, and let C a nonempty closed convex subset E. Let $N \geq 1$ be a positive integer, and let $\Lambda = \{1, 2, \ldots, N\}$. We denote by J the normalized duality map from E to 2^{E^*} defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2, \quad \forall x \in E\right\}. \tag{1.1}$$

A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. A mapping $f : C \to C$ is called *k-contraction* if there exists a constant $k \in (0, 1)$ such that

$$\left\|f(x) - f(y)\right\| \leq k\left\|x - y\right\|, \qquad \forall x, y \in C. \tag{1.2}$$

In the last ten years, many papers have been written on the approximation of fixed point for nonlinear mappings by using some iterative processes (see, e.g.,[1–20]).

An operator $A : D(A) \subset E \to E$ is said to be *accretive* if $\|x_1 - x_2\| \leq \|x_1 - x_2 + s(y_1 - y_2)\|$, for all $y_i \in Ax_i$, $i = 1, 2$ and $s > 0$. If A is accretive and I is identity mapping, then we define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by

$J_r := (I + rA)^{-1}$, which is called the *resolvent* of A. we also know that for an accretive operator A, $\mathcal{N}(A) = \mathrm{Fix}(J_r)$, where $\mathcal{N}(A) = \{x \in E : 0 \in Ax\}$ and $\mathrm{Fix}(J_r) = \{x \in E : J_r x = x\}$. An accretive operator A is said to be *m-accretive*, if $R(I + tA) = E$ for all $t > 0$. If E is a Hilbert space, then accretive operator is monotone operator. There are many papers throughout literature dealing with the solution of $0 \in Ax$ $(x \in E)$ by utilizing certain iterative sequence (see [1–3, 8–10, 13, 16, 20]).

In 2005, Kim and Xu [10] introduced the following Halpern type iterative sequence for m-accretive operator A: *Let C be a nonempty closed convex subset of E. For any $u, x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (\varepsilon, +\infty)$, *for some* $\varepsilon > 0$, *satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$,

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, and

(C4) $\sum_{n=1}^{\infty} |1 - r_{n+1}/r_n| < +\infty$.

They proved that the iterative sequence $\{x_n\}$ converges strongly to a zero of A.

Recently, Zegeye and Shahzad [20] proved a strong convergence theorem for a finite family of accretive operators by using the Halpern type iteration: *Let C be a nonempty closed convex subset of E. For any u, $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1, \tag{1.4}$$

where $S := a_0 I + a_1 J_{A_1} + \cdots + a_N J_{A_N}$ *with* $J_{A_i} = (I + A_i)^{-1}$, $a_i \in (0, 1)$, *for* $i = 0, 1, 2, \ldots, N$, $\sum_{i=0}^{N} a_i = 1$, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies the conditions*: (C1), (C2), (C3), or (C3*). $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| / \alpha_{n+1} = 0$).

More recently, Hu and Liu [8] proposed a generalized Halpern type iteration: *Let C be a nonempty closed convex subset of E. For any $u, x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{r_n} x_n, \quad n \geq 1, \tag{1.5}$$

where $S_{r_n} := a_0 I + a_1 J_{r_n}^1 + \cdots + a_N J_{r_n}^N$ *with* $J_{r_n}^i = (I + r_n A_i)^{-1}$, *for* $i = 1, 2, \ldots, N$, $a_i \in (0, 1)$ *and* $\sum_{i=0}^{N} a_i = 1$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, *and* $\{r_n\} \subset (0, +\infty)$ satisfy the following conditions: (C1), (C2),

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \qquad \lim_{n\to\infty} r_n = r, \quad \text{for some } r > 0, \qquad \alpha_n + \beta_n + \gamma_n = 1. \tag{1.6}$$

They proved that the sequence $\{x_n\}$ converges strongly to a common zero of $\{A_i : i \in \Lambda\}$.

In this paper, we introduce and study a new iterative sequence: *Let C be a nonempty closed convex subset of E and $f : C \to C$ a k-contraction. For any $x_1 \in C$, the sequence $\{x_n\}$ is defined by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S_{r_n} (\alpha_n f(x_n) + (1 - \alpha_n) x_n), \quad n \geq 1, \tag{1.7}$$

where $S_{r_n} := a_0 I + a_1 J_{r_n}^1 + \cdots + a_N J_{r_n}^N$ *with* $J_{r_n}^i = (I + r_n A_i)^{-1}$*, for* $i = 0, 1, 2, \ldots, N$, $a_i \in (0, 1)$ *and* $\sum_{i=0}^N a_i = 1$, $\{r_n\} \subset (0, +\infty)$ *and* $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$. The iterative sequence (1.7) is a natural generalization of all the above mentioned iterative sequences.

