Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 762478, 17 pages
doi:10.1155/2009/762478

Research Article

Global Attractivity Results for Mixed-Monotone Mappings in Partially Ordered Complete Metric Spaces

Dž. Burgić,[1] S. Kalabušić,[2] and M. R. S. Kulenović[3]

[1] *Department of Mathematics, University of Tuzla, 75000 Tuzla, Bosnia and Herzegovina*
[2] *Department of Mathematics, University of Sarajevo, 71000 Sarajevo, Bosnia and Herzegovina*
[3] *Department of Mathematics, University of Rhode Island, Kingston, R I 02881-0816, USA*

Correspondence should be addressed to M. R. S. Kulenović, mkulenovic@mail.uri.edu

Received 28 October 2008; Revised 17 January 2009; Accepted 9 February 2009

Recommended by Juan J. Nieto

We prove fixed point theorems for mixed-monotone mappings in partially ordered complete metric spaces which satisfy a weaker contraction condition than the classical Banach contraction condition for all points that are related by given ordering. We also give a global attractivity result for all solutions of the difference equation $z_{n+1} = F(z_n, z_{n-1})$, $n = 2, 3, \ldots$, where F satisfies mixed-monotone conditions with respect to the given ordering.

1. Introduction and Preliminaries

The following results were obtained first in [1] and were extended to the case of higher-order difference equations and systems in [2–6]. For the sake of completeness and the readers convenience, we are including short proofs.

Theorem 1.1. *Let $[a, b]$ be a compact interval of real numbers, and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{1.1}$$

is a continuous function satisfying the following properties:

(a) *$f(x, y)$ is nondecreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each $x \in [a, b]$;*

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m, \qquad f(M, m) = M, \tag{1.2}$$

then $m = M$.

Then

$$x_{n+1} = f\left(x_n, x_{n-1}\right), \quad n = 0, 1, \ldots \tag{1.3}$$

has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of* (1.3) *converges to* $\overline{x}$.

Proof. Set

$$m_0 = a, \qquad M_0 = b, \tag{1.4}$$

and for $i = 1, 2, \ldots$ set

$$M_i = f\left(M_{i-1}, m_{i-1}\right), \qquad m_i = f\left(m_{i-1}, M_{i-1}\right). \tag{1.5}$$

Now observe that for each $i \geq 0$,

$$\begin{gathered} m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0, \\ m_i \leq x_k \leq M_i, \quad \text{for } k \geq 2i + 1. \end{gathered} \tag{1.6}$$

Set

$$m = \lim_{i \to \infty} m_i, \qquad M = \lim_{i \to \infty} M_i. \tag{1.7}$$

Then

$$M \geq \limsup_{i \to \infty} x_i \geq \liminf_{i \to \infty} x_i \geq m \tag{1.8}$$

and by the continuity of f,

$$m = f(m, M), \qquad M = f(M, m). \tag{1.9}$$

Therefore in view of (b),

$$m = M \tag{1.10}$$

from which the result follows. $\square$

Theorem 1.2. *Let $[a,b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b] \tag{1.11}$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ *is nonincreasing in* $x \in [a,b]$ *for each* $y \in [a,b]$, *and* $f(x,y)$ *is nondecreasing in* $y \in [a,b]$ *for each* $x \in [a,b]$;

(b) *the difference equation* (1.3) *has no solutions of minimal period two in* $[a,b]$. *Then* (1.3) *has a unique equilibrium* $\overline{x} \in [a,b]$ *and every solution of* (1.3) *converges to* $\overline{x}$.

Proof. Set

$$m_0 = a, \qquad M_0 = b \tag{1.12}$$

and for $i = 1,2,\ldots$ set

$$M_i = f\left(m_{i-1}, M_{i-1}\right), \qquad m_i = f\left(M_{i-1}, m_{i-1}\right). \tag{1.13}$$

Now observe that for each $i \geq 0$,

$$\begin{gathered} m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0, \\ m_i \leq x_k \leq M_i, \quad \text{for } k \geq 2i+1. \end{gathered} \tag{1.14}$$

Set

$$m = \lim_{i\to\infty} m_i, \qquad M = \lim_{i\to\infty} M_i. \tag{1.15}$$

Then clearly (1.8) holds and by the continuity of f,

$$m = f(M,m), \qquad M = f(m,M). \tag{1.16}$$

In view of (b),

$$m = M \tag{1.17}$$

from which the result follows. □

These results have been very useful in proving attractivity results for equilibrium or periodic solutions of (1.3) as well as for higher-order difference equations and systems of difference equations; see [2, 7–12]. Theorems 1.1 and 1.2 have attracted considerable attention of the leading specialists in difference equations and discrete dynamical systems and have been generalized and extended to the case of maps in $\mathbf{R}^n$, see [3], and maps in Banach space with the cone see [4–6]. In this paper, we will extend Theorems 1.1 and 1.2 to the case of monotone mappings in partially ordered complete metric spaces.

