Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 190606, 16 pages
doi:10.1155/2010/190606

Research Article

Nonlinear Contractive Conditions for Coupled Cone Fixed Point Theorems

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Received 19 April 2010; Revised 8 June 2010; Accepted 5 July 2010

Academic Editor: Juan J. Nieto

We establish some new coupled fixed point theorems for various types of nonlinear contractive maps in the setting of quasiordered cone metric spaces which not only obtain several coupled fixed point theorems announced by many authors but also generalize them under weaker assumptions.

1. Introduction

The existence of fixed point in partially ordered sets has been studied and investigated recently in [1–13] and references therein. Since the various contractive conditions are important in metric fixed point theory, there is a trend to weaken the requirement on contractions. Nieto and Rodríguez-López in [8, 10] used Tarski's theorem to show the existence of solutions for fuzzy equations and fuzzy differential equations, respectively. The existence of solutions for matrix equations or ordinary differential equations by applying fixed point theorems are presented in [2, 6, 9, 11, 12]. In [3, 13], the authors proved some fixed point theorems for a mixed monotone mapping in a metric space endowed with partial order and applied their results to problems of existence and uniqueness of solutions for some boundary value problems.

In 2006, Bhaskar and Lakshmikantham [2] first proved the following interesting coupled fixed point theorem in partially ordered metric spaces.

Theorem BL (Bhaskar and Lakshmikantham). *Let $(X, \leqslant)$ be a partially ordered set and d a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall u \leqslant x,\ y \leqslant v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leqslant F(x_0, y_0)$ and $F(y_0, x_0) \leqslant y_0$, then, there exist $\widehat{x}, \widehat{y} \in X$, such that $\widehat{x} = F(\widehat{x}, \widehat{y})$ and $\widehat{y} = F(\widehat{y}, \widehat{x})$.

Let E be a topological vector space (t.v.s. for short) with its zero vector θ_E. A nonempty subset K of E is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be **pointed** if $K \cap (-K) = \{\theta_E\}$. For a given proper, pointed, and convex cone K in E, we can define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Longleftrightarrow y - x \in K. \tag{1.2}$$

$x \prec_K y$ will stand for $x \precsim_K y$ and $x \neq y$ while $x \ll_K y$ will stand for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of K.

In the following, unless otherwise specified, we always assume that Y is a locally convex Hausdorff t.v.s. with its zero vector θ, K a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K \neq \emptyset$, $\precsim_K$ a partial ordering with respect to K, and $e \in \operatorname{int} K$.

Very recently, Du [14] first introduced the concepts of TVS-cone metric and TVS-cone metric space to improve and extend the concept of cone metric space in the sense of Huang and Zhang [15].

Definition 1.1 (see [14]). Let X be a nonempty set. A vector-valued function $p : X^2 := X \times X \to Y$ is said to be a TVS-*cone metric* if the following conditions hold:

(C1) $\theta \precsim_K p(x, y)$ for all $x, y \in X$ and $p(x, y) = \theta$ if and only if $x = y$;

(C2) $p(x, y) = p(y, x)$ for all $x, y \in X$;

(C3) $p(x, z) \precsim_K p(x, y) + p(y, z)$ for all $x, y, z \in X$.

The pair (X, p) is then called a TVS-*cone metric space*.

Definition 1.2 (see [14]). Let (X, p) be a TVS-cone metric space, $x \in X$, and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X.

(i) $\{x_n\}$ is said to TVS-*cone converge to* x if for every $c \in Y$ with $\theta \ll_K c$ there exists a natural number $\mathbb{N}_0$ such that $p(x_n, x) \ll_K c$ for all $n \geq \mathbb{N}_0$. We denote this by *cone*-$\lim_{n \to \infty} x_n = x$ or $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$ and call x the TVS-*cone limit* of $\{x_n\}$.

(ii) $\{x_n\}$ is said to be a TVS-*cone Cauchy sequence* if for every $c \in Y$ with $\theta \ll_K c$ there is a natural number $\mathbb{N}_0$ such that $p(x_n, x_m) \ll_K c$ for all $n, m \geq \mathbb{N}_0$.

(iii) (X, p) is said to be TVS-*cone complete* if every TVS-cone Cauchy sequence in X is TVS-cone convergent in X.

In [14], the author proved the following important results.

