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Research Article

Fixed Point Theory for Contractive Mappings Satisfying Φ-Maps in G-Metric Spaces

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We prove some fixed point results for self-mapping $T : X \to X$ in a complete G-metric space X under some contractive conditions related to a nondecreasing map $\phi : [0, +\infty) \to [0, +\infty)$ with $\lim_{n\to+\infty}\phi^n(t) = 0$ for all $t \in (0, +\infty)$. Also, we prove the uniqueness of such fixed point, as well as studying the G-continuity of such fixed point.

1. Introduction

The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors. In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [1]. In fact, Mustafa et al. studied many fixed point results for a self-mapping in G-metric space under certain conditions; see[1–5]. In the present work, we study some fixed point results for self-mapping in a complete G-metric space X under some contractive conditions related to a nondecreasing map $\phi : [0, +\infty) \to [0, +\infty)$ with $\lim_{n\to+\infty}\phi^n(t) = 0$ for all $t \in (0, +\infty)$.

2. Basic Concepts

In this section, we present the necessary definitions and theorems in G-metric spaces.

Definition 2.1 (see [1]). Let X be a nonempty set and let $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties:

(1) (G_1) $G(x, y, z) = 0$ if $x = y = z$;

(2) (G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(3) (G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(4) (G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

(5) (G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2 (see [1]). Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, and we say that the sequence (x_n) is G-convergent to x or (x_n) G-converges to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition 2.3 (see [1]). *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) *(x_n) is G-convergent to x.*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$.

(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$.

(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 2.4 (see [1]). Let (X, G) be a G-metric space; a sequence (x_n) is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq k$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.5 (see [3]). *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*

(2) *For every $\epsilon > 0$, there is $k \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.*

Definition 2.6 (see [1]). Let (X, G) and (X', G') be G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 2.7 (see [1]). *Let (X, G) and (X', G') be G-metric spaces. Then $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 2.8 (see [1]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

The following are examples of G-metric spaces.

Example 2.9 (see [1]). Let $(\mathbf{R}, d)$ be the usual metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z) \tag{2.1}$$

for all $x, y, z \in \mathbf{R}$. Then it is clear that $(\mathbf{R}, G_s)$ is a G-metric space.

Example 2.10 (see [1]). Let $X = \{a, b\}$. Define G on $X \times X \times X$ by

$$\begin{gathered} G(a, a, a) = G(b, b, b) = 0, \\ G(a, a, b) = 1, \qquad G(a, b, b) = 2 \end{gathered} \tag{2.2}$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X, G) is a G-metric space.

Definition 2.11 (see [1]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

3. Main Results

Following to Matkowski [6], let Φ be the set of all functions ϕ such that $\phi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with $\lim_{n \to +\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called Φ-map. If ϕ is Φ-map, then it is an easy matter to show that

(1) $\phi(t) < t$ for all $t \in (0, +\infty)$;

(2) $\phi(0) = 0$.

From now unless otherwise stated we mean by ϕ the Φ-map. Now, we introduce and prove our first result.

Theorem 3.1. *Let X be a complete G-metric space. Suppose the map $T : X \to X$ satisfies*

$$G(T(x), T(y), T(z)) \leq \phi(G(x, y, z)) \tag{3.1}$$

for all $x, y, z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Choose $x_0 \in X$. Let $x_n = T(x_{n-1})$, $n \in \mathbf{N}$. Assume $x_n \neq x_{n-1}$, for each $n \in \mathbf{N}$. Claim (x_n) is a G-Cauchy sequence in X: for $n \in \mathbf{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T(x_{n-1}), T(x_n), T(x_n)) \\ &\leq \phi(G(x_{n-1}, x_n, x_n)) \\ &\leq \phi^2(G(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\ \ \vdots \\ &\leq \phi^n(G(x_0, x_1, x_1)). \end{aligned} \tag{3.2}$$

given $\epsilon > 0$, since $\lim_{n\to+\infty}\phi^n(G(x_0,x_1,x_1)) = 0$ and $\phi(\epsilon) < \epsilon$, there is an integer k_0 such that

$$\phi^n(G(x_0,x_1,x_1)) < \epsilon - \phi(\epsilon) \quad \forall n \geq k_0. \tag{3.3}$$