On the other hand, there has been recent interest in establishing fixed point theorems in partially ordered complete metric spaces with a contractivity condition which holds for all points that are related by partial ordering; see [13–20]. These fixed point results have been applied mainly to the existence of solutions of boundary value problems for differential equations and one of them, namely [20], has been applied to the problem of solving matrix equations. See also [21], where the application to the boundary value problems for integro-differential equations is given and [22] for application to some classes of nonexpansive mappings and [23] for the application of the Leray-Schauder theory to the problems of an impulsive boundary value problem under the condition of non-well-ordered upper and lower solutions. None of these results is global result, but they are rather existence results. In this paper, we combine the existence results with the results of the type of Theorems 1.1 and 1.2 to obtain global attractivity results.

2. Main Results: Mixed Monotone Case I

Let X be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Consider $X \times X$. We will use the following partial ordering.

For (x, y), $(u, v) \in X \times X$, we have

$$(x, y) \preceq (u, v) \Longleftrightarrow \{x \leq u, y \geq v\}. \tag{2.1}$$

This partial ordering is well known as "south-east ordering" in competitive systems in the plane; see [5, 6, 12, 24, 25].

Let d_1 be a metric on $X \times X$ defined as follows:

$$d_1((x, y), (u, v)) = d(x, u) + d(y, v). \tag{2.2}$$

Clearly

$$d_1((x, y), (u, v)) = d_1((y, x), (v, u)). \tag{2.3}$$

We prove the following theorem.

Theorem 2.1. *Let $F : X \times X \to X$ be a map such that $F(x, y)$ is nonincreasing in x for all $y \in X$, and nondecreasing in y for all $x \in X$. Suppose that the following conditions hold.*

(i) *There exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} d_1((x, y), (u, v)) \quad \forall (x, y) \preceq (u, v). \tag{2.4}$$

(ii) *There exists $x_0, y_0 \in X$ such that the following condition holds:*

$$x_0 \leq F(y_0, x_0), \qquad y_0 \geq F(x_0, y_0). \tag{2.5}$$

(iii) *If* $\{x_n\} \in X$ *is a nondecreasing convergent sequence such that* $\lim_{n\to\infty} x_n = x$, *then* $x_n \le x$, *for all* $n \in N$ *and if* $\{y_n\} \in Y$ *is a nonincreasing convergent sequence such that* $\lim_{n\to\infty} y_n = y$, *then* $y_n \ge y$, *for all* $n \in N$; *if* $x_n \le y_n$ *for every* n, *then* $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.

Then we have the following.

(a) *For every initial point* $(x_0, y_0) \in X \times X$ *such that condition* (2.5) *holds,* $F^n(x_0, y_0) \to x$, $F^n(y_0, x_0) \to y$, $n \to \infty$, *where* x, y *satisfy*

$$x = F(y, x), \qquad y = F(x, y). \tag{2.6}$$

If $x_0 \le y_0$ *in condition* (2.5), *then* $x \le y$. *If in addition* $x = y$, *then* $\{x_n\}, \{y_n\}$ *converge to the equilibrium of the equation*

$$x_{n+1} = F(y_n, x_n), \quad y_{n+1} = F(x_n, y_n), \quad n = 1, 2, \ldots. \tag{2.7}$$

(b) *In particular, every solution* $\{z_n\}$ *of*

$$z_{n+1} = F(z_n, z_{n-1}), \quad n = 2, 3, \ldots \tag{2.8}$$

such that $x_0 \le z_0,\ z_1 \le y_0$ *converges to the equilibrium of* (2.8).

(c) *The following estimates hold:*

$$d(F^n(y_0, x_0), x) \le \frac{1}{2}\frac{k^n}{1-k}\left[d(F(x_0, y_0), y_0) + d(F(y_0, x_0), x_0)\right], \tag{2.9}$$

$$d(F^n(x_0, y_0), y) \le \frac{1}{2}\frac{k^n}{1-k}\left[d(F(y_0, x_0), x_0) + d(F(x_0, y_0), y_0)\right]. \tag{2.10}$$

Proof. Let $x_1 = F(y_0, x_0)$ and $y_1 = F(x_0, y_0)$. Since $x_0 \le F(y_0, x_0) = x_1$ and $y_0 \ge F(x_0, y_0) = y_1$, for $x_2 = F(y_1, x_1),\ y_2 = F(x_1, y_1)$, we have

$$\begin{aligned} F^2(y_0, x_0) &:= F(F(x_0, y_0), F(y_0, x_0)) = F(y_1, x_1) = x_2, \\ F^2(x_0, y_0) &:= F(F(y_0, x_0), F(x_0, y_0)) = F(x_1, y_1) = y_2. \end{aligned} \tag{2.11}$$

