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Research Article

Convex Solutions of a Nonlinear Integral Equation of Urysohn Type

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We study the solvability of a nonlinear integral equation of Urysohn type. Using the technique of measures of noncompactness we prove that under certain assumptions this equation possesses solutions that are convex of order p for each $p \in \{-1, 0, \dots, r\}$, with $r \geq -1$ being a given integer. A concrete application of the results obtained is presented.

1. Introduction

Existence of solutions of differential and integral equations is subject of numerous investigations (see, e.g., the monographs [1–3] or [4]). Moreover, a lot of work in this domain is devoted to the existence of solutions in certain special classes of functions (e.g., positive functions or monotone functions). We merely mention here the result obtained by Caballero et al. [5] concerning the existence of nondecreasing solutions to the integral equation of Urysohn type

$$x(t) = a(t) + u(t, x(t)) \int_0^T v(t, s, x(s))\, ds, \quad t \in [0, T], \tag{1.1}$$

where T is a positive constant. In the special case when $u(t, x) := x^2$ (or even $u(t, x) := x^n$), the authors proved in [5] that if a is positive and nondecreasing, v is positive and nondecreasing in the first variable (when the other two variables are kept fixed), and they satisfy some additional assumptions, then there exists at least one positive nondecreasing solution $x : [0, T] \to \mathbb{R}$ to (1.1). A similar existence result, but involving a Volterra type integral equation, has been obtained by Banaś and Martinon [6].

It should be noted that both existence results were proved with the help of a measure of noncompactness related to monotonicity introduced by Banaś and Olszowy [7]. The reader is referred also to the paper by Banaś et al. [8], in which another measure of noncompactness is used to prove the solvability of an integral equation of Urysohn type on an unbounded interval.

The main purpose of the present paper is twofold. First, we generalize the result from the paper [5] to the framework of higher-order convexity. Namely, we show that given an integer $r \geq -1$, if a and v are convex of order p for each $p \in \{-1, 0, \ldots, r\}$, then (1.1) possesses at least one solution which is also convex of order p for each $p \in \{-1, 0, \ldots, r\}$. Second, we simplify the proof given in [5] by showing that it is not necessary to make use of the measure of noncompactness related to monotonicity introduced by Banaś and Olszowy [7].

2. Measures of Noncompactness

Measures of noncompactness are frequently used in nonlinear analysis, in branches such as the theory of differential and integral equations, the operator theory, or the approximation theory. There are several axiomatic approaches to the concept of a measure of noncompactness (see, e.g., [9–11] or [12]). In the present paper the definition of a measure of noncompactness given in the book by Banaś and Goebel [12] is adopted.

Let E be a real Banach space, let $\mathcal{M}_E$ be the family consisting of all nonempty bounded subsets of E, and let $\mathcal{N}_E$ be the subfamily of $\mathcal{M}_E$ consisting of all relatively compact sets. Given any subset X of E, we denote by $\operatorname{cl} X$ and $\operatorname{co} X$ the closure and the convex hull of X, respectively.

Definition 2.1 (see [12]). A function $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a *measure of noncompactness* in E if it satisfies the following conditions.

(1) The family $\ker \mu := \{X \in \mathcal{M}_E \mid \mu(X) = 0\}$ (called the *kernel* of μ) is nonempty and it satisfies $\ker\mu \subseteq \mathcal{N}_E$.

(2) $\mu(X) \leq \mu(Y)$ whenever $X, Y \in \mathcal{M}_E$ satisfy $X \subseteq Y$.

(3) $\mu(X) = \mu(\operatorname{cl} X) = \mu(\operatorname{co} X)$ for all $X \in \mathcal{M}_E$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0,1]$ and all $X, Y \in \mathcal{M}_E$.