(i) In contrast to the iterations (1.3)–(1.5), the convex composition of the iteration (1.7) deals with only x_n instead of u and x_n.

(ii) If we take $\alpha_n \equiv 0$, for all $n \geq 1$, in (1.7), then (1.7) reduces to Mann iteration. In 2000, Kamimura and Takahashi [9] proved that if E is a Hilbert space and $\{\beta_n\}$ and $\{r_n\}$ are chosen such that $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = +\infty$ and $\lim_{n\to\infty} r_n = +\infty$, then the Mann iterative sequence,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \quad \forall n \geq 1, \tag{1.8}$$

converges weakly to a zero of A. However, the Mann iteration scheme has only weak convergence for nonexpansive mappings even in a Hilbert space (see [4]).

Our main purpose is to prove strong convergence theorems for a finite family of accretive operators on a strictly convex Banach space with uniformly Gâteaux differentiable norm by using viscosity approximation methods. Our theorems extend the comparable results in the following three aspects.

(1) In contrast to weak convergence results on a Hilbert Space in [9], strong convergence of the iterative sequence is obtained in the general setup of a Banach space.

(2) The restrictions (C3), (C3*), and (C4) on the results in [10, 20] are dropped.

(3) A single mapping of the results in [3] is replaced by a finite family of mappings.

2. Preliminaries and Lemmas

A Banach space E is said to have *Gâteaux differentiable norm* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$, where $U = \{x \in E : \|x\| = 1\}$. The norm of E is *uniformly Gâteaux differentiable* if for each $y \in U$, the limit is attained uniformly for $x \in U$. The norm of E is *uniformly Fréchet differentiable* (E is also called *uniformly smooth*) if the limit is attained uniformly for each $x, y \in U$. It is well known that if E is uniformly Gâteaux differentiable norm, then the duality mapping J is single-valued and norm-to-weak* uniformly continuous on each bounded subset of E.

A Banach space E is called *strictly convex* if for $i \in \Lambda$, $a_i \in (0, 1)$, and $\sum_{i=1}^N a_i = 1$, we have $\|a_1 x_1 + a_2 x_2 + \cdots + a_N x_N\| < 1$ for $x_i \in E$, $i \in \Lambda$ and $x_i \neq x_j$ for $i \neq j$. In a strictly convex Banach space E, we have that if $\|x_1\| = \|x_2\| = \cdots = \|x_N\| = \|a_1 x_1 + a_2 x_2 + \cdots + a_N x_N\|$, for $x_i \in E$, $a_i \in (0, 1)$, $i \in \Lambda$ and $\sum_{i=1}^N a_i = 1$, then $x_1 = x_2 = \cdots = x_N$.

Lemma 2.1 (The Resolvent Identity). *For $\lambda, \mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big). \tag{2.2}$$

We denote by $\mathbb{N}$ the set of all natural numbers, and let μ be a mean on $\mathbb{N}$, that is, a continuous linear functional μ on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf_{n\in\mathbb{N}} b_n \leq \mu(f) \leq \sup_{n\in\mathbb{N}} b_n, \tag{2.3}$$

for each $f = (b_1, b_2, \ldots) \in l^\infty$. In general, we use $\operatorname{LIM}(b_n)$ instead of $\mu(f)$. Let $f = (b_1, b_2, \ldots) \in l^\infty$ with $b_n \to b$, and let μ be a Banach limit on $\mathbb{N}$. Then $\mu(f) = \operatorname{LIM}(b_n) = b$. Further, we know the following result.

Lemma 2.2 (see [15, 16]). *Let C be a nonempty closed convex subset of a Banach space E with uniformly Gâteaux differentiable norm. Assume that $\{x_n\}$ is a bounded sequence in C. Let $z \in C$, and let*LIM *a Banach limit. Then $\operatorname{LIM}\|x_n - z\|^2 = \min_{x\in C} \operatorname{LIM}\|x_n - x\|^2$ if and only if $\operatorname{LIM}\langle x - z, j(x_n - z)\rangle \leq 0$,* for all $x \in C$.

Let $C \subseteq E$ be a closed convex and, let Q a mapping of E onto C. Then Q is said to be *sunny* [12, 13] if $Q(x + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E onto C is said to be *retraction* if $Q^2 = Q$; If a mapping Q is a retraction then $Qx = x$ for any $x \in R(Q)$, the range of Q. A subset C of E is said to be a *sunny nonexpansive retraction* of E if there exists a sunny nonexpansive retraction of E onto C, and it is said to be a *nonexpansive retraction* of E if there exists a nonexpansive retraction of E onto C. In a smooth Banach space E, it is known ([5, Page 48]) that $Q : E \to C$ is a sunny nonexpansive retraction if and only if the following condition holds: $\langle x - Q(x), J(z - Q(x))\rangle \leq 0$, $x \in E$ and $z \in C$.