Now, we have

$$\begin{aligned} x_2 &= F^2(y_0, x_0) = F(y_1, x_1) \ge F(y_0, x_0) = x_1, \\ y_2 &= F^2(x_0, y_0) = F(x_1, y_1) \le F(x_0, y_0) = y_1. \end{aligned} \tag{2.12}$$

For $n = 1, 2, \dots$, we let

$$\begin{aligned} x_{n+1} &= F^{n+1}(y_0, x_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)), \\ y_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)). \end{aligned} \tag{2.13}$$

By using the monotonicity of F, we obtain

$$\begin{aligned} &x_0 \le F(y_0, x_0) = x_1 \le F^2(y_0, x_0) = x_2 \le \cdots \le F^{n+1}(y_0, x_0) \le \cdots, \\ &y_0 \ge F(x_0, y_0) = y_1 \ge F^2(x_0, y_0) = y_2 \ge \cdots \ge F^{n+1}(x_0, y_0) \ge \cdots \end{aligned} \tag{2.14}$$

that is

$$\begin{aligned} &x_0 \le x_1 \le x_2 \le \cdots \\ &y_0 \ge y_1 \ge y_2 \ge \cdots. \end{aligned} \tag{2.15}$$

We claim that for all $n \in \mathbb{N}$ the following inequalities hold:

$$d(x_{n+1}, x_n) = d(F^{n+1}(y_0, x_0), F^n(y_0, x_0)) \le \frac{k^n}{2} d_1((x_1, y_1), (x_0, y_0)), \tag{2.16}$$

$$d(y_{n+1}, y_n) = d(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \le \frac{k^n}{2} d_1((x_1, y_1), (x_0, y_0)). \tag{2.17}$$

Indeed, for $n = 1$, using $x_0 \le F(y_0, x_0)$, $y_0 \ge F(x_0, y_0)$, and (2.3), we obtain

$$\begin{gathered} d(x_2, x_1) = d(F(y_1, x_1), F(y_0, x_0)) \le \frac{k}{2} d_1((y_1, x_1), (y_0, x_0)) = \frac{k}{2} d_1((x_1, y_1), (x_0, y_0)), \\ d(y_2, y_1) = d(F(x_1, y_1), F(x_0, y_0)) \le \frac{k}{2} d_1((x_1, y_1), (x_0, y_0)). \end{gathered} \tag{2.18}$$

Assume that (2.16) holds. Using the inequalities

$$\begin{aligned} &F^{n+1}(y_0, x_0) \ge F^n(y_0, x_0), \\ &F^{n+1}(x_0, y_0) \le F^n(x_0, y_0), \end{aligned} \tag{2.19}$$

and the contraction condition (2.4), we have

$$
\begin{aligned}
d(x_{n+2},x_{n+1}) &= d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0))\\
&= d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^{n}(x_0,y_0),F^{n}(y_0,x_0)))\\
&\le \frac{k}{2}[d(F^{n+1}(x_0,y_0),F^{n}(x_0,y_0))+d(F^{n+1}(y_0,x_0),F^{n}(y_0,x_0))]\\
&\le \frac{k}{2}\bigg[\frac{k^n}{2}(d(F(x_0,y_0),y_0)+d(F(y_0,x_0),x_0)+d(F(y_0,x_0),x_0)\\
&\qquad +d(F(x_0,y_0),y_0))\bigg]\\
&= \frac{k^{n+1}}{2}d_1((x_1,y_1),(x_0,y_0)).
\end{aligned}
\tag{2.20}
$$

Similarly,

$$
d(y_{n+2},y_{n+1}) = d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)) \le \frac{k^{n+1}}{2}d_1((x_1,y_1),(x_0,y_0)). \tag{2.21}
$$

This implies that $\{x_n\}=\{F^n(y_0,x_0)\}$ and $\{y_n\}=\{F^n(x_0,y_0)\}$ are Cauchy sequences in X.

Indeed,

$$
\begin{aligned}
d(F^n(y_0,x_0),F^{n+p}(y_0,x_0)) &\le d(F^n(y_0,x_0),F^{n+1}(y_0,x_0))+\cdots\\
&\quad + d(F^{n+p-1}(y_0,x_0),F^{n+p}(y_0,x_0))\\
&\le \frac{k^n}{2}[d(F(x_0,y_0),y_0)+d(F(y_0,x_0),x_0)]+\cdots+\\
&\quad + \frac{k^{n+p-1}}{2}[d(F(x_0y_0),y_0)+d(F(y_0,x_0),x_0)]\\
&= \frac{k^n}{2}(1+k+k^2+\cdots+k^{p-1})[d(F(x_0,y_0),y_0)+d(F(y_0,x_0),x_0)]\\
&= \frac{k^n}{2}\frac{1-k^p}{1-k}[d(F(x_0,y_0),y_0)+d(F(y_0,x_0),x_0)].
\end{aligned}
\tag{2.22}
$$