(5) If (X_n) is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subseteq X_n$ for each positive integer n and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An important and very convenient measure of noncompactness is the so-called *Hausdorff measure of noncompactness* $\chi : \mathcal{M}_E \to [0, \infty)$, defined by

$$\chi(X) := \inf \{\varepsilon \in (0, \infty) \mid X \text{ possesses a finite } \varepsilon-\text{net in } X\}. \tag{2.1}$$

The importance of this measure of noncompactness is given by the fact that in certain Banach spaces it can be expressed by means of handy formulas. For instance, consider the Banach space $C := C[a,b]$ consisting of all continuous functions $x : [a,b] \to \mathbb{R}$, endowed with the standard maximum norm

$$\|x\| := \max \{|x(t)| \mid t \in [a,b]\}. \tag{2.2}$$

Given $X \in \mathcal{M}_C$, $x \in X$, and $\varepsilon > 0$, let

$$\omega(x,\varepsilon) := \sup\{|x(t) - x(s)| \mid t,\, s \in [a,b],\, |t-s| \le \varepsilon\} \tag{2.3}$$

be the usual modulus of continuity of x. Further, let

$$\omega(X,\varepsilon) := \sup\{\omega(x,\varepsilon) \mid x \in X\}, \tag{2.4}$$

and $\omega_0(X) := \lim_{\varepsilon\to 0+}\omega(X,\varepsilon)$. Then it can be proved (see Banaś and Goebel [12, Theorem 7.1.2]) that

$$\chi(X) = \frac{1}{2}\omega_0(X) \quad \forall X \in \mathcal{M}_C. \tag{2.5}$$

For further facts concerning measures of noncompactness and their properties the reader is referred to the monographs [9, 11] or [12]. We merely recall here the following fixed point theorem.

Theorem 2.2 (see [12, Theorem 5.1]). *Let E be a real Banach space, let $\mu : \mathcal{M}_E \to [0,\infty)$ be a measure of noncompactness in E, and let Q be a nonempty bounded closed convex subset of E. Further, let $F : Q \to Q$ be a continuous operator such that $\mu(F(X)) \le k\mu(X)$ for each subset X of Q, where $k \in [0,1)$ is a constant. Then F has at least one fixed point in Q.*

3. Convex Functions of Higher Orders

Let $I \subseteq \mathbb{R}$ be a nondegenerate interval. Given an integer $p \ge -1$, a function $x : I \to \mathbb{R}$ is said to be *convex of order p* or *p-convex* if

$$[t_0, t_1, \ldots, t_{p+1};\ x] \ge 0 \tag{3.1}$$

for any system $t_0 < t_1 < \cdots < t_{p+1}$ of $p+2$ points in I, where

$$\begin{aligned}
[t_0, t_1, \ldots, t_{p+1};\ x] := {}& \frac{1}{(t_0-t_1)(t_0-t_2)\cdots(t_0-t_{p+1})}\, x(t_0) \\
&+ \frac{1}{(t_1-t_0)(t_1-t_2)\cdots(t_1-t_{p+1})}\, x(t_1) + \cdots \\
&+ \frac{1}{(t_{p+1}-t_0)(t_{p+1}-t_1)\cdots(t_{p+1}-t_p)}\, x(t_{p+1})
\end{aligned} \tag{3.2}$$

is called the *divided difference* of x at the points $t_0, t_1, \ldots, t_{p+1}$. With the help of the polynomial function defined by

$$\omega(t) := (t-t_0)(t-t_1)\cdots(t-t_{p+1}), \tag{3.3}$$

the previous divided difference can be written as

$$[t_0, t_1, \ldots, t_{p+1}; x] = \sum_{k=0}^{p+1} \frac{x(t_k)}{\omega'(t_k)}. \tag{3.4}$$

An alternative way to define the divided difference $[t_0, t_1, \ldots, t_{p+1}; x]$ is to set

$$\begin{gathered}
[t_i; x] := x(t_i) \quad \text{for each } i \in \{0, 1, \ldots, p+1\}, \\
[t_i, t_{i+1}, \ldots, t_{i+j}; x] := \frac{[t_i, \ldots, t_{i+j-1}; x] - [t_{i+1}, \ldots, t_{i+j}; x]}{t_i - t_{i+j}},
\end{gathered} \tag{3.5}$$

whenever $j \in \{0, 1, \ldots, p+1-i\}$. Finally, we mention a representation of the divided difference by means of two determinants. It can be proved that

$$[t_0, t_1, \ldots, t_{p+1}; x] = \frac{U(t_0, t_1, \ldots, t_{p+1}; x)}{V(t_0, t_1, \ldots, t_{p+1})}, \tag{3.6}$$