Lemma 2.3 (see [14]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \qquad n \geq 0, \tag{2.4}$$

where $\{\beta_n\}$ is a sequence in $(0,\ 1)$ such that $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Assume

$$\limsup_{n\to\infty}\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0. \tag{2.5}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4. *Let E be a real Banach space. Then for all x, y in E and $j(x + y) \in J(x + y)$, the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y,\ j(x + y)\rangle. \tag{2.6}$$

Lemma 2.5 ([18]). *Let $\{a_n\}$ is a sequence of nonnegative real number such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n\xi_n, \qquad \forall n \geq 0, \tag{2.7}$$

where $\{\delta_n\}$ *is a sequence in* $[0, 1]$ *and* $\{\xi_n\}$ *is a sequence in* $\mathbb{R}$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \delta_n = +\infty$;

(ii) $\limsup_{n\to\infty} \xi_n \le 0$ *or* $\sum_{n=1}^{\infty} \delta_n |\xi_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [8]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space* E. *Suppose that* $\{A_i : 1 \le i \le N\} : C \to E$ *is a finite family of accretive operators such that* $\bigcap_{i=1}^{N} \mathcal{N}(A_i) \neq \emptyset$ *and satisfies the range conditions:*

$$\mathrm{cl}\,(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \qquad i = 1, 2, \ldots, N. \tag{2.8}$$

Let $\{a_i : i \in \{0\} \cup \Lambda\}$ *be real numbers in* $(0, 1)$ *with* $\sum_{i=0}^{N} a_i = 1$ *and* $S_r = a_0 I + a_1 J_r^1 \cdots + a_N J_r^N$, *where* $J_r^i := (I + rA_i)^{-1}$ *and* $r > 0$. *Then* S_r *is nonexpansive and* $\mathrm{Fix}(S_r) = \bigcap_{i=1}^{N} \mathcal{N}(A_i)$.

3. Main Results

For the sake of convenience, we list the assumptions to be used in this paper as follows.

(i) E is a strictly convex Banach space which has uniformly Gâteaux differentiable norm, and C is a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings.

(ii) The real sequence $\{\alpha_n\}$ satisfies the conditions: (C1). $\lim_{n\to\infty} \alpha_n = 0$ and (C2). $\sum_{n=0}^{\infty} \alpha_n = +\infty$.

We will employ the viscosity approximation methods [11, 19] to obtain a strong convergence theorem. The method of proof is closely related to [2, 3, 19].

Theorem 3.1. *Let* $\{A_i : i \in \Lambda\} : C \to E$ *be a finite family of accretive operators satisfying the following range conditions:*

$$\mathrm{cl}(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \ldots, N. \tag{3.1}$$

Assume that $F := \bigcap_{i=1}^{N} \mathcal{N}(A_i) \neq \emptyset$. *Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$. *For* $t \in (0,1)$, *the net* $\{x_t\}$ *is generated by*

$$x_t = t f(x_t) + (1 - t) S_{r_t} x_t, \tag{I}$$

where $S_{r_t} := a_0 I + a_1 J_{r_t}^1 + \cdots + a_N J_{r_t}^N$ *with* $J_{r_t}^i := (I + r_t A_i)^{-1}$, *for* $i = 0, 1, \ldots, N$, $a_i \in (0, 1)$ *and* $\sum_{i=0}^{N} a_i = 1$. *If* $\lim_{t\to 0} r_t = r$, *then the net* $\{x_t\}$ *converges strongly to* $v \in F$, *as* $t \to 0$, *where* v *is the unique solution of a variational inequality:*

$$\langle v - f(v), J(v - p) \rangle \le 0, \quad \forall p \in F. \tag{VI}$$

Proof. Put $W_t x := tf(x) + (1-t)S_{r_t}x$, for all $x \in C$ and $t \in (0,1)$. Then we have

$$\begin{aligned}\|W_t x - W_t y\| &= \|tf(x) + (1-t)S_{r_t}x - tf(y) - (1-t)S_{r_t}y\| \\ &\le t\|f(x) - f(y)\| + (1-t)\|S_{r_t}x - S_{r_t}y\| \\ &\le (1 - t(1-k))\|x - y\|,\end{aligned} \tag{3.2}$$

and so W_t is a contraction of C into itself. Hence, for each $t \in (0,1)$, there exists a unique element $x_t \in C$ such that

$$x_t = tf(x_t) + (1-t)S_{r_t}x_t. \tag{3.3}$$

Thus the net $\{x_t\}$ is well defined.