Since $k\in[0,1)$, we have

$$
d(x_n,x_{n+p}) = d(F^n(y_0,x_0),F^{n+p}(y_0,x_0)) \le \frac{k^n}{2(1-k)}d_1((x_1,y_1),(x_0,y_0)). \tag{2.23}
$$

Using (2.23), we conclude that $\{x_n\} = \{F^n(y_0, x_0)\}$ is a Cauchy sequence. Similarly, we conclude that $\{y_n\} = \{F^n(x_0, y_0)\}$ is a Cauchy sequence. Since X is a complete metric space, then there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} F^n(y_0, x_0) = x, \qquad \lim_{n\to\infty} y_n = \lim_{m\to\infty} F^m(x_0, y_0) = y. \tag{2.24}$$

Using the continuity of F, which follows from contraction condition (2.4), the equations

$$x_{n+1} = F(y_n, x_n), \qquad y_{n+1} = F(x_n, y_n) \tag{2.25}$$

imply (2.6).

Assume that $x_0 \leq y_0$. Then, in view of the monotonicity of F

$$\begin{aligned} x_1 &= F(y_0, x_0) \leq F(x_0, y_0) = y_1, \\ x_2 &= F(y_1, x_1) \leq F(x_1, y_1) = y_2, \\ x_3 &= F(y_2, x_2) \leq F(x_2, y_2) = y_3. \end{aligned} \tag{2.26}$$

By using induction, we can show that $x_n \leq y_n$ for all n. Assume that $x_0 \leq z_0,\ z_1 \leq y_0$. Then, in view of the monotonicity of F, we have

$$\begin{aligned} x_1 &= F(y_0, x_0) \leq F(z_1, z_0) = z_2 \leq F(x_0, y_0) = y_1, \\ x_1 &= F(y_0, x_0) \leq F(z_2, z_1) = z_3 \leq F(x_0, y_0) = y_1. \end{aligned} \tag{2.27}$$

Continuing in a similar way we can prove that $x_i \leq z_k \leq y_i$ for all $k \geq 2i+1$. By using condition (iii) we conclude that whenever $\lim_{n\to\infty} z_k$ exists we must have

$$x \leq \lim_{k\to\infty} z_k \leq y \tag{2.28}$$

which in the case when $x = y$ implies $\lim_{k\to\infty} z_k = x$.

By letting $p \to \infty$ in (2.23), we obtain the estimate (2.9). □

Remark 2.2. Property (iii) is usually called closedness of the partial ordering, see [6], and is an important ingredient of the definition of an ordered L-space; see [17, 19].

Theorem 2.3. *Assume that along with conditions (i) and (ii) of Theorem 2.1, the following condition is satisfied:*

(iv) *every pair of elements has either a lower or an upper bound.*

Then, the fixed point (x, y) is unique and $x = y$.

Proof. First, we prove that the fixed point (x, y) is unique. Condition (iv) is equivalent to the following. For every (x, y), $(x^*, y^*) \in X \times X$, there exists $(z_1, z_2) \in X \times X$ that is comparable to (x, y), (x^*, y^*). See [16].

Let (x, y) and (x^*, y^*) be two fixed points of the map F.

We consider two cases.

Case 1. If (x,y) is comparable to (x^*,y^*), then for all $n = 0,1,2,\dots$ $(F^n(y,x),F^n(x,y))$ is comparable to $(F^n(y^*,x^*),F^n(x^*,y^*)) = (x^*,y^*)$. We have to prove that

$$d_1((x,y),(x^*,y^*)) = 0. \tag{2.29}$$

Indeed, using (2.2), we obtain

$$\begin{aligned} d_1((x,y),(x^*,y^*)) &= d(x,x^*) + d(y,y^*) \\ &= d(F^n(y,x),F^n(y^*,x^*)) + d(F^n(x,y),F^n(x^*,y^*)). \end{aligned} \tag{2.30}$$

We estimate $d(F^n(y,x),F^n(y^*,x^*))$, and $d(F^n(x,y),F^n(x^*,y^*))$.

First, by using contraction condition (2.4), we have

$$\begin{aligned} d(F(y,x),F(y^*,x^*)) &\le \frac{k}{2}[d(y,y^*) + d(x,x^*)] = \frac{k}{2}d_1((x,y),(x^*,y^*)), \\ d(F(x,y),F(x^*,y^*)) &\le \frac{k}{2}[d(x,x^*) + d(y,y^*)] = \frac{k}{2}d_1((x,y),(x^*,y^*)). \end{aligned} \tag{2.31}$$

Now, by using (2.31) and (2.30), we have

$$d_1((x,y),(x^*,y^*)) \le k d_1((x,y),(x^*,y^*)) < d_1((x,y),(x^*,y^*)), \tag{2.32}$$

which implies that

$$d_1((x,y),(x^*,y^*)) = 0. \tag{2.33}$$

Case 2. If (x,y) is not comparable to (x^*,y^*), then there exists an upper bound or a lower bound (z_1,z_2) of (x,y) and (x^*,y^*). Then, $(F^n(z_2,z_1),F^n(z_1,z_2))$ is comparable to $(F^n(y,x),F^n(x,y))$ and $(F^n(y^*,x^*),F^n(x^*,y^*))$.