Theorem 1.3 (see [14]). *Let (X, p) be a TVS-cone metric space. Then $d_p : X^2 \to [0, \infty)$ defined by $d_p := \xi_e \circ p$ is a metric, where $\xi_e : Y \to \mathbb{R}$ is defined by*

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \quad \forall y \in Y. \tag{1.3}$$

Theorem 1.4 (see [14]). *Let (X,p) be a TVS-cone metric space, $x \in X$, and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. Then the following statements hold:*

(a) *if $\{x_n\}$ TVS-cone converges to x (i.e., $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$), then $d_p(x_n,x) \to 0$ as $n \to \infty$ (i.e., $x_n \xrightarrow{d_p} x$ as $n \to \infty$);*

(b) *if $\{x_n\}$ is a TVS-cone Cauchy sequence in (X,p), then $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X,d_p).*

In this paper, we establish some new coupled fixed point theorems for various types of nonlinear contractive maps in the setting of quasiordered cone metric spaces. Our results generalize and improve some results in [2, 4, 9, 11] and references therein.

2. Preliminaries

Let X be a nonempty set and "$\preceq$" a quasiorder (preorder or pseudoorder, i.e., a reflexive and transitive relation) on X. Then $(X,\preceq)$ is called a quasiordered set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $\preceq$-*nondecreasing* (resp., $\preceq$-*nonincreasing*) if $x_n \preceq x_{n+1}$ (resp., $x_{n+1} \preceq x_n$) for each $n \in \mathbb{N}$. In this paper, we endow the product space $X^2 := X \times X$ with the following quasiorder $\preccurlyeq$:

$$(u,v) \preccurlyeq (x,y) \Longleftrightarrow u \preceq x,\ y \preceq v \quad \text{for any } (x,y),(u,v) \in X^2. \tag{2.1}$$

Recall that the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \quad \forall y \in Y. \tag{2.2}$$

Theorem 2.1 (see [14, 16, 17]). *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \le r \Leftrightarrow y \in re - K$;

(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - K$;

(iii) $\xi_e(y) \ge r \Leftrightarrow y \notin re - \operatorname{int} K$;

(iv) $\xi_e(y) < r \Leftrightarrow y \in re - \operatorname{int} K$;

(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* Y;

(vi) *if* $y_1 \in y_2 + K$, *then* $\xi_e(y_2) \le \xi_e(y_1)$;

(vii) $\xi_e(y_1 + y_2) \le \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1, y_2 \in Y$.

Remark 2.2. (a) Clearly, $\xi_e(\theta) = 0$.

(b) The reverse statement of (vi) in Theorem 2.1 (i.e., $\xi_e(y_2) \le \xi_e(y_1) \Rightarrow y_1 \in y_2 + K$) does not hold in general. For example, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x,y \ge 0\}$, and

$e = (1,1)$. Then K is a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K = \{(x,y) \in \mathbb{R}^2 : x, y > 0\} \neq \emptyset$ and $e \in \operatorname{int} K$. For $r = 1$, it is easy to see that $y_1 = (6,-25) \notin re - \operatorname{int} K$, and $y_2 = (0,0) \in re - \operatorname{int} K$. By applying (iii) and (iv) of Theorem 2.1, we have $\xi_e(y_2) < 1 \leq \xi_e(y_1)$ but indeed $y_1 \notin y_2 + K$.

For any TVS-cone metric space (X,p), we can define the map $\rho : X^2 \times X^2 \to Y$ by

$$\rho((x,y),(u,v)) = p(x,u) + p(y,v) \quad \text{for any } (x,y),(u,v) \in X^2. \tag{2.3}$$

It is obvious that ρ is also a TVS-cone metric on $X^2 \times X^2$, and if $x_n \xrightarrow{\text{cone}} a$ and $y_n \xrightarrow{\text{cone}} b$ as $n \to \infty$, then $(x_n, y_n) \xrightarrow{\text{cone}} (a,b)$ (i.e., $\{(x_n,y_n)\}$ TVS-cone converges to (a,b)).

By Theorem 1.3, we know that $d_p := \xi_e \circ p$ is a metric on X. Hence the function σ_p: $X^2 \times X^2 \to [0,\infty)$, defined by

$$\sigma_p((x,y),(u,v)) = d_p(x,u) + d_p(y,v) \quad \text{for any } (x,y),(u,v) \in X^2, \tag{2.4}$$

is a metric on $X^2 \times X^2$.

A map $F : X^2 \to X$ is said to be *d_p-continuous* at $(\hat{x},\hat{y}) \in X^2$ if any sequence $\{(x_n,y_n)\} \subset X^2$ with $(x_n,y_n) \xrightarrow{\sigma_p} (\hat{x},\hat{y})$ implies that $F(x_n,y_n) \xrightarrow{d_p} F(\hat{x},\hat{y})$. F is said to be *d_p-continuous* on (X^2,σ_p) if F is continuous at every point of X^2.