where

$$\begin{gathered}
U(t_0, t_1, \ldots, t_{p+1}; x) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{p+1} \\ t_0^2 & t_1^2 & \cdots & t_{p+1}^2 \\ \vdots & \vdots & \cdots & \vdots \\ t_0^p & t_1^p & \cdots & t_{p+1}^p \\ x(t_0) & x(t_1) & \cdots & x(t_{p+1}) \end{vmatrix}, \\
V(t_0, t_1, \ldots, t_{p+1}) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{p+1} \\ t_0^2 & t_1^2 & \cdots & t_{p+1}^2 \\ \vdots & \vdots & \cdots & \vdots \\ t_0^{p+1} & t_1^{p+1} & \cdots & t_{p+1}^{p+1} \end{vmatrix}.
\end{gathered} \tag{3.7}$$

Note that a convex function of order -1 is a nonnegative function, a convex function of order 0 is a nondecreasing function, while a convex function of order 1 is an ordinary convex function.

Let $I \subseteq \mathbb{R}$ be a nondegenerate interval, let $x : I \to \mathbb{R}$ be an arbitrary function, and let $h \in \mathbb{R}$. The *difference operator* Δ_h with the span h is defined by

$$(\Delta_h x)(t) := x(t+h) - x(t) \tag{3.8}$$

for all $t \in I$ for which $t+h \in I$. The iterates Δ_h^p $(p = 0, 1, 2, \dots)$ of Δ_h are defined recursively by

$$\Delta_h^0 x := x, \quad \Delta_h^{p+1} x := \Delta_h\left(\Delta_h^p x\right) \quad \text{for } p = 0, 1, 2, \dots. \tag{3.9}$$

It can be proved (see, e.g., [13, page 368, Corollary 3]) that

$$\left(\Delta_h^p x\right)(t) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} x(t+kh) \tag{3.10}$$

for every $t \in I$ for which $t + ph \in I$. On the other hand, the equality

$$\left[t, t+h, t+2h, \dots, t+ph;\ x\right] = \frac{\left(\Delta_h^p x\right)(t)}{p!h^p} \tag{3.11}$$

holds for every nonnegative integer p and every $t \in I$ for which $t + ph \in I$.

Let $I \subseteq \mathbb{R}$ be a nondegenerate interval. Given an integer $p \geq -1$, a function $x : I \to \mathbb{R}$ is called *Jensen convex of order* p or *Jensen* p*-convex* if

$$\left(\Delta_h^{p+1} x\right)(t) \geq 0 \tag{3.12}$$

for all $t \in I$ and all $h > 0$ such that $t + (p+1)h \in I$. Due to (3.11), it is clear that every convex function of order p is also Jensen convex of order p. In general, the converse does not hold. However, under the additional assumption that x is continuous, the two notions turn out to be equivalent.

Theorem 3.1 (see [13, page 387, Theorem 1]). *Let $I \subseteq \mathbb{R}$ be a nondegenerate interval, let $p \geq -1$ be an integer, and let $x : I \to \mathbb{R}$ be a continuous function. Then x is convex of order p if and only if it is Jensen convex of order p.*

Finally, we mention the following result concerning the difference of order p of a product of two functions:

Lemma 3.2. *Let $I \subseteq \mathbb{R}$ be a nondegenerate interval, and let p be a nonnegative integer. Given two functions $x, y : I \to \mathbb{R}$, the equality*

$$\left(\Delta_h^p xy\right)(t) = \sum_{k=0}^{p} \binom{p}{k} \left(\Delta_h^k x\right)(t) \cdot \left(\Delta_h^{p-k} y\right)(t+kh) \tag{3.13}$$

holds for every $t \in I$ such that $t + ph \in I$.