Therefore, we have

$$\begin{aligned} d_1((x,y),(x^*,y^*)) &= d_1((F^n(y,x),F^n(x,y)),(F^n(y^*,x^*),F^n(x^*,y^*))) \\ &\le d_1((F^n(y,x),F^n(x,y)),(F^n(z_2,z_1),F^n(z_1,z_2))) \\ &\quad + d_1((F^n(z_2,z_1),F^n(z_1,z_2)),(F^n(y^*,x^*),F^n(x^*,y^*))) \\ &= d((F^n(y,x),F^n(z_2,z_1))) + d(F^n(z_2,z_1),F^n(y^*,x^*)) \\ &\quad + d(F^n(z_1,z_2),F^n(x^*,y^*)) + d(F^n(z_2,z_1),F^n(y^*,x^*)). \end{aligned} \tag{2.34}$$

Now, we obtain

$$d_1((x,y),(x^*,y^*)) = d((F^n(y,x),F^n(z_2,z_1))) + d(F^n(z_2,z_1),F^n(y^*,x^*)) \\ + d(F^n(z_1,z_2),F^n(x^*,y^*)) + d(F^n(z_2,z_1),F^n(y^*,x^*)). \tag{2.35}$$

We now estimate the right-hand side of (2.35).

First, by using

$$d(F(y,x),F(z_2,z_1)) \le \frac{k}{2}(d(y,z_2) + d(x,z_1)), \tag{2.36}$$

we have

$$\begin{aligned} d(F^2(y,x),F^2(z_2,z_1)) &= d(F(F(x,y),F(y,x)),F(F(z_1,z_2),F(z_2,z_1))) \\ &\le \frac{k}{2}[d(F(x,y),F(z_1,z_2)) + d(F(y,x),F(z_2,z_1))] \\ &\le \frac{k}{2}\left[\frac{k}{2}(d(x,z_1) + d(y,z_2)) + \frac{k}{2}(d(y,z_2) + d(x,z_1))\right] \\ &= \frac{k^2}{2}(d(x,z_1) + d(y,z_2)). \end{aligned} \tag{2.37}$$

Similarly,

$$\begin{aligned} d(F^2(x,y),F^2(z_1,z_2)) &= d(F(F(y,x),F(x,y)),F(F(z_2,z_1),F(z_1,z_2))) \\ &\le \frac{k}{2}[d(F(y,x),F(z_2,z_1)) + d(F(x,y),F(z_1,z_2))] \\ &\le \frac{k}{2}\left[\frac{k}{2}(d(y,z_2) + d(x,z_1)) + \frac{k}{2}(d(y,z_2) + d(x,z_1))\right] \\ &= \frac{k^2}{2}(d(x,z_1) + d(y,z_2)). \end{aligned} \tag{2.38}$$

So,

$$\begin{aligned} d(F^2(y,x),F^2(z_2,z_1)) &\le \frac{k^2}{2}(d(x,z_1) + d(y,z_2)), \\ d(F^2(x,y),F^2(z_1,z_2)) &\le \frac{k^2}{2}(d(x,z_1) + d(y,z_2)). \end{aligned} \tag{2.39}$$

Using induction, we obtain

$$
\begin{aligned}
d\big(F^n(y,x),F^n(z_2,z_1)\big) &\le \frac{k^n}{2}\big(d(x,z_1)+d(y,z_2)\big),\\
d\big(F^n(x,y),F^n(z_1,z_2)\big) &\le \frac{k^n}{2}\big(d(x,z_1)+d(y,z_2)\big),\\
d\big(F^n(z_2,z_1),F^n(y^*,x^*)\big) &\le \frac{k^n}{2}\big(d(z_1,x^*),d(z_2,y^*)\big),\\
d\big(F^n(z_1,z_2),F^n(x^*,y^*)\big) &\le \frac{k^n}{2}\big(d(z_1,x^*),d(z_2,y^*)\big).
\end{aligned}
\tag{2.40}
$$

Using (2.40), relation (2.35) becomes

$$
\begin{aligned}
d_1\big((x,y),(x^*,y^*)\big) \le{}& \frac{k^n}{2}\big(d(x,z_1)+d(y,z_2)\big)+\frac{k^n}{2}\big(d(x,z_1)+d(y,z_2)\big)\\
&+\frac{k^n}{2}\big(d(z_1,x^*)+d(z_2,y^*)\big)+\frac{k^n}{2}\big(d(z_1,x^*)+d(z_2,y^*)\big)\\
={}& k^n\big(d(x,z_1)+d(y,z_2)+d(z_1,x^*)+d(z_2,y^*)\big)\longrightarrow 0,\quad n\longrightarrow\infty.
\end{aligned}
\tag{2.41}
$$

So,

$$
d_1\big((x,y),(x^*,y^*)\big)=0. \tag{2.42}
$$

Finally, we prove that $x=y$. We will consider two cases.