Hence

$$G(x_n,x_{n+1},x_{n+1}) < \epsilon - \phi(\epsilon) \quad \forall\ n \geq k_0. \tag{3.4}$$

For $m,n \in \mathbf{N}$ with $m > n$, we claim that

$$G(x_n,x_m,x_m) < \epsilon \quad \text{for all } m \geq n \geq k_0. \tag{3.5}$$

We prove Inequality (3.5) by induction on m. Inequality (3.5) holds for $m = n+1$ by using Inequality (3.4) and the fact that $\epsilon - \phi(\epsilon) < \epsilon$. Assume Inequality (3.5) holds for $m = k$. For $m = k+1$, we have

$$\begin{aligned} G(x_n,x_{k+1},x_{k+1}) &\leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{k+1},x_{k+1}) \\ &< \epsilon - \phi(\epsilon) + \phi(G(x_n,x_k,x_k)) \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon. \end{aligned} \tag{3.6}$$

By induction on m, we conclude that Inequality (3.5) holds for all $m \geq n \geq k_0$. So (x_n) is G-Cauchy and hence (x_n) is G-convergent to some $u \in X$. For $n \in \mathbf{N}$, we have

$$\begin{aligned} G(u,u,T(u)) &\leq G(u,u,x_{n+1}) + G(x_{n+1},x_{n+1},T(u)) \\ &\leq G(u,u,x_{n+1}) + \phi(G(x_n,x_n,u)) \\ &< G(u,u,x_{n+1}) + G(x_n,x_n,u). \end{aligned} \tag{3.7}$$

Letting $n \to +\infty$, and using the fact that G is continuous on its variable, we get that $G(u,u,T(u)) = 0$. Hence $T(u) = u$. So u is a fixed point of T. Now, let v be another fixed point of T with $v \neq u$. Since ϕ is a ϕ-map, we have

$$\begin{aligned} G(u,u,v) &= G(T(u),T(u),T(v)) \\ &\leq \phi(G(u,u,v)) \\ &< G(u,u,v) \end{aligned} \tag{3.8}$$

which is a contradiction. So $u = v$, and hence Thas a unique fixed point. To Show that T is

G-continuous at u, let (y_n) be any sequence in X such that (y_n) is G-convergent to u. For $n \in \mathbf{N}$, we have

$$\begin{aligned} G(u,u,T(y_n)) &= G(T(u),T(u),T(y_n)) \\ &\le \phi(G(u,u,y_n)) \\ &< G(u,u,y_n). \end{aligned} \tag{3.9}$$

Letting $n \to +\infty$, we get $\lim_{n\to+\infty} G(u,u,T(y_n)) = 0$. Hence $T(y_n)$ is G-convergent to $u = T(u)$. So T is G-continuous at u. □

As an application of Theorem 3.1, we have the following results.

Corollary 3.2. *Let X be a complete G-metric space. Suppose that the map $T : X \to X$ satisfies for $m \in \mathbf{N}$:*

$$G(T^m(x),T^m(y),T^m(z)) \le \phi(x,y,z) \tag{3.10}$$

for all $x,y,z \in X$. Then T has a unique fixed point (say u).