4. Main Results

Throughout this section T is a positive real number. In the space $C[0,T]$, consisting of all continuous functions $x : [0,T] \to \mathbb{R}$, we consider the usual maximum norm

$$\|x\| := \max\{|x(t)| \mid t \in [0,T]\}. \tag{4.1}$$

Our first main result concerns the integral equation of Urysohn type (1.1) in which a, u, and v are given functions, while x is the unknown function. We assume that the functions a, u, and v satisfy the following conditions:

(C_1) $r \geq -1$ is a given integer number;

(C_2) $a : [0,T] \to \mathbb{R}$ is a continuous function which is convex of order p for each $p \in \{-1,0,\dots,r\}$;

(C_3) $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u(t,0) = 0$ for all $t \in [0,T]$ and the function

$$t \in [0,T] \longmapsto u(t,x(t)) \in \mathbb{R} \tag{4.2}$$

is convex of order p for each $p \in \{-1,0,\dots,r\}$ whenever $x \in C[0,T]$ is convex of order p for each $p \in \{-1,0,\dots,r\}$;

(C_4) there exists a continuous function $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ which is nondecreasing in each variable and satisfies

$$\left|u(t,x) - u(t,y)\right| \leq \left|x-y\right|\varphi(x,y) \tag{4.3}$$

for all $t \in [0,T]$ and all $x, y \in [0,\infty)$;

(C_5) $v : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that the function $v(\cdot,s,x) : [0,T] \to \mathbb{R}$ is convex of order p for each $p \in \{-1,0,\dots,r\}$ whenever $s \in [0,T]$ and $x \in [0,\infty)$;

(C_6) there exists a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$|v(t,s,x)| \leq \psi(|x|) \quad \forall t,s \in [0,T],\ x \in \mathbb{R}; \tag{4.4}$$

(C_7) there exists $r_0 > 0$ such that

$$\|a\| + Tr_0\varphi(r_0,0)\psi(r_0) \leq r_0, \qquad T\varphi(r_0,r_0)\psi(r_0) < 1. \tag{4.5}$$

Theorem 4.1. *If the conditions (C_1)–(C_7) are satisfied, then* (1.1) *possesses at least one solution $x \in C[0,T]$ which is convex of order p for each $p \in \{-1,0,\dots,r\}$.*

Proof. Consider the operator F, defined on $C[0,T]$ by

$$(Fx)(t) := a(t) + u(t,x(t))\int_0^T v(t,s,x(s))ds, \quad t \in [0,T]. \tag{4.6}$$

Then $Fx \in C[0,T]$ whenever $x \in C[0,T]$ (see [5, the proof of Theorem 3.2]).

We claim that F is continuous on $C[0,T]$. To this end we fix any x_0 in $C[0,T]$ and prove that F is continuous at x_0. Let $c := \|x_0\| + 1$, and let

$$\begin{aligned} M_1 &:= \max\{|u(t,x)| \mid t \in [0,T],\ x \in [-\|x_0\|, \|x_0\|]\} \\ M_2 &:= \max\{|v(t,s,x)| \mid t,s \in [0,T],\ x \in [-c,c]\}. \end{aligned} \tag{4.7}$$

Further, let $\varepsilon > 0$. The uniform continuity of u on $[0,T] \times [-c,c]$ as well as that of v on $[0,T] \times [0,T] \times [-c,c]$ ensures the existence of a real number $\delta > 0$ such that

$$\left|u(t,x) - u(t,y)\right| < \varepsilon, \qquad \left|v(t,s,x) - v(t,s,y)\right| < \varepsilon \tag{4.8}$$

for all $t,s \in [0,T]$ and all $x,y \in [-c,c]$ satisfying $|x-y| < \delta$. Then for every $x \in C[0,T]$ such that $\|x - x_0\| < \min\{1,\varepsilon,\delta\}$ and every $t \in [0,T]$ we have

$$\begin{aligned} |(Fx)(t) - (Fx_0)(t)| &\le \left|[u(t,x(t)) - u(t,x_0(t))]\int_0^T v(t,s,x(s))ds\right| \\ &\quad + \left|u(t,x_0(t))\int_0^T [v(t,s,x(s)) - v(t,s,x_0(s))]ds\right| \\ &\le |u(t,x(t)) - u(t,x_0(t))|\int_0^T |v(t,s,x(s))|ds \\ &\quad + |u(t,x_0(t))|\int_0^T |v(t,s,x(s)) - v(t,s,x_0(s))|ds \\ &\le \varepsilon T(M_1 + M_2). \end{aligned} \tag{4.9}$$

Therefore, the inequality $\|Fx - Fy\| \le \varepsilon T(M_1 + M_2)$ holds for every x in $C[0,T]$ satisfying $\|x - x_0\| < \min\{1,\varepsilon,\delta\}$. This proves the continuity of F at x_0.