Case A. If x is comparable to y, then $F(y,x)=x$ is comparable to $F(x,y)=y$. Now, we obtain

$$
d(x,y)=d(F(y,x),F(x,y))\le\frac{k}{2}[d(x,y)+d(y,x)]=kd(x,y), \tag{2.43}
$$

since $k\in[0,1)$, this implies

$$
d(x,y)=0\Longleftrightarrow x=y. \tag{2.44}
$$

Case B. If x is not comparable to y, then there exists an upper bound or alower bound of x and y, that is, there exists $z\in X$ such that $x\le z,\ \ y\le z$. Then by using monotonicity character of F, we have

$$\begin{aligned} F(x,y) &\le F(x,z), \qquad F(y,x) \le F(y,z), \\ F(x,y) &\ge F(z,y), \qquad F(y,x) \ge F(z,x). \end{aligned} \tag{2.45}$$

Now,

$$F^2(x,y) = F(F(y,x),F(x,y)) \le F(F(z,x),F(x,z)) = F^2(x,z), \tag{2.46}$$

that is

$$F^2(x,y) \le F^2(x,z). \tag{2.47}$$

Furthermore,

$$F^2(x,y) = F(F(y,x),F(x,y)) \ge F(F(y,z),F(z,y)) = F^2(y,z), \tag{2.48}$$

that is

$$F^2(x,y) \ge F^2(y,z). \tag{2.49}$$

Similarly,

$$F^2(y,x) = F(F(x,y),F(y,x)) \le F(F(z,y),F(y,z)) = F^2(y,z), \tag{2.50}$$

that is

$$F^2(y,x) \le F^2(y,z), \tag{2.51}$$

and

$$F^2(y,x) = F(F(x,y),F(y,x)) \ge F(F(x,z),F(z,x)) = F^2(z,x). \tag{2.52}$$

By using induction, we have

$$\begin{aligned} F^{n+1}(x,y) &\le F^{n+1}(x,z), \\ F^{n+1}(x,y) &\ge F^{n+1}(y,z), \\ F^{n+1}(y,x) &\le F^{n+1}(y,z), \\ F^{n+1}(y,x) &\ge F^{n+1}(z,x). \end{aligned} \tag{2.53}$$

Since (x,y) is a fixed point, we obtain

$$\begin{aligned}
d(x,y) &= d\left(F^{n+1}(y,x),F^{n+1}(x,y)\right)\\
&= d\left(F\left(F^{n}(x,y),F^{n}(y,x)\right),F\left(F^{n}(y,x),F^{n}(x,y)\right)\right)\\
&\le d\left(F\left(F^{n}(x,y),F^{n}(y,x)\right),F\left(F^{n}(x,z),F^{n}(z,x)\right)\right)\\
&\quad + d\left(F\left(F^{n}(x,z),F^{n}(z,x)\right),F\left(F^{n}(y,x),F^{n}(x,y)\right)\right)\\
&\le d\left(F\left(F^{n}(x,y),F^{n}(y,x)\right),F\left(F^{n}(x,z),F^{n}(z,x)\right)\right)\\
&\quad + d\left(F\left(F^{n}(z,x),F^{n}(x,z)\right),F\left(F^{n}(x,z),F^{n}(z,x)\right)\right)\\
&\quad + d\left(F\left(F^{n}(y,x),F^{n}(x,y)\right),F\left(F^{n}(z,x),F^{n}(x,z)\right)\right).
\end{aligned}\tag{2.54}$$

Using the contractivity condition (2.4) on F, we have

$$\begin{aligned}
d(x,y) &\le \frac{k}{2}\left[d\left(F^{n}(x,y),F^{n}(x,z)\right)+d\left(F^{n}(y,x),F^{n}(z,x)\right)\right]\\
&\quad + \frac{k}{2}\left[d\left(F^{n}(z,x),F^{n}(x,z)\right)+d\left(F^{n}(x,z),F^{n}(z,x)\right)\right]\\
&\quad + \frac{k}{2}\left[d\left(F^{n}(y,x),F^{n}(z,x)\right)+d\left(F^{n}(y,x),F^{n}(x,y)\right)\right]\\
&= \frac{k}{2}\left[2d\left(F^{n}(x,y),F^{n}(x,z)\right)+2d\left(F^{n}(y,x),F^{n}(z,x)\right)+2d\left(F^{n}(x,z),F^{n}(z,x)\right)\right]\\
&= k\left[d\left(F^{n}(x,y),F^{n}(x,z)\right)+d\left(F^{n}(y,x),F^{n}(z,x)\right)+d\left(F^{n}(x,z),F^{n}(z,x)\right)\right].
\end{aligned}\tag{2.55}$$