Proof. From Theorem 3.1, we conclude that T^m has a unique fixed point say u. Since

$$T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)), \tag{3.11}$$

we have that $T(u)$ is also a fixed point to T^m. By uniqueness of u, we get $T(u) = u$. □

Corollary 3.3. *Let X be a complete G-metric space. Suppose that the map $T : X \to X$ satisfies*

$$G(T(x),T(y),T(y)) \le \phi(G(x,y,y)), \tag{3.12}$$

for all $x,y \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. follows from Theorem 3.1 by taking $z = y$. □

Corollary 3.4. *Let X be a complete G-metric space. Suppose there is $k \in [0,1)$ such that the map $T : X \to X$ satisfies*

$$G(T(x),T(y),T(z)) \le kG(x,y,z), \tag{3.13}$$

for all $x,y,z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Define $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(w) = kw$. Then it is clear that ϕ is a nondecreasing function with $\lim_{n\to+\infty}\phi^n(t) = 0$ for all $t > 0$. Since

$$G(T(x),T(y),T(z)) \le \phi(G(x,y,z)) \quad \forall x,y,z \in X, \tag{3.14}$$

the result follows from Theorem 3.1. □

The above corollary has been stated in [7, Theorem 5.1.7], and proved by a different way.

Corollary 3.5. *Let X be a complete G-metric space. Suppose the map $T : X \to X$ satisfies*

$$G(T(x), T(y), T(z)) \leq \frac{G(x, y, z)}{1 + G(x, y, z)}, \tag{3.15}$$

for all $x, y, z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(w) = w/(1 + w)$. Then it is clear that ϕ is a nondecreasing function with $\lim_{n\to+\infty}\phi^n(t) = 0$ for all $t > 0$. Since

$$G(T(x), T(y), T(z)) \leq \phi(G(x, y, z)) \quad \forall x, y, z \in X, \tag{3.16}$$

the result follows from Theorem 3.1. □

Theorem 3.6. *Let X be a complete G-metric space. Suppose that the map $T : X \to X$ satisfies*

$$\begin{aligned} &G(T(x), T(y), T(z)) \\ &\leq \phi(\max\{G(x, y, z), G(x, T(x), T(x)), G(y, T(y), T(y)), G(T(x), y, z)\}) \end{aligned} \tag{3.17}$$

for all $x, y, z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Choose $x_0 \in X$. Let $x_n = T(x_{n-1})$, $n \in \mathbf{N}$. Assume $x_n \neq x_{n-1}$, for each $n \in \mathbf{N}$. Thus for $n \in \mathbf{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T(x_{n-1}), T(x_n), T(x_n)) \\ &\leq \phi(\max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\}. \end{aligned} \tag{3.18}$$

If

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} = G(x_n, x_{n+1}, x_{n+1}), \tag{3.19}$$

then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \phi(G(x_n, x_{n+1}, x_{n+1})) < G(x_n, x_{n+1}, x_{n+1}), \tag{3.20}$$

which is impossible. So it must be the case that

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} = G(x_{n-1}, x_n, x_n), \tag{3.21}$$

and hence

$$G(x_n, x_{n+1}, x_{n+1}) \le \phi(G(x_{n-1}, x_n, x_n)). \tag{3.22}$$

Thus for $n \in \mathbf{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T(x_{n-1}), T(x_n), T(x_n)) \\ &\le \phi(G(x_{n-1}, x_n, x_n)) \\ &\le \phi^2(G(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\vdots \\ &\le \phi^n(G(x_0, x_1, x_1)). \end{aligned} \tag{3.23}$$

The same argument is similar to that in proof of Theorem 3.1; one can show that (x_n) is a G-Cauchy sequence. Since X is G-complete, we conclude that (x_n) is G-convergent to some $u \in X$. For $n \in \mathbf{N}$, we have

$$\begin{aligned} G(u, u, T(u)) \le G(u, u, x_n) + G(x_n, x_n, T(u)) \le G(u, u, x_n) \\ + \phi(\max\{G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n-1}, u)\}). \end{aligned} \tag{3.24}$$

Case 1.

$$\max\{G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, x_n, x_n), G(x_n, x_{n-1}, u)\} = G(x_{n-1}, x_n, x_n), \tag{3.25}$$

then we have

$$G(u, u, T(u)) < G(u, u, x_n) + G(x_{n-1}, x_n, x_n). \tag{3.26}$$

Letting $n \to +\infty$, we conclude that $G(u, u, T(u)) = 0$, and hence $T(u) = u$.