Next, let r_0 be the positive real number whose existence is assured by (C_7), and let Q be the subset of $C[0,T]$, consisting of all functions x such that $\|x\| \le r_0$ and x is convex of order p for each $p \in \{-1,0,\ldots,r\}$. Obviously, Q is a nonempty bounded closed convex subset of $C[0,T]$. We claim that F maps Q into itself. To prove this, let $x \in Q$ be arbitrarily chosen. For every $t \in [0,T]$ we have

$$|(Fx)(t)| \le |a(t)| + |u(t,x(t))|\int_0^T |v(t,s,x(s))|ds. \tag{4.10}$$

Since x is convex of order -1 (i.e., nonnegative), according to (C_3) and (C_4) we also have

$$|u(t,x(t))| = |u(t,x(t)) - u(t,0)| \le x(t)\varphi(x(t),0) \le \|x\|\varphi(\|x\|,0). \tag{4.11}$$

This inequality and (C_6) yield

$$\begin{aligned} |(Fx)(t)| &\le \|a\| + \|x\|\varphi(\|x\|,0)\int_0^T \psi(|x(s)|)ds \\ &\le \|a\| + T\|x\|\varphi(\|x\|,0)\psi(\|x\|). \end{aligned} \tag{4.12}$$

Taking into account that $\|x\| \le r_0$, by (C_4), (C_6), and (C_7) we conclude that

$$\|Fx\| \le \|a\| + Tr_0\varphi(r_0,0)\psi(r_0) \le r_0. \tag{4.13}$$

On the other hand, for every $t \in [0,T]$ we have

$$(Fx)(t) = a(t) + x_u(t)x_v(t), \tag{4.14}$$

where $x_u, x_v : [0,T] \to \mathbb{R}$ are the functions defined by

$$x_u(t) := u(t,x(t)), \qquad x_v(t) := \int_0^T v(t,s,x(s))ds, \tag{4.15}$$

respectively. According to Lemma 3.2, we have

$$\left(\Delta_h^{p+1}(Fx)\right)(t) = \left(\Delta_h^{p+1}a\right)(t) + \sum_{k=0}^{p+1}\binom{p+1}{k}\left(\Delta_h^k x_v\right)(t)\left(\Delta_h^{p+1-k}x_u\right)(t+kh) \tag{4.16}$$

for every $p \in \{-1,0,\ldots,r\}$ and every $t \in [0,T]$ such that $t + ph \in [0,T]$. But

$$\begin{aligned} \left(\Delta_h^k x_v\right)(t) &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}x_v(t+ih) \\ &= \int_0^T \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}v(t+ih,s,x(s))ds \\ &= \int_0^T \left(\Delta_h^k x_{v,s}\right)(t)ds, \end{aligned} \tag{4.17}$$

where $x_{v,s}(t) := v(t,s,x(s))$. By virtue of (C_5) we have $(\Delta_h^k x_{v,s})(t) \ge 0$, whence

$$\left(\Delta_h^k x_v\right)(t) \ge 0 \quad \text{for each } k \in \{0,1,\ldots,r+1\}. \tag{4.18}$$

This inequality together with (4.16), (C_2), and (C_3) ensures that the function Fx is Jensen convex of order p for each $p \in \{-1, 0, \ldots, r\}$. Since Fx is continuous on $[0, T]$, by Theorem 3.1 it follows that Fx is convex of order p for each $p \in \{-1, 0, \ldots, r\}$. Taking into account (4.13), we conclude that F maps Q into itself, as claimed.