Lemma 2.6 implies that $F = \mathrm{Fix}(S_{r_t}) = \bigcap_{i=1}^{N} \mathcal{N}(A_i) \neq \emptyset$. Taking $p \in F$, we have for any $t \in (0,1)$

$$\begin{aligned}\|x_t - p\| &\le t\|f(x_t) - p\| + (1-t)\|S_{r_t}x_t - p\| \\ &\le tk\|x_t - p\| + t\|f(p) - p\| + (1-t)\|x_t - p\|.\end{aligned} \tag{3.4}$$

Consequently, we get

$$\|x_t - p\| \le \frac{1}{1-k}\|f(p) - p\|, \tag{3.5}$$

that is, the net $\{x_t\}$ is bounded, and so are $\{f(x_t)\}$ and $\{S_{r_t}x_t\}$. Rewriting (I) to find

$$x_t - f(x_t) = -\frac{1-t}{t}(x_t - S_{r_t}x_t), \tag{3.6}$$

and hence for any $p \in F$, it yields that

$$\begin{aligned}\langle x_t - f(x_t), J(x_t - p)\rangle &= -\frac{1-t}{t}\langle x_t - S_{r_t}x_t, J(x_t - p)\rangle \\ &= -\frac{1-t}{t}\langle (I - S_{r_t})x_t - (I - S_{r_t})p, J(x_t - p)\rangle \\ &\le 0 \quad (\text{ Since } (I - S_{r_t}) \text{ is monotone}).\end{aligned} \tag{3.7}$$

Obviously, estimate (I) yields

$$\begin{aligned}\|x_t - S_{r_t}x_t\| &\le t\|f(x_t) - S_{r_t}x_t\| \\ &\le t\big((1+k)\|x_t - p\| + \|f(p) - p\|\big) \longrightarrow 0, \quad \text{as} \quad t \longrightarrow 0.\end{aligned} \tag{3.8}$$

In view of the Resolvent Identity, we deduce

$$\begin{aligned}\left\|J_{r_t}^i x_t - J_r^i x_t\right\| &= \left\|J_r^i\left(\frac{r}{r_t}x_t + \left(1-\frac{r}{r_t}\right)J_{r_t}^i x_t\right) - J_r^i x_t\right\| \\ &\leq \left\|\frac{r}{r_t}x_t + \left(1-\frac{r}{r_t}\right)J_{r_t}^i x_t - x_t\right\| \leq \left|1-\frac{r}{r_t}\right|\left\|x_t - J_{r_t}^i x_t\right\|,\end{aligned} \tag{3.9}$$

and so

$$\begin{aligned}\|S_{r_t}x_t - S_r x_t\| &= \left\|\sum_{i=1}^N a_i\left(J_{r_t}^i x_t - J_r^i x_t\right)\right\| \\ &\leq \sum_{i=1}^N a_i\left|1-\frac{r}{r_t}\right|\left\|x_t - J_{r_t}^i x_t\right\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0.\end{aligned} \tag{3.10}$$

Combining (3.8) and the above inequality, we obtain

$$\|x_t - S_r x_t\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0. \tag{3.11}$$

Assume $t_n \to 0$, as $n \to \infty$. Set $x_n := x_{t_n}$ and define $\mu : C \to \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers) by

$$\mu(x) = \text{LIM}\|x_n - x\|^2, \quad x \in C, \tag{3.12}$$

where LIM is a Banach limit on l^∞. Let

$$K = \left\{q \in C : \mu(q) = \min_{x\in C} \text{LIM}\|x_n - x\|^2\right\}. \tag{3.13}$$

It is easy to see that K is a nonempty closed convex and bounded subset of E and K is invariant under S_r. Indeed, as $n \to \infty$, we have for any $q \in K$,

$$\mu(S_r q) = \text{LIM}\left\|x_n - S_r q\right\|^2 = \text{LIM}\left\|S_r x_n - S_r q\right\|^2 \leq \text{LIM}\left\|x_n - q\right\|^2 = \mu(q), \tag{3.14}$$

and so $S_r q$ is an element of K. Since C has the fixed point property for nonexpansive mappings, S_r has a fixed point v in K. Using Lemma 2.2, we have

$$\text{LIM}\langle x - v, J(x_n - v)\rangle \leq 0, \quad x \in C. \tag{3.15}$$

Clearly

$$\begin{aligned}\|x_t - v\|^2 &= t\langle f(x_t) - v, J(x_t - v)\rangle + (1-t)\langle S_{r_t}x_t - v, J(x_t - v)\rangle \\ &\le t\langle f(x_t) - f(v), J(x_t - v)\rangle + t\langle f(v) - v, J(x_t - v)\rangle + (1-t)\|x_t - v\|^2 \\ &\le (1 - t(1-k))\|x_t - v\|^2 + t\langle f(v) - v, J(x_t - v)\rangle.\end{aligned} \tag{3.16}$$

Consequently, by (3.15), we obtain

$$\mathrm{LIM}\|x_n - v\|^2 \le \ \mathrm{LIM}\ \frac{1}{1-k}\langle f(v) - v, J(x_t - v)\rangle \le 0, \tag{3.17}$$

, that is,

$$\mathrm{LIM}\|x_n - v\|^2 = 0, \tag{3.18}$$

and there exists a subsequence which is still denoted by $\{x_n\}$ such that $x_n \to v$.

