Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 918785, 24 pages
doi:10.1155/2009/918785

Research Article

A Fixed Point Approach to the Fuzzy Stability of an Additive-Quadratic-Cubic Functional Equation

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Received 23 August 2009; Revised 18 October 2009; Accepted 23 October 2009

Recommended by Fabio Zanolin

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic functional equation $f(x+2y)+f(x-2y)=2f(x+y)-2f(-x-y)+2f(x-y)-2f(y-x)+f(2y)+f(-2y)+4f(-x)-2f(x)$ in fuzzy Banach spaces.

1. Introduction and Preliminaries

Katsaras [1] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [2–4]. In particular, Bag and Samanta [5], following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michálek type [7]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [8].

We use the definition of fuzzy normed spaces given in [5, 9, 10] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)-2f(-x-y)+2f(x-y)-2f(y-x) \\ &\quad + f(2y)+f(-2y)+4f(-x)-2f(x) \end{aligned} \tag{1.1}$$

in the fuzzy normed vector space setting.

Definition 1.1 (see [5, 9–11]). Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N_3) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N_5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [9, 12].

Definition 1.2 (see [5, 9–11]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is wellknown that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [8]).

In 1940, Ulam [13] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We will call such an $f : G \to G'$ an *approximate homomorphism.*

In 1941, Hyers [14] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies the *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.2}$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.3}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \varepsilon \tag{1.4}$$

for all $x \in E$.

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then $L : E \to E'$ is $\mathbb{R}$-linear, and if f is continuous at a single point of E, then $L : E \to E'$ is also continuous.

Hyers' theorem was generalized by Aoki [15] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [16] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

In 1982–1994, a generalization of the Hyers's result was proved by J. M. Rassias. He introduced the following *weaker condition*:

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.5}$$

for all $x, y \in E$, controlled by a product of different powers of norms, where $\theta \geq 0$ and real numbers $p, q, r := p + q \neq 1$, and retained the condition of continuity of $f(tx)$ in $t \in \mathbb{R}$ for each fixed $x \in E$. Besides he investigated that it is possible to replace ε in the above Hyers inequality by a nonnegative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove *asymptotic type formulas* of the form

$$L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \quad \text{or} \quad L(x) = \lim_{n \to \infty} 2^n f(2^{-n} x). \tag{1.6}$$

Theorem 1.3 (see [18–23]). *Let X be a real normed linear space and Y a real Banach space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the Cauchy-Rassias inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.7}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r \tag{1.8}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L : X \to Y$ is an $\mathbb{R}$-linear mapping.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.9}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [24] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [25] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [26] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [27–69]).

In [70], Jun and Kim considered the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.10}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional (1.10), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4 (see [71, 72]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.11}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \ge n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \le (1/(1-L)) d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th. M. Rassias [73] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [74–78]).

This paper is organized as follows. In Section 2, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic functional (1.1) in fuzzy Banach spaces for an odd case. In Section 3, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic functional (1.1) in fuzzy Banach spaces for an even case.

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

2. Generalized Hyers-Ulam Stability of the Functional Equation (1.1): An Odd Case

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.1) if and only if the odd mapping mapping $f : X \to Y$ is an additive-cubic mapping, that is,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x). \tag{2.1}$$

It was shown in [79, Lemma 2.2] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.1) if and only if the even mapping $f : X \to Y$ is a quadratic mapping, that is,

$$f(x+2y) + f(x-2y) = 2f(x) + 2f(2y). \tag{2.2}$$

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) := f(x+2y) &+ f(x-2y) - 2f(x+y) + 2f(-x-y) - 2f(x-y) \\ &+ 2f(y-x) - f(2y) - f(-2y) - 4f(-x) + 2f(x) \end{aligned} \tag{2.3}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in fuzzy Banach spaces, an odd case.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y) \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N(Df(x,y),t) \ge \frac{t}{t+\varphi(x,y)} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then

$$C(x) := N\text{-}\lim_{n\to\infty} 8^n\left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right) \tag{2.6}$$

exists for each $x \in X$ *and defines a cubic mapping* $C : X \to Y$ *such that*

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(8-8L)t}{(8-8L)t + 5L(\varphi(x,x) + \varphi(2x,x))} \tag{2.7}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $x = y$ in (2.5), we get

$$N(f(3y) - 4f(2y) + 5f(y), t) \geq \frac{t}{t + \varphi(y,y)} \tag{2.8}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (2.5), we get

$$N(f(4y) - 4f(3y) + 6f(2y) - 4f(y), t) \geq \frac{t}{t + \varphi(2y,y)} \tag{2.9}$$

for all $y \in X$ and all $t > 0$.