Finally, we prove that the operator F satisfies the Darbo condition with respect to the Hausdorff measure of noncompactness χ. To this end let X be an arbitrary nonempty subset of Q and let $x \in X$. Further, let $\varepsilon > 0$ and let $t_1, t_2 \in [0, T]$ be such that $|t_1 - t_2| \le \varepsilon$. We have

$$\begin{aligned}
&|(Fx)(t_1) - (Fx)(t_2)| \\
&\quad \le |a(t_1) - a(t_2)| + \left| u(t_1, x(t_1)) \int_0^T v(t_1, s, x(s))ds - u(t_2, x(t_2)) \int_0^T v(t_2, s, x(s))ds \right| \\
&\quad \le |a(t_1) - a(t_2)| + |u(t_1, x(t_1)) - u(t_1, x(t_2))| \left| \int_0^T |v(t_1, s, x(s))| ds \right. \\
&\qquad + |u(t_1, x(t_2)) - u(t_2, x(t_2))| \left| \int_0^T |v(t_1, s, x(s))| ds \right. \\
&\qquad + |u(t_2, x(t_2))| \left| \int_0^T |v(t_1, s, x(s)) - v(t_2, s, x(s))| ds \right. \\
&\quad \le \omega(a, \varepsilon) + |x(t_1) - x(t_2)| \varphi(x(t_1), x(t_2)) T \psi(\|x\|) \\
&\qquad + |u(t_1, x(t_2)) - u(t_2, x(t_2))| T \psi(\|x\|) \\
&\qquad + |x(t_2)| \varphi(x(t_2), 0) \int_0^T |v(t_1, s, x(s)) - v(t_2, s, x(s))| ds.
\end{aligned} \tag{4.19}$$

Letting

$$\begin{aligned}
\omega_{r_0}(u, \varepsilon) &:= \sup\{ |u(t, y) - u(t', y)| : \ t, t' \in [0, T],\ |t - t'| \le \varepsilon, y \in [0, r_0] \ \}, \\
\omega_{r_0}(v, \varepsilon) &:= \sup\{ |v(t, s, y) - v(t', s, y)| : \ s, t, t' \in [0, T],\ |t - t'| \le \varepsilon, y \in [0, r_0] \ \},
\end{aligned} \tag{4.20}$$

we get

$$\begin{aligned}
|(Fx)(t_1) - (Fx)(t_2)| \le{}& \omega(a, \varepsilon) + T\varphi(r_0, r_0)\psi(r_0)\omega(x, \varepsilon) \\
&+ T\psi(r_0)\omega_{r_0}(u, \varepsilon) + Tr_0\varphi(r_0, 0)\omega_{r_0}(v, \varepsilon).
\end{aligned} \tag{4.21}$$

Thus

$$\begin{aligned}
\omega(Fx, \varepsilon) \le{}& \omega(a, \varepsilon) + T\varphi(r_0, r_0)\psi(r_0)\omega(x, \varepsilon) \\
&+ T\psi(r_0)\omega_{r_0}(u, \varepsilon) + Tr_0\varphi(r_0, 0)\omega_{r_0}(v, \varepsilon),
\end{aligned} \tag{4.22}$$

whence

$$\begin{aligned} \omega(FX,\varepsilon) &\le \omega(a,\varepsilon) + T\varphi(r_0,r_0)\psi(r_0)\omega(X,\varepsilon) \\ &\quad + T\psi(r_0)\omega_{r_0}(u,\varepsilon) + Tr_0\varphi(r_0,0)\omega_{r_0}(v,\varepsilon). \end{aligned} \tag{4.23}$$

Taking into account that a is uniformly continuous on $[0,T]$, u is uniformly continuous on $[0,T]\times[0,r_0]$ and v is uniformly continuous on $[0,T]\times[0,T]\times[0,r_0]$, we have that $\omega(a,\varepsilon) \to 0$, $\omega_{r_0}(u,\varepsilon) \to 0$ and $\omega_{r_0}(v,\varepsilon) \to 0$ as $\varepsilon \to 0+$. So letting $\varepsilon \to 0+$ we obtain $\omega_0(FX) \le T\varphi(r_0,r_0)\psi(r_0)\omega_0(X)$, that is,

$$\chi(FX) \le T\varphi(r_0,r_0)\psi(r_0)\chi(X) \tag{4.24}$$

by virtue of (2.5).