On the other hand, let $\{x_{t_j}\}$ of $\{x_t\}$ be such that $x_{t_j} \to \overline{v} \in F$. Now (3.7) implies

$$\left\langle x_{t_j} - f\left(x_{t_j}\right), J\left(x_{t_j} - v\right)\right\rangle \le 0, \quad v \in F. \tag{3.19}$$

Thus

$$\langle \overline{v} - f(\overline{v}), J(\overline{v} - v)\rangle \le 0, \quad v \in F. \tag{3.20}$$

Interchange $\overline{v}$ and v to get

$$\langle v - f(v), J(v - \overline{v})\rangle \le 0, \quad v \in F. \tag{3.21}$$

Addition of (3.20) and (3.21) yields

$$\langle \overline{v} - f(\overline{v}) - v + f(v), J(\overline{v} - v)\rangle \le 0, \tag{3.22}$$

and so we have

$$\|\overline{v} - v\|^2 \le \langle f(\overline{v}) - f(v), J(\overline{v} - v)\rangle \le k\|\overline{v} - v\|^2. \tag{3.23}$$

Since $k \in (0,1)$, it follows that $\overline{v} = v$. Consequently $x_t \to v$ as $t \to 0$. Likewise, using (3.7), it implies for all $p \in F$

$$\langle x_t - f(x_t), J(x_t - p)\rangle \le 0. \tag{3.24}$$

Letting $t \to 0$ yields

$$\langle v - f(v), J(v-p)\rangle \le 0, \tag{3.25}$$

for all $p \in F$. □

Remark 3.2. In addition, if E is a uniformly smooth Banach space in Theorem 3.1 and we define $Q(f) := \lim_{t\to 0} x_t$, then we obtain from Theorem 3.1 and [19, Theorem 4.1] that the net $\{x_t\}$ converges strongly to $v \in F$, as $t \to 0$, where $v = Q_F f(v)$ and Q_F is a sunny nonexpansive retraction of C onto F.

Theorem 3.3. *Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:*

$$\mathrm{cl}(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \qquad i = 1,\ 2,\ \ldots,\ N. \tag{3.26}$$

Assume that $F := \bigcap_{i=1}^{N} \mathcal{N}(A_i) \ne \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. For any $x_1 \in C$, the sequence $\{x_n\}$ is generated by (1.7)*. Suppose further that sequences in the iterative sequence* (1.7) *satisfy the conditions:*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, \qquad \lim_{n\to\infty} r_n = r, \qquad r > 0. \tag{3.27}$$

Then the sequence $\{x_n\}$ converges strongly to $v \in F$, where v is the unique solution of a variational inequality (VI).

Proof. Lemma 2.6 implies that $F = \mathrm{Fix}(S_{r_n}) = \bigcap_{i=1}^{N} \mathcal{N}(A_i) \ne \emptyset$. Rewrite (1.7) as follows:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_{r_n} y_n, \tag{3.28}$$

where

$$y_n = \alpha_n f(x_n) + (1-\alpha_n) x_n, \quad \forall n \ge 1. \tag{3.29}$$

Taking $p \in F$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \beta_n \|x_n - p\| + (1-\beta_n)\|S_{r_n} y_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)(\alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\|) \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)(\alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\|) \\
&= (1 - (1-\beta_n)\alpha_n(1-k))\|x_n - p\| + (1-\beta_n)\alpha_n(1-k)\frac{1}{1-k}\|f(p) - p\| \\
&\le \max\left\{ \|x_1 - p\|,\ \frac{1}{1-k}\|f(p) - p\| \right\}.
\end{aligned} \tag{3.30}$$

Therefore, the sequence $\{x_n\}$ is bounded, and so are the sequences $\{f(x_n)\}$, $\{S_{r_n}x_n\}$, $\{y_n\}$, $\{J_{r_n}^i y_n\}$ and, $\{S_{r_n}y_n\}$. We estimate from (3.29)

$$\begin{aligned}\left\|y_{n+1}-y_n\right\| &\leq \alpha_{n+1}\left\|f(x_{n+1})-f(x_n)\right\| + (1-\alpha_{n+1})\|x_{n+1}-x_n\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\left\|f(x_n)-x_n\right\| \\ &\leq (1-\alpha_{n+1}(1-k))\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\left\|f(x_n)-x_n\right\|.\end{aligned} \tag{3.31}$$