By (2.8) and (2.9),

$$\begin{aligned} &N(f(4y) - 10f(2y) + 16f(y), 4t + t) \\ &\quad \geq \min\{N(4(f(3y) - 4f(2y) + 5f(y)), 4t),\ N(f(4y) - 4f(3y) + 6f(2y) - 4f(y), t)\} \\ &\quad \geq \frac{t}{t + \varphi(y,y) + \varphi(2y,y)} \end{aligned} \tag{2.10}$$

for all $y \in X$ and all $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$N\left(g(x) - 8g\left(\frac{x}{2}\right), 5t\right) \geq \frac{t}{t + \varphi(x/2, x/2) + \varphi(x, x/2)} \tag{2.11}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \longrightarrow Y\} \tag{2.12}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)},\ \forall x \in X,\ \forall t > 0\right\}, \tag{2.13}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 of [80].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right) \tag{2.14}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \tag{2.15}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{8}\varepsilon t\right) \\ &\geq \frac{Lt/8}{Lt/8 + \varphi(x/2, x/2) + \varphi(x, x/2)} \\ &\geq \frac{Lt/8}{Lt/8 + (L/8)(\varphi(x,x) + \varphi(2x,x))} \\ &= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \end{aligned} \tag{2.16}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h) \tag{2.17}$$

for all $g, h \in S$.

It follows from (2.11) that

$$N\left(g(x) - 8g\left(\frac{x}{2}\right), \frac{5L}{8}t\right) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \tag{2.18}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq 5L/8$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.19}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{2.20}$$

This implies that C is a unique mapping satisfying (2.19) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.21}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n \to \infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{2.22}$$

for all $x \in X$.

(3) $d(g, C) \leq (1/(1-L)) d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5L}{8 - 8L}. \tag{2.23}$$

This implies that inequality (2.7) holds.

By (2.5),

$$N\left(8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 8^n t\right) \geq \frac{t}{t + \varphi(x/2^n, x/2^n)} \tag{2.24}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. So

$$N\left(8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \frac{t/8^n}{t/8^n + (L^n/8^n)\varphi(x, y)} \tag{2.25}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} (t/8^n)/(t/8^n + (L^n/8^n)\varphi(x, y)) = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(DC(x, y), t) = 1 \tag{2.26}$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N(Df(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.27}$$

for all $x, y \in X$ and all $t > 0$. Then

$$C(x) := N\text{-}\lim_{n \to \infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right) \right) \tag{2.28}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(3 + 2^p)\theta\|x\|^p} \tag{2.29}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.30}$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.31}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then

$$C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right) \tag{2.32}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5\varphi(x, x) + 5\varphi(2x, x)} \tag{2.33}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x) \tag{2.34}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.35}$$

for all $x \in X$ and all $t > 0$. Hence

$$
\begin{aligned}
N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{8}g(2x) - \frac{1}{8}h(2x), L\varepsilon t\right) \\
&= N(g(2x) - h(2x), 8L\varepsilon t) \\
&\geq \frac{8Lt}{8Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \\
&\geq \frac{8Lt}{8Lt + 8L(\varphi(x, x) + \varphi(2x, x))} \\
&= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\end{aligned}
\tag{2.36}
$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.37}$$

for all $g, h \in S$.

It follows from (2.11) that

$$N\left(g(x) - \frac{1}{8}g(2x), \frac{5}{8}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.38}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq 5/8$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C(2x) = 8C(x) \tag{2.39}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.40}$$

This implies that C is a unique mapping satisfying (2.39) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.41}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n \to \infty} \frac{1}{8^n} g(2^n x) = C(x) \tag{2.42}$$

for all $x \in X$.

(3) $d(g, C) \leq (1/(1-L))d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5}{8-8L}. \tag{2.43}$$

This implies that the inequality (2.33) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (2.27). Then*

$$C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right) \tag{2.44}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(8-2^p)t}{(8-2^p)t + 5(3+2^p)\theta\|x\|^p} \tag{2.45}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.46}$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y) \tag{2.47}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n\left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{2.48}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(2x) - 8f(x) - A(x), t) \geq \frac{(2-2L)t}{(2-2L)t + 5L(\varphi(x, x) + \varphi(2x, x))} \tag{2.49}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $y := x/2$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.10), we get

$$N\left(h(x) - 2h\left(\frac{x}{2}\right), 5t\right) \geq \frac{t}{t + \varphi(x/2, x/2) + \varphi(x, x/2)} \tag{2.50}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{2.51}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.52}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= -N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{Lt/2}{Lt/2 + \varphi(x/2, x/2) + \varphi(x, x/2)} \\ &\geq \frac{Lt/2}{Lt/2 + (L/2)(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned} \tag{2.53}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.54}$$

for all $g, h \in S$.