Definition 2.3 (see [2, 4]). Let $(X,\preceq)$ be a quasiordered set and $F : X \times X \to X$ a map. one says that F has *the mixed monotone property* on X if $F(x,y)$ is monotone nondecreasing in $x \in X$ and is monotone nonincreasing in $y \in X$, that is, for any x, $y \in X$,

$$\begin{aligned} &x_1, x_2 \in X \quad \text{with } x_1 \preceq x_2 \Longrightarrow F(x_1,y) \preceq F(x_2,y), \\ &y_1, y_2 \in X \quad \text{with } y_1 \preceq y_2 \Longrightarrow F(x,y_2) \preceq F(x,y_1). \end{aligned} \tag{2.5}$$

Definition 2.4 (see [2, 4]). Let X be a nonempty set and $F : X \times X \to X$ a map. One calls an element $(x,y) \in X^2$ a *coupled fixed point* of F if

$$F(x,y) = x, \qquad F(y,x) = y. \tag{2.6}$$

Definition 2.5. Let $(X,p,\preceq)$ be a TVS-cone metric space with a quasi-order $\preceq$ ($(X,p,\preceq)$ for short). A nonempty subset M of X is said to be

(i) TVS-*cone sequentially $\preceq^{\uparrow}$-complete* if every $\preceq$-nondecreasing TVS-cone Cauchy sequence in M converges,

(ii) TVS-*cone sequentially $\preceq_{\downarrow}$-complete* if every $\preceq$-nonincreasing TVS-cone Cauchy sequence in M converges,

(iii) TVS-*cone sequentially $\preceq^{\uparrow}_{\downarrow}$-complete* if it is both TVS-cone sequentially $\preceq^{\uparrow}$-complete and TVS-cone sequentially *$\preceq_{\downarrow}$-complete*.

Definition 2.6 (see [4, 18]). A function $\varphi : [0,\infty) \to [0,1)$ is said to be a $\mathcal{MT}$*-function* if it satisfies Mizoguchi-Takahashi's condition (i.e., $\limsup_{s\to t+0}\varphi(s) < 1$ for all $t \in [0,\infty)$).

Clearly, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function, then φ is a $\mathcal{MT}$-function. Notice that $\varphi : [0,\infty) \to [0,1)$ is a $\mathcal{MT}$-function *if and only if* for each $t \in [0,\infty)$ there exist $r_t \in [0,1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \le r_t$ for all $s \in [t, t+\varepsilon_t)$; for more detail, see [4, Remark 2.5 (iii)].

Very recently, Du and Wu [5] introduced and studied the concept of functions of contractive factor.

Definition 2.7 (see [5]). One says that $\varphi : [0,\infty) \to [0,1)$ is a *function of contractive factor* if for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, one has

$$0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1. \tag{2.7}$$

The following result tells us the relationship between $\mathcal{MT}$-functions and functions of contractive factor.

Theorem 2.8. *Any $\mathcal{MT}$-function is a function of contractive factor.*

Proof. Let $\varphi : [0,\infty) \to [0,1)$ be a $\mathcal{MT}$-function, and let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $[0,\infty)$. Then $t_0 := \lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n \ge 0$ exists. Since φ is a $\mathcal{MT}$-function, there exist $r_{t_0} \in [0,1)$ and $\varepsilon_{t_0} > 0$ such that $\varphi(s) \le r_{t_0}$ for all $s \in [t_0, t_0+\varepsilon_{t_0})$. On the other hand, there exists $\ell \in \mathbb{N}$, such that

$$t_0 \le x_n < t_0 + \varepsilon_{t_0} \tag{2.8}$$

for all $n \in \mathbb{N}$ with $n \ge \ell$. Hence $\varphi(x_n) \le r_{t_0}$ for all $n \ge \ell$. Let

$$\eta := \max\{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{\ell-1}), r_{t_0}\} < 1. \tag{2.9}$$

Then $\varphi(x_n) \le \eta$ for all $n \in \mathbb{N}$, and hence $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) \le \eta < 1$. Therefore φ is a function of contractive factor. □

3. Coupled Fixed Point Theorems for Various Types of Nonlinear Contractive Maps

Definition 3.1. One says that $\kappa : [0,\infty) \to (0,1)$ is a *function of strong contractive factor* if for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, one has

$$0 < \sup_{n\in\mathbb{N}} \kappa(x_n) < 1. \tag{3.1}$$

It is quite obvious that if κ is a function of strong contractive factor, then κ is a function of contractive factor but the reverse is not always true.

The following results are crucial to our proofs in this paper.