In view of the Resolvent Identity, we get

$$\begin{aligned}\left\|J_{r_{n+1}}^i y_{n+1} - J_{r_n}^i y_n\right\| &= \left\|J_{r_n}^i\left(\frac{r_n}{r_{n+1}}y_{n+1} + \left(1-\frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}^i y_{n+1}\right) - J_{r_n}^i y_n\right\| \\ &\leq \left\|\frac{r_n}{r_{n+1}}(y_{n+1}-y_n) + \left(1-\frac{r_n}{r_{n+1}}\right)\left(J_{r_{n+1}}^i y_{n+1} - y_n\right)\right\| \\ &\leq \frac{r_n}{r_{n+1}}\left\|y_{n+1}-y_n\right\| + \left|1-\frac{r_n}{r_{n+1}}\right| M_1,\end{aligned} \tag{3.32}$$

where

$$M_1 = \sup_{n\geq 1}\left\{\left\|y_n - J_{r_{n+1}}^i y_{n+1}\right\|,\ i\in\Lambda\right\}. \tag{3.33}$$

Since $S_{r_n} = a_0 I + \sum_{i=1}^N a_i J_{r_n}^i$, we have

$$\begin{aligned}\left\|S_{r_{n+1}}y_{n+1} - S_{r_n}y_n\right\| &\leq a_0\left\|y_{n+1}-y_n\right\| + \sum_{i=1}^N a_i\left\|J_{r_{n+1}}^i y_{n+1} - J_{r_n}^i y_n\right\| \\ &\leq \left[\frac{r_n}{r_{n+1}} + a_0\left(1-\frac{r_n}{r_{n+1}}\right)\right]\left\|y_{n+1}-y_n\right\| + \left|1-\frac{r_n}{r_{n+1}}\right| M \\ &\leq \left[\frac{r_n}{r_{n+1}} + a_0\left(1-\frac{r_n}{r_{n+1}}\right)\right](1-\alpha_{n+1}(1-k))\|x_{n+1}-x_n\| \\ &\quad + \left[\frac{r_n}{r_{n+1}} + a_0\left(1-\frac{r_n}{r_{n+1}}\right)\right]|\alpha_{n+1}-\alpha_n|\left\|f(x_n)-x_n\right\| \\ &\quad + \left|1-\frac{r_n}{r_{n+1}}\right| M_1.\end{aligned} \tag{3.34}$$

$\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty} r_n = r$ imply

$$\limsup_{n\to\infty}\left(\left\|S_{r_{n+1}}y_{n+1} - S_{r_n}y_n\right\| - \|x_{n+1}-x_n\|\right) \leq 0. \tag{3.35}$$

Consequently, by Lemma 2.3, we obtain

$$\lim_{n\to\infty}\left\|S_{r_n}y_n - x_n\right\| = 0. \tag{3.36}$$

From (3.29), we get

$$\lim_{n\to\infty}\|y_n - x_n\| = \alpha_n\|f(x_n) - x_n\| \longrightarrow 0, \tag{3.37}$$

and so it follows from (3.36) and (3.37) that

$$\lim_{n\to\infty}\|y_n - S_{r_n}y_n\| = 0. \tag{3.38}$$

Using the Resolvent Identity and $S_{r_n} = a_0 I + \sum_{i=1}^{N} a_i J_{r_n}^i$, we discover

$$\begin{aligned}\|S_{r_n}y_n - S_r y_n\| &= \left\|\sum_{i=1}^{N} a_i\left(J_{r_n}^i y_n - J_r^i y_n\right)\right\| \\ &\le \sum_{i=1}^{N} a_i\left\|J_r^i\left(\frac{r}{r_n}y_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}^i y_n\right) - J_r^i y_n\right\| \\ &\le \sum_{i=1}^{N} a_i\left|1 - \frac{r}{r_n}\right|\left\|y_n - J_{r_n}^i y_n\right\| \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{3.39}$$

Hence, we have

$$\|y_n - S_r y_n\| \le \|y_n - S_{r_n}y_n\| + \|S_{r_n}y_n - S_r y_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.40}$$