It follows from (2.50) that

$$N\left(h(x) - 2h\left(\frac{x}{2}\right), \frac{5L}{2}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.55}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq 5L/2$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.56}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.57}$$

This implies that A is a unique mapping satisfying (2.56) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.58}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) = A(x) \tag{2.59}$$

for all $x \in X$;

(3) $d(h, A) \leq (1/(1 - L))d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{5L}{2 - 2L}. \tag{2.60}$$

This implies that inequality (2.49) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.6. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.27). *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{2.61}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(2x) - 8f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 5(3 + 2^p)\theta\|x\|^p} \tag{2.62}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.63}$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.64}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{2.65}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(2x) - 8f(x) - A(x), t\right) \ge \frac{(2-2L)t}{(2-2L)t + 5\varphi(x,x) + 5\varphi(2x,x)} \tag{2.66}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x) \tag{2.67}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\left(g(x) - h(x), \varepsilon t\right) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \tag{2.68}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), L\varepsilon t\right) \\
&= N\left(g(2x) - h(2x), 2L\varepsilon t\right) \\
&\ge \frac{2Lt}{2Lt + \varphi(2x,2x) + \varphi(4x,2x)} \\
&\ge \frac{2Lt}{2Lt + 2L\left(\varphi(x,x) + \varphi(2x,x)\right)} \\
&= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}
\end{aligned} \tag{2.69}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.70}$$

for all $g, h \in S$.

It follows from (2.50) that

$$N\left(h(x) - \frac{1}{2}h(2x), \frac{5}{2}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.71}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq 5/2$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following.

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{2.72}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.73}$$

This implies that A is a unique mapping satisfying (2.72) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \tag{2.74}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{2^n}h(2^n x) = A(x) \tag{2.75}$$

for all $x \in X$.

(3) $d(h, A) \leq (1/(1 - L))d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{5}{2 - 2L}. \tag{2.76}$$

This implies that inequality (2.66) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.27)*. Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{2.77}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(2x) - 8f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 5(3+2^p)\theta\|x\|^p} \tag{2.78}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.79}$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3. Generalized Hyers-Ulam Stability of the Functional Equation (1.1): An Even Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in fuzzy Banach spaces, an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x,2y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5)*. Then*

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \tag{3.2}$$

exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(16-16L)t}{(16-16L)t + L^2\varphi(2x,x)} \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing x by $2y$ in (2.5), we get

$$N(f(4y) - 4f(2y), t) \geq \frac{t}{t + \varphi(2y, y)} \tag{3.4}$$

for all $y \in X$ and all $t > 0$.

It follows from (3.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L^2}{16}t\right) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \longrightarrow Y\} \tag{3.6}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(2x, x)},\ \forall x \in X,\ \forall t > 0\right\}, \tag{3.7}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 of [80].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right) \tag{3.8}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.9}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{Lt/4}{Lt/4 + \varphi(x, x/2)} \\ &\geq \frac{Lt/4}{Lt/4 + (L/4)\varphi(2x, x)} \\ &= \frac{t}{t + \varphi(2x, x)} \end{aligned} \tag{3.10}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h) \tag{3.11}$$

for all $g, h \in S$.

It follows from (3.5) that $d(f, Jf) \leq L^2/16$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.12}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{3.13}$$

This implies that Q is a unique mapping satisfying (3.12) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.14}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x) \tag{3.15}$$

for all $x \in X$.

(3) $d(f,Q) \leq (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L^2}{16 - 16L}. \tag{3.16}$$

This implies that inequality (3.3) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.27). Then*

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \tag{3.17}$$

exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{2^p(2^p - 4)t}{2^p(2^p - 4)t + (1 + 2^p)\theta\|x\|^p} \tag{3.18}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{3.19}$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.20}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then

$$Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x) \tag{3.21}$$

exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(16 - 16L)t}{(16 - 16L)t + L\varphi(2x, x)} \tag{3.22}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.
Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{3.23}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.24}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{4}g(2x) - \frac{1}{4}h(2x), L\varepsilon t\right) \\ &= N(g(2x) - h(2x), 4L\varepsilon t) \\ &\geq \frac{4Lt}{4Lt + \varphi(4x, 2x)} \\ &\geq \frac{4Lt}{4Lt + 4L\varphi(2x, x)} \\ &= \frac{t}{t + \varphi(2x, x)} \end{aligned} \tag{3.25}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h) \tag{3.26}$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{L}{16}t\right) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.27}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq L/16$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following.

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{3.28}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{3.29}$$

This implies that Q is a unique mapping satisfying (3.28) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(2x, x)} \tag{3.30}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x) = Q(x) \tag{3.31}$$

for all $x \in X$.

(3) $d(f, Q) \le (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{L}{16 - 16L}. \tag{3.32}$$

This implies that inequality (3.22) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.27)*. Then*

$$Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x) \tag{3.33}$$

exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{16(4 - 2^p)t}{16(4 - 2^p)t + 2^p(1 + 2^p)\theta\|x\|^p} \tag{3.34}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{3.35}$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

Acknowledgment

This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

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