By (C_7) and Theorem 2.2 we conclude the existence of at least one fixed point of F in Q. This fixed point is obviously a solution of (1.1) which (in view of the definition of Q) is convex of order p for each $p \in \{-1,0,\dots,r\}$. □

Theorem 4.1 can be further generalized as follows. Given an integer number $r \ge -1$ and a sequence $\xi := (\xi_{-1},\xi_0,\dots,\xi_r) \in \{-1,1\}^{r+2}$, we denote by $\mathrm{Conv}_{r,\xi}[0,T]$ the set consisting of all functions $x \in C[0,T]$ with the property that for each $p \in \{-1,0,\dots,r\}$ the function $\xi_p x$ is convex of order p. For instance, if $r = 1$ and $\xi = (1,-1,1)$, then $\mathrm{Conv}_{r,\xi}[0,T]$ consists of all functions in $C[0,T]$ that are nonnegative, nonincreasing, and convex on $[0,T]$.

Recall (see, e.g., Roberts and Varberg [14, pages 233-234]) that a function $x : [0,T] \to \mathbb{R}$ is called *absolutely monotonic* (resp., *completely monotonic*) if it possesses derivatives of all orders on $[0,T]$ and

$$x^{(k)}(t) \ge 0 \ \left(\text{resp., } (-1)^k x^{(k)}(t) \ge 0\right) \tag{4.25}$$

for each $t \in [0,T]$ and each integer $k \ge 0$. By [13, Theorem 6, page 392] it follows that if $x : [0,T] \to \mathbb{R}$ is an absolutely monotonic (resp., a completely monotonic) function, then x belongs to every set $\mathrm{Conv}_{r,\xi}[0,T]$ with $r \ge -1$ and $\xi_k = 1$ (resp., $\xi_k = (-1)^{k+1}$) for each $k \in \{-1,0,\dots,r\}$.

Instead of the conditions (C_1), (C_2), (C_3), and (C_5) we consider the following conditions.

(C_1') $r \ge -1$ is a given integer number and $\xi := (\xi_{-1},\xi_0,\dots,\xi_r) \in \{-1,1\}^{r+2}$ is a sequence such that either

$$\xi_k = 1 \quad \text{for each } k \in \{-1,0,\dots,r\} \tag{4.26}$$

or

$$\xi_k = (-1)^{k+1} \quad \text{for each } k \in \{-1,0,\dots,r\}. \tag{4.27}$$

(C_2') $a : [0,T] \to \mathbb{R}$ belongs to $\mathrm{Conv}_{r,\xi}[0,T]$.

(C_3') $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u(t,0) = 0$ for all $t \in [0,T]$ and the function

$$t \in [0,T] \longmapsto u(t,x(t)) \in \mathbb{R} \tag{4.28}$$

belongs to $\mathrm{Conv}_{r,\xi}[0,T]$ whenever $x \in \mathrm{Conv}_{r,\xi}[0,T]$.

(C_5') $v : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that the function $v(\cdot,s,x) : [0,T] \to \mathbb{R}$ belongs to $\mathrm{Conv}_{r,\xi}[0,T]$ whenever $s \in [0,T]$ and $x \in [0,\infty)$.

Theorem 4.2. *If the conditions (C_1')–(C_3'), (C_4), (C_5'), and (C_6)-(C_7) are satisfied, then* (1.1) *possesses at least one solution $x \in \mathrm{Conv}_{r,\xi}[0,T]$.*

Proof. Consider the operator F, defined on $C[0,T]$, as in the proof of Theorem 4.1. As we have already seen in the proof of Theorem 4.1 we have $Fx \in C[0,T]$ whenever $x \in C[0,T]$ and F is continuous on $C[0,T]$.