Now, we estimate the terms on the right-hand side

$$\begin{aligned}
d\left(F^{n}(x,y),F^{n}(x,z)\right) &= d\left(F\left(F^{n-1}(y,x),F^{n-1}(x,y)\right),F\left(F^{n-1}(z,x),F^{n-1}(x,z)\right)\right)\\
&\le \frac{k}{2}\left[d\left(F^{n-1}(y,x),F^{n-1}(z,x)\right)+d\left(F^{n-1}(x,y),F^{n-1}(x,z)\right)\right],\\
d\left(F^{n}(y,x),F^{n}(z,x)\right) &= d\left(F\left(F^{n-1}(x,y),F^{n-1}(y,x)\right),F\left(F^{n-1}(x,z),F^{n-1}(z,x)\right)\right)\\
&\le \frac{k}{2}\left[d\left(F^{n-1}(x,y),F^{n-1}(x,z)\right)+d\left(F^{n-1}(y,x),F^{n-1}(z,x)\right)\right],\\
d\left(F^{n}(x,z),F^{n}(z,x)\right) &= d\left(F\left(F^{n-1}(z,x),F^{n-1}(x,z)\right),F\left(F^{n-1}(x,z),F^{n-1}(z,x)\right)\right)\\
&\le \frac{k}{2}\left[d\left(F^{n-1}(z,x),F^{n-1}(x,z)\right)+d\left(F^{n-1}(x,z),F^{n-1}(z,x)\right)\right].
\end{aligned}\tag{2.56}$$

Now, we have

$$d(x,y) \le k^2\left[d\left(F^{n-1}(y,x),F^{n-1}(z,x)\right)+d\left(F^{n-1}(x,y),F^{n-1}(x,z)\right)+d\left(F^{n-1}(z,x),F^{n-1}(x,z)\right)\right]. \tag{2.57}$$

Continuing this process, we obtain

$$d(x,y) \le k^n[d(F(y,x),F(z,x))+d(F(x,y),F(x,z))+d(F(z,x),F(x,z))]. \tag{2.58}$$

Using the contractivity of F, we have

$$\begin{aligned} d(x,y) &\le k^n\left[\frac{k}{2}(d(x,x)+d(y,z)+d(y,z)+d(x,x)+d(x,z)+d(z,x))\right] \\ &= k^{n+1}(d(y,z)+d(z,x)). \end{aligned} \tag{2.59}$$

That is

$$d(x,y) \le k^{n+1}(d(y,z)+d(z,x)) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.60}$$

So,

$$d(x,y)=0 \Longleftrightarrow x=y. \tag{2.61}$$

□

3. Main Results: Mixed Monotone Case II

Let X be a partially ordered set and let d be a metric on X such that (X,d) is a complete metric space. Consider $X\times X$. We will use the following partial order.

For (x,y), $(u,v)\in X\times X$, we have

$$(x,y) \preceq (u,v) \Longleftrightarrow \{x \ge u, y \le v\}. \tag{3.1}$$

Let d_1 be a metric on $X\times X$ defined as follows:

$$d_1((x,y),(u,v)) = d(x,u)+d(y,v). \tag{3.2}$$

The following two theorems have similar proofs to the proofs of Theorems 2.1 and 2.3, respectively, and so their proofs will be skipped. Significant parts of these results have been included in [14] and applied successfully to some boundary value problems in ordinary differential equations.

Theorem 3.1. *Let $F : X \times X \to X$ be a map such that $F(x, y)$ is nondecreasing in x for all $y \in X$, and nonincreasing in y for all $x \in X$. Suppose that the following conditions hold.*

(i) *There exists $k \in [0, 1)$ with*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2} d_1((x,y),(u,v)) \quad \forall (x,y) \preceq (u,v). \tag{3.3}$$

(ii) *There exists $x_0, y_0 \in X$ such that the following condition holds:*

$$x_0 \leq F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0). \tag{3.4}$$

(iii) *If $\{x_n\} \in X$ is a nondecreasing convergent sequence such that $\lim_{n\to\infty} x_n = x$, then $x_n \leq x$, for all $n \in N$ and if $\{y_n\} \in Y$ is a nonincreasing convergent sequence such that $\lim_{n\to\infty} y_n = y$, then $y_n \geq y$, for all $n \in N$; if $x_n \leq y_n$ for every n, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.*

Then we have the following.