It follows from Theorem 3.1 that $\{x_t\}$ generated by $x_t = tf(x_t) + (1-t)S_r x_t$ converges strongly to $v \in F$, as $t \to 0$, where v is the unique solution of a variational inequality (VI). Furthermore,

$$x_t - y_n = (1-t)(S_r x_t - y_n) + t(f(x_t) - y_n). \tag{3.41}$$

In view of Lemma 2.4, we find

$$\begin{aligned}\|x_t - y_n\|^2 &\le (1-t)^2\|S_r x_t - y_n\|^2 + 2t\langle f(x_t) - y_n,\ J(x_t - y_n)\rangle \\ &\le \left(1 - 2t + t^2\right)(\|S_r x_t - S_r y_n\| + \|S_r y_n - y_n\|)^2 + 2t\langle f(x_t) - x_t,\ J(x_t - y_n)\rangle \\ &\quad + 2t\|x_t - y_n\|^2 \\ &\le \left(1 + t^2\right)\|x_t - y_n\|^2 + \left(1 + t^2\right)\|y_n - S_r y_n\|(2\|x_t - y_n\| + \|y_n - S_r y_n\|) \\ &\quad + 2t\langle f(x_t) - x_t,\ J(x_t - y_n)\rangle,\end{aligned} \tag{3.42}$$

and hence

$$\langle f(x_t) - x_t,\ J(y_n - x_t)\rangle \leq \frac{t}{2}\|x_t - y_n\|^2 + \frac{(1+t^2)\|y_n - S_r y_n\|}{2t}\left(2\|x_t - y_n\| + \|y_n - S_r y_n\|\right). \tag{3.43}$$

Since the sequences $\{y_n\}$, $\{x_t\}$, and $\{S_r y_n\}$ are bounded and $\lim_{n\longrightarrow\infty}\|y_n - S_r y_n\|/2t = 0$, we obtain

$$\limsup_{n\to\infty}\langle f(x_t) - x_t,\ J(y_n - x_t)\rangle \leq \frac{t}{2}M_2, \tag{3.44}$$

where $M_2 = \sup_{n\geq 1, t\in(0,1)}\{\|x_t - y_n\|^2\}$. We also know that

$$\begin{aligned}\langle f(v) - v,\ J(y_n - v)\rangle &= \langle f(x_t) - x_t,\ J(y_n - x_t)\rangle + \langle f(v) - f(x_t) + x_t - v,\ J(y_n - x_t)\rangle \\ &\quad + \langle f(v) - v,\ j(y_n - v) - J(y_n - x_t)\rangle.\end{aligned} \tag{3.45}$$

From the facts that $x_t \to v \in F$, as $t \to 0$, $\{y_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, it follows that

$$\begin{aligned}&\langle f(v) - v,\ j(y_n - v) - J(y_n - x_t)\rangle \longrightarrow 0,\ \text{as}\ \ t \longrightarrow 0, \\ &\langle f(v) - f(x_t) + x_t - v,\ J(y_n - x_t)\rangle \longrightarrow 0,\ \text{as}\ \ t \longrightarrow 0.\end{aligned} \tag{3.46}$$

Combining (3.44), (3.45), and the two results mentioned above, we get

$$\limsup_{n\to\infty}\langle f(v) - v,\ J(y_n - v)\rangle \leq 0. \tag{3.47}$$

Similarly, from (3.29) and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, it follows that

$$\lim_{n\to\infty}\left|\langle f(x_n) - f(v), J(y_n - v) - J(x_n - v)\rangle\right| = 0. \tag{3.48}$$

Write

$$x_{n+1} - v = \beta_n(x_n - v) + (1-\beta_n)S_{r_n}(y_n - v), \tag{3.49}$$

and apply Lemma 2.4 to find

$$
\begin{aligned}
\|x_{n+1}-v\|^2 &\le \beta_n\|x_n-v\|^2+(1-\beta_n)\left\|S_{r_n}y_n-v\right\|^2\\
&\le \beta_n\|x_n-v\|^2+(1-\beta_n)\left\|\alpha_n(f(x_n)-v)+(1-\alpha_n)(x_n-v)\right\|^2\\
&\le \beta_n\|x_n-v\|^2+(1-\beta_n)(1-\alpha_n)^2\|x_n-v\|^2\\
&\quad +2(1-\beta_n)\alpha_n\langle f(x_n)-v, J(y_n-v)\rangle\\
&\le \beta_n\|x_n-v\|^2+(1-\beta_n)(1-\alpha_n)^2\|x_n-v\|^2+2(1-\beta_n)\alpha_n k\|x_n-v\|^2\\
&\quad +2(1-\beta_n)\alpha_n\langle f(v)-v, J(y_n-v)\rangle\\
&\quad +2(1-\beta_n)\alpha_n\langle f(x_n)-f(v), J(y_n-v)-J(x_n-v)\rangle\\
&\le \left[1-2(1-\beta_n)(1-k)\alpha_n\right]\|x_n-v\|^2+2(1-\beta_n)\alpha_n\\
&\quad \times\left[\alpha_n\|x_n-v\|+\left|\langle f(x_n)-f(v), J(y_n-v)-J(x_n-v)\rangle\right|\right.\\
&\quad \left.+\langle f(v)-v, J(y_n-v)\rangle\right]\\
&= [1-(1-k)\delta_n]\|x_n-v\|^2+\delta_n\xi_n,
\end{aligned}
\tag{3.50}
$$

where

$$
\begin{gathered}
\delta_n=2(1-\beta_n)\alpha_n,\\
\xi_n=\alpha_n\|x_n-v\|+\left|\langle f(x_n)-f(v), J(y_n-v)-J(x_n-v)\rangle\right|+\langle f(v)-v, J(y_n-v)\rangle.
\end{gathered}
\tag{3.51}
$$