Lemma 3.2. *A function of strong contractive factor can be structured by a function of contractive factor.*

Proof. Let $\varphi : [0,\infty) \to [0,1)$ be a function of contractive factor. Define $\kappa(t) = (1+\varphi(t))/2$, $t \in [0,\infty)$. We claim that κ is a function of strong contractive factor. Clearly, $0 \le \varphi(t) < \kappa(t) < 1$ for all $t \in [0,\infty)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $[0,\infty)$. Since φ is a function of contractive factor, $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$. Thus it follows that

$$0 < \sup_{n\in\mathbb{N}} \kappa(x_n) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi(x_n)\right] < 1. \tag{3.2}$$

Hence κ is a function of strong contractive factor. □

Lemma 3.3. *Let E be a t.v.s., K a convex cone with* $\operatorname{int} K \neq \emptyset$ *in E, and $a, b, c \in E$. Then the following statements hold.*

(i) *If $a \precsim_K b$ and $b \ll_K c$, then $a \ll_K c$;*

(ii) *If $a \ll_K b$ and $b \precsim_K c$, then $a \ll_K c$;*

(iii) *If $a \ll_K b$ and $b \ll_K c$, then $a \ll_K c$.*

Proof. To see (i), since the set $\operatorname{int} K + K$ is open in E and K is a convex cone, we have

$$\operatorname{int} K + K = \operatorname{int}(\operatorname{int} K + K) \subseteq \operatorname{int} K. \tag{3.3}$$

Since $a \precsim_K b \Longleftrightarrow b - a \in K$ and $b \ll_K c \Longleftrightarrow c - b \in \operatorname{int} K$, it follows that

$$c - a = (c - b) + (b - a) \in \operatorname{int} K + K \subseteq \operatorname{int} K, \tag{3.4}$$

which means that $a \ll_K c$. The proofs of conclusions (ii) and(iii) are similar to (i). □

Lemma 3.4 (see [4]). *Let $(X, \preceq)$ be a quasiordered set and $F : X^2 \to X$ a multivalued map having the mixed monotone property on X. Let $x_0, y_0 \in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}), \\ y_n &= F(y_{n-1}, x_{n-1}) \end{aligned} \tag{3.5}$$

for each $n \in \mathbb{N}$. If $x_0 \preceq x_1$ and $y_1 \preceq y_0$, then $\{x_n\}$ is $\preceq$-nondecreasing and $\{y_n\}$ is $\preceq$-nonincreasing.

In this section, we first present the following new coupled fixed point theorem for functions of contractive factor in quasiordered cone metric spaces which is one of the main results of this paper.

Theorem 3.5. *Let* $(X, p, \preceq)$ *be a TVS-cone sequentially* $\preceq^{\uparrow}_{\downarrow}$*-complete metric space,* $F : X^2 \to X$ *a map having the mixed monotone property on* X*, and* $d_p := \xi_e \circ p$*. Assume that there exists a function of contractive factor* $\varphi : [0, \infty) \to [0, 1)$ *such that for any* $(x, y), (u, v) \in X^2$ *with* $(u, v) \preccurlyeq (x, y)$*,*

$$p(F(x, y), F(u, v)) \precsim_K \frac{1}{2}\varphi(d_p(x, u) + d_p(y, v))\rho((x, y), (u, v)), \tag{3.6}$$

and there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq y_0$*. Define the iterative sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ *in* X^2 *by* $x_n = F(x_{n-1}, y_{n-1})$ *and* $y_n = F(y_{n-1}, x_{n-1})$ *for* $n \in \mathbb{N}$*. Then the following statements hold.*

(a) *There exists a nonempty subset* $\mathfrak{D}$ *of* X*, such that* $(\mathfrak{D}, d_p)$ *is a complete metric space.*

(b) *There exists a nonempty subset* Ω *of* X^2*, such that* (Ω, σ_p) *is a complete metric space, where* $\sigma_p((x, y), (u, v)) := d_p(x, u) + d_p(y, v)$ *for any* $(x, y), (u, v) \in X^2$*. Moreover, if* F *is* d_p*-continuous on* (Ω, σ_p)*, then* $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ *TVS-cone converges to a coupled fixed point in* Ω *of* F*.*

Proof. Since Y is a locally convex Hausdorff t.v.s. with its zero vector θ, let τ denote the topology of Y and let $\mathcal{U}_\tau$ be the base at θ consisting of all absolutely convex neighborhood of θ. Let

$$\mathcal{L} = \{\ell : \ell \text{ is a Minkowski functional of } U \text{ for } U \in \mathcal{U}_\tau\}. \tag{3.7}$$

Then $\mathcal{L}$ is a family of seminorms on Y. For each $\ell \in \mathcal{L}$, let

$$V(\ell) = \{y \in Y : \ell(y) < 1\}, \tag{3.8}$$

and let

$$\mathcal{U}_{\mathcal{L}} = \{U : U = r_1V(\ell_1) \cap r_2V(\ell_2) \cap \cdots \cap r_nV(\ell_n),\ r_k > 0,\ \ell_k \in \mathcal{L},\ 1 \le k \le n,\ n \in \mathbb{N}\}. \tag{3.9}$$

Then $\mathcal{U}_{\mathcal{L}}$ is a base at θ, and the topology $\Gamma_{\mathcal{L}}$ generated by $\mathcal{U}_{\mathcal{L}}$ is the weakest topology for Y such that all seminorms in $\mathcal{L}$ are continuous and $\tau = \Gamma_{\mathcal{L}}$. Moreover, given any neighborhood $\mathcal{O}_\theta$ of θ, there exists $U \in \mathcal{U}_{\mathcal{L}}$ such that $\theta \in U \subset \mathcal{O}_\theta$ (see, e.g., [19, Theorem 12.4 in II.12, Page 113]).