(a) *For every initial point $(x_0, y_0) \in X \times X$ such that the condition (3.2) holds, $F^n(x_0, y_0) \to x$, $F^n(y_0, x_0) \to y$, $n \to \infty$, where x, y satisfy*

$$x = F(x, y), \qquad y = F(y, x). \tag{3.5}$$

If $x_0 \leq y_0$ in condition (3.4), then $x \leq y$. If in addition $x = y$, then $\{x_n\}, \{y_n\}$ converge to the equilibrium of the equation

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \quad n = 1, 2, \ldots. \tag{3.6}$$

(b) *In particular, every solution $\{z_n\}$ of*

$$z_{n+1} = F(z_n, z_{n-1}), \quad n = 2, 3, \ldots \tag{3.7}$$

such that $x_0 \leq z_0, z_1 \leq y_0$ converges to the equilibrium of (3.7).

(c) *The following estimates hold:*

$$\begin{aligned} d(F^n(x_0,y_0),x) &\leq \frac{1}{2}\frac{k^n}{1-k}\left[d(F(x_0,y_0),x_0) + d(F(y_0,x_0),y_0)\right], \\ d(F^n(y_0,x_0),y) &\leq \frac{1}{2}\frac{k^n}{1-k}\left[d(F(x_0,y_0),x_0) + d(F(y_0,x_0),y_0)\right]. \end{aligned} \tag{3.8}$$

Theorem 3.2. *Assume that along with conditions (i) and (ii) of Theorem 3.1, the following condition is satisfied:*

(iv) *every pair of elements has either a lower or an upper bound.*

Then, the fixed point (x, y) is unique and $x = y$.

Remark 3.3. Theorems 3.1 and 3.2 generalize and extend the results in [14]. The new feature of our results is global attractivity part that extends Theorems 1.1 and 1.2. Most of presented ideas were presented for the first time in [14].

Acknowledgment

The authors are grateful to the referees for pointing out few fine details that improved the presented results.

References

[1] M. R. S. Kulenović, G. Ladas, and W. S. Sizer, "On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$," *Mathematical Sciences Research Hot-Line*, vol. 2, no. 5, pp. 1–16, 1998.

[2] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjecture*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[3] M. R. S. Kulenović and O. Merino, "A global attractivity result for maps with invariant boxes," *Discrete and Continuous Dynamical Systems. Series B*, vol. 6, no. 1, pp. 97–110, 2006.

[4] R. D. Nussbaum, "Global stability, two conjectures and Maple," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 5, pp. 1064–1090, 2007.

[5] H. L. Smith, "The discrete dynamics of monotonically decomposable maps," *Journal of Mathematical Biology*, vol. 53, no. 4, pp. 747–758, 2006.

[6] H. L. Smith, "Global stability for mixed monotone systems," *Journal of Difference Equations and Applications*, vol. 14, no. 10-11, pp. 1159–1164, 2008.

[7] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*, vol. 5 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2008.

[8] C. H. Gibbons, M. R. S. Kulenović, and G. Ladas, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-1})/(\gamma + x_n)$," *Mathematical Sciences Research Hot-Line*, vol. 4, no. 2, pp. 1–11, 2000.

[9] C. H. Gibbons, M. R. S. Kulenović, G. Ladas, and H. D. Voulov, "On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$," *Journal of Difference Equations and Applications*, vol. 8, no. 1, pp. 75–92, 2002.

[10] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, vol. 4 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.

[11] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[12] M. R. S. Kulenović and M. Nurkanović, "Asymptotic behavior of a system of linear fractional difference equations," *Journal of Inequalities and Applications*, vol. 2005, no. 2, pp. 127–143, 2005.

[13] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[14] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[15] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[16] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[17] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, "Fixed point theorems in ordered abstract spaces," *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.

[18] D. O'Regan and A. Petruşel, "Fixed point theorems for generalized contractions in ordered metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1241–1252, 2008.

[19] A. Petruşel and I. A. Rus, "Fixed point theorems in ordered L-spaces," *Proceedings of the American Mathematical Society*, vol. 134, no. 2, pp. 411–418, 2006.

[20] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[21] B. Ahmad and J. J. Nieto, "The monotone iterative technique for three-point second-order integrodifferential boundary value problems with p-Laplacian," *Boundary Value Problems*, vol. 2007, Article ID 57481, 9 pages, 2007.

[22] Y. Su, D. Wang, and M. Shang, "Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 284613, 8 pages, 2008.

[23] X. Xian, D. O'Regan, and R. P. Agarwal, "Multiplicity results via topological degree for impulsive boundary value problems under non-well-ordered upper and lower solution conditions," *Boundary Value Problems*, vol. 2008, Article ID 197205, 21 pages, 2008.

[24] M. R. S. Kulenović and O. Merino, "Competitive-exclusion versus competitive-coexistence for systems in the plane," *Discrete and Continuous Dynamical Systems. Series B*, vol. 6, no. 5, pp. 1141–1156, 2006.

[25] H. L. Smith, "Planar competitive and cooperative difference equations," *Journal of Difference Equations and Applications*, vol. 3, no. 5-6, pp. 335–357, 1998.