Case 2.

$$\max\{G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, x_n, x_n), G(x_n, x_{n-1}, u)\} = G(x_{n-1}, x_{n-1}, u), \tag{3.27}$$

then we have

$$G(u, u, T(u)) < G(u, u, x_n) + G(x_{n-1}, x_{n-1}, u). \tag{3.28}$$

Letting $n \to +\infty$, we conclude that $G(u, u, T(u)) = 0$, and hence $T(u) = u$.

Case 3.

$$\max\{G(x_{n-1},x_{n-1},u),G(x_{n-1},x_n,x_n),G(x_{n-1},x_n,x_n),G(x_n,x_{n-1},u)\} = G(x_n,x_{n-1},u), \tag{3.29}$$

then we have

$$\begin{aligned} G(u,u,T(u)) &< G(u,u,x_n) + G(x_n,x_{n-1},u) \\ &\leq G(u,u,x_n) + G(x_n,x_{n-1},x_{n-1}) + G(x_{n-1},x_{n-1},u). \end{aligned} \tag{3.30}$$

Letting $n \to +\infty$, we conclude that $G(u,u,T(u)) = 0$, and hence $T(u) = u$. In all cases, we conclude that u is a fixed point of T. Let v be any other fixed point of T such that $v \neq u$. Then

$$\begin{aligned} G(u,v,v) &\leq \phi(\max\{G(u,v,v),G(u,u,u),G(v,v,v),G(u,v,v)\}) \\ &= \phi(G(u,v,v)) < G(u,v,v)), \end{aligned} \tag{3.31}$$

which is a contradiction since $\phi(G(u,v,v)) < G(u,v,v)$. Therefore, $G(u,v,v) = 0$ and hence $u = v$. To show that T is G-continuous at u, let (y_n) be any sequence in X such that (y_n) is G-convergent to u. Then

$$\begin{aligned} G(u,u,T(y_n)) &\leq \phi(\max\{G(u,u,y_n),G(u,u,u),G(u,u,u),G(u,u,y_n)\}) \\ &= \phi(G(u,u,y_n)) < G(u,u,y_n). \end{aligned} \tag{3.32}$$

Let $n \to +\infty$, we get that $T(y_n)$ is G-convergent to $T(u) = u$. Hence T is G-continuous at u.

□

As an application to Theorem 3.6, we have the following results.

Corollary 3.7. *Let X be a complete G-metric space. Suppose there is $k \in [0,1)$ such that the map $T : X \to X$ satisfies*

$$G((Tx),T(y),T(z)) \leq k \max\{G(x,y,z),G(x,T(x),T(x)),G(y,T(y),T(y)),G(T(x),y,z)\} \tag{3.33}$$

for all $x,y,z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Define $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(w) = kw$. Then it is clear that ϕ is a nondecreasing function with $\lim_{n\to+\infty}\phi^n(t) = 0$ for all $t > 0$. Since

$$G(T(x),T(y),T(z)) \leq \phi(\max\{G(x,y,z),G(x,T(x),T(x)),G(y,T(y),T(y)),G(T(x),y,z)\}) \tag{3.34}$$

for all $x,y,z \in X$, the result follows from Theorem 3.6. □

Corollary 3.8. *Let X be a complete G-metric space. Suppose that the map $T : X \to X$ satisfies:*

$$G(T(x),T(y),T(y)) \leq \phi(\max\{G(x,y,y),G(x,T(x),T(x)),G(y,T(y),T(y)),G(T(x),y,y)\}) \tag{3.35}$$

for all $x,y \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. It follows from Theorem 3.6 by replacing $z = y$. □

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