By Lemma 3.2, we can define a function of strong contractive factor $\kappa : [0, \infty) \to [0, 1)$ by $\kappa(t) = (\varphi(t) + 1)/2$. Then $0 \le \varphi(t) < \kappa(t) < 1$ for all $t \in [0, \infty)$. For any $n \in \mathbb{N}$, let $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$. Then, by Lemma 3.4, $\{x_n\}$ is $\preceq$-nondecreasing and

$\{y_n\}$ is $\preceq$-nonincreasing. So $(x_n, y_n) \preccurlyeq (x_{n+1}, y_{n+1})$ and $(y_{n+1}, x_{n+1}) \preccurlyeq (y_n, x_n)$ for each $n \in \mathbb{N}$. By (3.6), we obtain

$$\begin{aligned} p(x_2, x_1) &= p(F(x_1, y_1), F(x_0, y_0)) \\ &\precsim_K \frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))\sigma((x_1, y_1), (x_0, y_0)) \\ &= \frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))[p(x_1, x_0) + p(y_1, y_0)], \end{aligned} \tag{3.10}$$

$$\begin{aligned} p(y_2, y_1) &= p(y_1, y_2) \\ &= p(F(y_0, x_0), F(y_1, x_1)) \\ &\precsim_K \frac{1}{2}\varphi(d_p(y_0, y_1) + d_p(x_0, x_1))[p(y_0, y_1) + p(x_0, x_1)] \\ &= \frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))[p(x_1, x_0) + p(y_1, y_0)]. \end{aligned} \tag{3.11}$$

By (3.10) and Theorem 2.1,

$$\begin{aligned} d_p(x_2, x_1) &= \xi_e(p(x_2, x_1)) \\ &\leq \xi_e\left(\frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))[p(x_1, x_0) + p(y_1, y_0)]\right) \\ &= \frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))[d_p(x_1, x_0) + d_p(y_1, y_0)] \\ &< \frac{1}{2}\kappa(d_p(x_1, x_0) + d_p(y_1, y_0))[d_p(x_1, x_0) + d_p(y_1, y_0)]. \end{aligned} \tag{3.12}$$

Similarly, by (3.11) and Theorem 2.1, we also have

$$\begin{aligned} d_p(y_2, y_1) &= \xi_e(p(x_2, x_1)) \\ &\leq \frac{1}{2}\varphi(d_p(x_1, x_0) + d_p(y_1, y_0))[d_p(x_1, x_0) + d_p(y_1, y_0)] \\ &< \frac{1}{2}\kappa(d_p(x_1, x_0) + d_p(y_1, y_0))[d_p(x_1, x_0) + d_p(y_1, y_0)]. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we get

$$d_p(x_2, x_1) + d_p(y_2, y_1) < \kappa(d_p(x_1, x_0) + d_p(y_1, y_0))[d_p(x_1, x_0) + d_p(y_1, y_0)]. \tag{3.14}$$

For each $n \in \mathbb{N}$, let $\xi_n = d_p(x_n, x_{n-1}) + d_p(y_n, y_{n-1})$. Then $\xi_2 < \kappa(\xi_1)\xi_1$. By induction, we can obtain the following. For each $n \in \mathbb{N}$,

$$p(x_{n+1}, x_n) \precsim_K \frac{1}{2}\varphi(\xi_n)\left[p(x_n, x_{n-1}) + p(y_n, y_{n-1})\right]; \tag{3.15}$$

$$p(y_{n+1}, y_n) \precsim_K \frac{1}{2}\varphi(\xi_n)\left[p(x_n, x_{n-1}) + p(y_n, y_{n-1})\right]; \tag{3.16}$$

$$d_p(x_{n+1}, x_n) < \frac{1}{2}\kappa(\xi_n)\xi_n; \tag{3.17}$$

$$d_p(y_{n+1}, y_n) < \frac{1}{2}\kappa(\xi_n)\xi_n; \tag{3.18}$$

$$\xi_{n+1} < \kappa(\xi_n)\xi_n. \tag{3.19}$$

Since $0 < \kappa(t) < 1$ for all $t \in [0, \infty)$, the sequence $\{\xi_n\}$ is strictly decreasing in $[0, \infty)$ from (3.19). Since κ is a function of strong contractive factor, we have

$$0 < \lambda := \sup_{n \in \mathbb{N}} \kappa(\xi_n) < 1. \tag{3.20}$$

So $\varphi(\xi_n) < \kappa(\xi_n) \leq \lambda$ for all $n \in \mathbb{N}$. We want to prove that $\{x_n\}$ is a $\preceq$-nondecreasing TVS-cone Cauchy sequence and $\{y_n\}$ is a $\preceq$-nonincreasing TVS-cone Cauchy sequence in X. For each $n \in \mathbb{N}$, by (3.15), we have

$$p(x_{n+2}, x_{n+1}) \precsim_K \frac{1}{2}\lambda\left[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\right]. \tag{3.21}$$