From (3.47), (3.48), (C1), (C2), and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, it follows that $\sum_{n=1}^{\infty}\delta_n=+\infty$ and $\limsup_{n\to\infty}\xi_n\le 0$. Consequently applying Lemma 2.5 to (3.50), we conclude that $\lim_{n\to\infty}\|x_n-v\|=0$. □

If we take $f(x)\equiv u$, for all $x\in C$, in the iteration (1.7), then, from Theorem 3.3, we have what follows

Corollary 3.4. *Let $\{A_i : i\in\Lambda\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ be as in Theorem 3.3. For any $u, x_1\in C$, the sequence $\{x_n\}$ is generated by*

$$
x_{n+1}=\beta_n x_n+(1-\beta_n)S_{r_n}(\alpha_n u+(1-\alpha_n)x_n),\qquad n\ge 1,
\tag{3.52}
$$

where $S_{r_n} := a_0 I+a_1 J_{r_n}^1+\cdots+a_N J_{r_n}^N$ with $J_{r_n}^i=(I+r_nA_i)^{-1}$, for $i=0, 1, 2, \ldots, N$, $a_i\in(0, 1)$ and $\sum_{i=0}^{N}a_i=1$. Then the sequence $\{x_n\}$ converges strongly to $v\in F$.

Remark 3.5. Theorem 3.3 and Corollary 3.4 prove strong convergence results of the new iterative sequences which are different from the iterative sequences (1.4) and (1.5). In contrast to [20], the restriction: (C3). $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<+\infty$ or (C3*) $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n|/\alpha_{n+1}=0$ is removed.

If we consider the case of an accretive operator A, then as a direct consequence of Theorem 3.1 and Theorem 3.3, we have the following corollaries.

Corollary 3.6 ([3, Theorem 3.1]). *Let $A : C \to E$ (not strictly convex) be an accretive operator satisfying the following range condition:*

$$\mathrm{cl}(D(A)) \subseteq C \subset \bigcap_{r>0} R(I + rA). \tag{3.53}$$

Assume that $\mathcal{F} := \mathcal{N}(A) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. For $t \in (0,1)$, the net $\{x_t\}$ is given by:

$$x_t = tf(x_t) + (1-t)J_{r_t}x_t, \tag{3.54}$$

where $J_{r_t} := (I + r_tA)^{-1}$. If $\inf_{t\in(0,1)} r_t \geq \varepsilon$, for some $\varepsilon > 0$, then $\{x_t\}$ converges strongly to $v \in \mathcal{F}$, as $t \to 0$, where v is the unique solution of a variational inequality:

$$\langle v - f(v), J(v-p)\rangle \leq 0, \quad \forall p \in \mathcal{F}. \tag{VI$'$}$$

Corollary 3.7. *Let $A : C \to E$ (not strictly convex) be an accretive operator satisfying the following range condition:*

$$\mathrm{cl}(D(A)) \subseteq C \subset \bigcap_{r>0} R(I + rA). \tag{3.55}$$

Assume that $\mathcal{F} := \mathcal{N}(A) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $\mathbb{R}^+$, satisfying the conditions: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\inf_{n\geq 1} r_n \geq \varepsilon$, for some $\varepsilon > 0$. For any $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{r_n}(\alpha_n f(x_n) + (1-\alpha_n)x_n), \quad n \geq 1, \tag{3.56}$$

where $J_{r_n} = (I + r_nA)^{-1}$. Then the sequence $\{x_n\}$ converges strongly to $v \in \mathcal{F}$, where v is the unique solution of a variational inequality (VI$'$).

Remark 3.8.

(i) Corollary 3.7 describes strong convergence result in Banach spaces for a modification of Mann iteration scheme in contrast to the weak convergence result on Hilbert spaces given in [9, Theorem 3].

(ii) In contrast to the result [10, Theorem 4.2], the iterative sequence in Corollary 3.7 is different from the iteration (1.3), and the conditions $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ and $\sum_{n=1}^{\infty} |1 - r_{n-1}/r_n| < +\infty$ are not required.

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