Instead of the set Q, considered in the proof of Theorem 4.1, we take now Q to be the subset of $\mathrm{Conv}_{r,\xi}[0,T]$ consisting of all functions x such that $\|x\| \le r_0$. Then Q is a nonempty bounded closed convex subset of $C[0,T]$. We claim that F maps Q into itself. Indeed, according to (4.13) we have $\|Fx\| \le r_0$ whenever $x \in C[0,T]$ satisfies $\|x\| \le r_0$. On the other hand, Fx admits the representation (4.14), where $x_u, x_v : [0,T] \to \mathbb{R}$ are defined by (4.15). Given any $p \in \{-1,0,\ldots,r\}$, note that

$$\xi_p = \xi_{k-1}\xi_{p-k} \quad \text{for each } k \in \{0,1,\ldots,p+1\}, \tag{4.29}$$

whence

$$\left(\Delta_h^{p+1}(\xi_p Fx)\right)(t) = \left(\Delta_h^{p+1}(\xi_p a)\right)(t) + \sum_{k=0}^{p+1}\binom{p+1}{k}\left(\Delta_h^{k}(\xi_{k-1}x_v)\right)(t)\left(\Delta_h^{p+1-k}(\xi_{p-k}x_u)\right)(t+kh) \tag{4.30}$$

for every $t \in [0,T]$ such that $t + ph \in [0,T]$. By proceeding as in the proof of Theorem 4.1 one can show that

$$\left(\Delta_h^{p+1}(\xi_p Fx)\right)(t) \ge 0 \quad \text{whenever } t \in [0,T] \text{ satisfies } t + ph \in [0,T]. \tag{4.31}$$

Therefore $Fx \in \mathrm{Conv}_{r,\xi}[0,T]$.

The rest of the proof is similar to the corresponding part in the proof of Theorem 4.1 and we omit it. □

5. An Application

As an application of the results established in the previous section, in what follows we study the solvability of the integral equation

$$x(t) = 1 + \lambda x^n(t)\int_0^1 \frac{1}{t+s+1}x(s)ds, \quad t \in [0,1], \tag{5.1}$$

in which n is a given positive integer and λ is a positive real parameter. Note that (5.1) is similar to the Chandrasekhar equation, arising in the theory of radiative transfer (see, e.g., Chandrasekhar [15] or Banaś et al. [16], and the references therein).

We are going to prove that if $0 < \lambda < 1/n(1+1/n)^{n+1}$, then (5.1) possesses at least one continuous nonnegative solution, which is nonincreasing and convex. To this end, we apply Theorem 4.2 for $r := 1$ and $\xi := (1,-1,1)$. Take $T := 1$, $a(t) \equiv 1$, $u(t,x) := x^n$ and $v(t,s,x) := \lambda/(t+s+1)x$. It is immediately seen that all the conditions (C_1')–(C_3'), (C_4), (C_5'), and (C_6) are satisfied if the functions φ and ψ are defined by

$$\varphi(x,y) := x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}, \qquad \psi(x) := \lambda x, \tag{5.2}$$

respectively. It remains to show that (C_7) is satisfied, too. Taking into account the expressions of φ and ψ, condition (C_7) is equivalent to the following statement. If $0 < \lambda < 1/n(1+1/n)^{n+1}$, then there exists an $r_0 > 0$ such that

$$1 + \lambda r_0^{n+1} \le r_0, \qquad n\lambda r_0^n < 1. \tag{5.3}$$

Clearly, such an r_0 must satisfy $r_0 > 1$. Let $f, g : (1,\infty) \to \mathbb{R}$ be the functions defined by

$$f(r) := \frac{r-1}{r^{n+1}}, \qquad g(r) := \frac{1}{nr^n}, \tag{5.4}$$

respectively. Since

$$f'(r) = \frac{n+1-nr}{r^{n+2}}, \tag{5.5}$$

one can see that f attains a maximum at $r_n := (n+1)/n$, the maximum value being $\lambda_n := 1/n(1+1/n)^{n+1}$. On the other hand, we have

$$g(r) - f(r) = \frac{n-r(n-1)}{nr^{n+1}}. \tag{5.6}$$

If $n = 1$, then $g(r) > f(r)$ for all $r \in (0,\infty)$. If $n > 1$ and $0 < r < n/(n-1)$, then $g(r) > f(r)$, while if $n > 1$ and $r \ge n/(n-1)$, then $g(r) \le f(r)$. Note that $0 < r_n < n/(n-1)$.

Assume now that $0 < \lambda < \lambda_n$. Then we can select an r_0 sufficiently close to r_n such that $\lambda < f(r_0) < g(r_0)$. Obviously, r_0 satisfies (5.3).

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