Similarly, by (3.16), we obtain

$$p(y_{n+2}, y_{n+1}) \precsim_K \frac{1}{2}\lambda\left[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\right]. \tag{3.22}$$

From (3.21) and (3.22), we get

$$p(x_{n+2}, x_{n+1}) + p(y_{n+2}, y_{n+1}) \precsim_K \lambda\left[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\right] \quad \text{for each } n \in \mathbb{N}. \tag{3.23}$$

Hence it follows from (3.21), (3.22), and (3.23) that

$$\begin{aligned} p(x_{n+2}, x_{n+1}) &\precsim_K \frac{1}{2}\lambda\left[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\right] \\ &\precsim_K \frac{1}{2}\lambda^2\left[p(x_n, x_{n-1}) + p(y_n, y_{n-1})\right] \\ &\precsim_K \cdots \\ &\precsim_K \frac{1}{2}\lambda^n\left[p(x_2, x_1) + p(y_2, y_1)\right], \\ p(y_{n+2}, y_{n+1}) &\precsim_K \frac{1}{2}\lambda^n\left[p(x_2, x_1) + p(y_2, y_1)\right] \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{3.24}$$

Therefore, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$p(x_m, x_n) \precsim_K \sum_{j=n}^{m-1} p(x_{j+1}, x_j) \precsim_K \frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2, x_1) + p(y_2, y_1)\right], \tag{3.25}$$

$$p(y_m, y_n) \precsim_K \sum_{j=n}^{m-1} p(y_{j+1}, y_j) \precsim_K \frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2, x_1) + p(y_2, y_1)\right]. \tag{3.26}$$

Given $c \in Y$ with $\theta \ll_K c$ (i.e., $c \in \operatorname{int} K = \operatorname{int}(\operatorname{int} K)$), there exists a neighborhood N_θ of θ such that $c + N_\theta \subseteq \operatorname{int} K$. Therefore, there exists $U_c \in \mathcal{U}_{\mathcal{L}}$ with $U_c \subseteq N_\theta$ such that $c + U_c \subseteq c + N_\theta \subseteq \operatorname{int} K$, where

$$U_c = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_s V(\ell_s), \tag{3.27}$$

for some $r_i > 0$ and $\ell_i \in \mathcal{L}$, $1 \le i \le s$. Let

$$\begin{gathered} \delta_c = \min\{r_i : 1 \le i \le s\} > 0, \\ \eta = \max\{\ell_i(p(x_2, x_1) + p(y_2, y_1)) : 1 \le i \le s\}. \end{gathered} \tag{3.28}$$

If $\eta = 0$, since each ℓ_i is a seminorm, we have $\ell_i(p(x_2, x_1) + p(y_2, y_1)) = 0$ and

$$\ell_i\left(-\frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2, x_1) + p(y_2, y_1)\right]\right) = \frac{\lambda^{n-1}}{2(1-\lambda)}\ell_i(p(x_2, x_1) + p(y_2, y_1)) = 0 < r_i \tag{3.29}$$

for all $1 \le i \le s$ and all $n \in \mathbb{N}$. If $\eta > 0$, since $\lambda \in (0,1)$, $\lim_{n\to\infty}(\lambda^{n-1}/2(1-\lambda)) = 0$, and hence there exists $n_0 \in \mathbb{N}$ such that $\lambda^{n-1}/2(1-\lambda) < \delta_c/\eta$ for all $n \ge n_0$. So, for each $i \in \{1,2,\ldots,s\}$ and any $n \ge n_0$, we obtain

$$\begin{aligned}
\ell_i\left(-\frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2,x_1)+p(y_2,y_1)\right]\right) &= \frac{\lambda^{n-1}}{2(1-\lambda)}\ell_i\left(p(x_2,x_1)+p(y_2,y_1)\right) \\
&< \frac{\delta_c}{\eta}\ell_i\left(p(x_2,x_1)+p(y_2,y_1)\right) \\
&\le \delta_c \\
&\le r_i.
\end{aligned} \tag{3.30}$$

Therefore for any $n \ge n_0$, $-(\lambda^{n-1}/2(1-\lambda))[p(x_2,x_1)+p(y_2,y_1)] \in r_i V(\ell_i)$ for all $1 \le i \le s$, and hence $-(\lambda^{n-1}/2(1-\lambda))[p(x_2,x_1)+p(y_2,y_1)] \in U_c$. So we obtain

$$c - \frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2,x_1)+p(y_2,y_1)\right] \in c + U_c \subseteq \operatorname{int} K \tag{3.31}$$

or

$$\frac{\lambda^{n-1}}{2(1-\lambda)}\left[p(x_2,x_1)+p(y_2,y_1)\right] \ll_K c \tag{3.32}$$

for all $n \ge n_0$. For $m,n \in \mathbb{N}$ with $m > n \ge n_0$, by (3.25), (3.26), (3.32), and Lemma 3.3, we obtain

$$\begin{aligned}
p(x_m,x_n) &\ll_K c, \\
p(y_m,y_n) &\ll_K c.
\end{aligned} \tag{3.33}$$

Hence $\{x_n\}$ is a $\preceq$-nondecreasing TVS-cone Cauchy sequence and $\{y_n\}$ is a $\preceq$-nonincreasing TVS-cone Cauchy sequence in X. By the TVS-cone sequential $\preceq_{\downarrow}^{\uparrow}$-completeness of X, there exist $\widehat{x}, \widehat{y} \in X$ such that $\{x_n\}$ TVS-cone converges to $\widehat{x}$ and $\{y_n\}$ TVS-cone converges to $\widehat{y}$. Therefore $\{(x_n,y_n)\}$ TVS-cone converges to $(\widehat{x},\widehat{y})$.

On the other hand, applying Theorem 1.4, we have the following:

$$\{x_n\} \text{ is a } \preceq \text{-nondecreasing Cauchy sequence in } (X,d_p); \tag{3.34}$$

$$\{y_n\} \text{ is a } \preceq \text{-nonincreasing Cauchy sequence in } (X,d_p); \tag{3.35}$$

$$d_p(x_n,\widehat{x}) \longrightarrow 0 \left(\text{or } x_n \xrightarrow{d_p} \widehat{x}\right) \quad \text{as } n \longrightarrow \infty; \tag{3.36}$$

$$d_p(y_n,\widehat{y}) \longrightarrow 0 \left(\text{or } y_n \xrightarrow{d_p} \widehat{y}\right) \quad \text{as } n \longrightarrow \infty. \tag{3.37}$$

Since $\sigma_p((x_n, y_n), (\widehat{x}, \widehat{y}) = d_p(x_n, \widehat{x}) + d_p(y_n, \widehat{y})$ for all $n \in \mathbb{N}$, by (3.36) and (3.37), we have $(x_n, y_n) \xrightarrow{\sigma_p} (\widehat{x}, \widehat{y})$ as $n \to \infty$. Let $\mathfrak{D}_1 = \{x_n\}_{n\in\mathbb{N}\cup\{0\}} \cup \{\widehat{x}\}$, $\mathfrak{D}_2 = \{y_n\}_{n\in\mathbb{N}\cup\{0\}} \cup \{\widehat{y}\}$, and $\Omega = \mathfrak{D}_1 \times \mathfrak{D}_2$. Then $(\mathfrak{D}_1, d_p)$, $(\mathfrak{D}_2, d_p)$, and (Ω, σ_p) are also complete metric spaces. Hence conclusion (a) holds.

Finally, in order to complete the proof of conclusion (b), we need to verify that $(\widehat{x}, \widehat{y}) \in \Omega$ is a coupled fixed point of F. Let $\varepsilon > 0$ be given. Since F is d_p-continuous on (Ω, σ_p) and $(\widehat{x}, \widehat{y}) \in \Omega$, F is d_p-continuous at $(\widehat{x}, \widehat{y})$. So there exists $\delta > 0$ such that

$$d_p(F(\widehat{x}, \widehat{y}), F(u, v)) < \frac{\varepsilon}{2} \tag{3.38}$$

whenever $(u, v) \in \Omega$ with $\sigma_p((\widehat{x}, \widehat{y}), (u, v)) < \delta$. Since $x_n \xrightarrow{d_p} \widehat{x}$ and $y_n \xrightarrow{d_p} \widehat{y}$ as $n \to \infty$, for $\zeta = \min\{\varepsilon/2, \delta/2\} > 0$, there exists $v_0 \in \mathbb{N}$ such that

$$d_p(x_n, \widehat{x}) < \zeta, \quad d_p(y_n, \widehat{y}) < \zeta \quad \forall n \in \mathbb{N} \text{ with } n \geq v_0. \tag{3.39}$$

So, for each $n \in \mathbb{N}$ with $n \geq v_0$, by (3.39),

$$\sigma_p((\widehat{x}, \widehat{y}), (x_n, y_n)) = d_p(x_n, \widehat{x}) + d_p(y_n, \widehat{y}) < \delta, \tag{3.40}$$

and hence we have from (3.38) that

$$d_p(F(\widehat{x}, \widehat{y}), F(x_n, y_n)) < \frac{\varepsilon}{2}. \tag{3.41}$$

Therefore

$$\begin{aligned} d_p(F(\widehat{x}, \widehat{y}), \widehat{x}) &\leq d_p(F(\widehat{x}, \widehat{y}), x_{v_0+1}) + d_p(x_{v_0+1}, \widehat{x}) \\ &= d_p(F(\widehat{x}, \widehat{y}), F(x_{v_0}, y_{v_0})) + d_p(x_{v_0+1}, \widehat{x}) \\ &< \frac{\varepsilon}{2} + \zeta \quad (\text{by (3.39) and (3.41)}) \\ &\leq \varepsilon. \end{aligned} \tag{3.42}$$

Since ε is arbitrary, $d_p(F(\widehat{x}, \widehat{y}), \widehat{x}) = 0$ or $F(\widehat{x}, \widehat{y}) = \widehat{x}$. Similarly, we can also prove that $F(\widehat{y}, \widehat{x}) = \widehat{y}$. So $(\widehat{x}, \widehat{y}) \in \Omega$ is a coupled fixed point of F. The proof is finished. □

The following conclusions are immediate from Theorems 2.8 and 3.5.

Theorem 3.6. *Let $(X, p, \preceq)$ be a TVS-cone sequentially $\preceq_{\downarrow}^{\uparrow}$-complete metric space, $F : X^2 \to X$ a map having the mixed monotone property on X, and $d_p := \xi_e \circ p$. Assume that there exists a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that for any $(x, y), (u, v) \in X^2$ with $(u, v) \preccurlyeq (x, y)$,*

$$p(F(x, y), F(u, v)) \precsim_K \frac{1}{2}\varphi(d_p(x, u) + d_p(y, v))\rho((x, y), (u, v)), \tag{3.43}$$

and there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$. Define the iterative sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ in X^2 by $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$ for $n \in \mathbb{N}$. Then the following statements hold.

(a) *There exists a nonempty subset $\mathfrak{D}$ of X, such that $(\mathfrak{D}, d_p)$ is a complete metric space.*

(b) *There exists a nonempty subset Ω of X^2, such that (Ω, σ_p) is a complete metric space. Moreover, if F is d_p-continuous on (Ω, σ_p), then $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to a coupled fixed point in Ω of F.*

Theorem 3.7. *Let $(X, p, \preceq)$ be a TVS-cone sequentially $\preceq_{\downarrow}^{\uparrow}$-complete metric space, $F : X^2 \to X$ a map having the mixed monotone property on X, and $d_p := \xi_e \circ p$. Assume that there exists a nonnegative number $\gamma < 1$ such that for any $(x, y), (u, v) \in X^2$ with $(u, v) \preccurlyeq (x, y)$,*

$$p\big(F(x,y), F(u,v)\big) \precapprox_K \frac{\gamma}{2}\rho\big((x,y),(u,v)\big), \tag{3.44}$$

and there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$. Define the iterative sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ in X^2 by $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$ for $n \in \mathbb{N}$. Then the following statements hold.

(a) *There exists a nonempty subset $\mathfrak{D}$ of X, such that $(\mathfrak{D}, d_p)$ is a complete metric space.*

(b) *There exists a nonempty subset Ω of X^2, such that (Ω, σ_p) is a complete metric space. Moreover, if F is d_p-continuous on (Ω, σ_p), then $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to a coupled fixed point in Ω of F.*

Remark 3.8. (a) Theorems 3.5 and 3.6 all generalize and improve [4, Theorem 2.8] and some results in [2, 9, 11].

(b) Theorems 3.5–3.7 all generalize Bhaskar-Lakshmikantham's coupled fixed points theorem (i.e., Theorem BL).

Finally, we focus our research on TVS-cone metric spaces.

Theorem 3.9. *Let (X, p) be a TVS-cone complete metric space, $F : X^2 \to X$ a map, and $d_p := \xi_e \circ p$. Assume that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that for any $(x, y), (u, v) \in X^2$*

$$p\big(F(x,y), F(u,v)\big) \precapprox_K \frac{1}{2}\varphi\big(d_p(x,u) + d_p(y,v)\big)\rho\big((x,y),(u,v)\big). \tag{3.45}$$

Let $x_0, y_0 \in X$. Define the iterative sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ in X^2 by $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$ for $n \in \mathbb{N}$. Then the following statements hold.

(a) *There exists a nonempty subset $\mathfrak{D}$ of X, such that $(\mathfrak{D}, d_p)$ is a complete metric space.*

(b) *There exists a nonempty subset Ω of X^2, such that (Ω, σ_p) is a complete metric space.*

(c) *F has a unique coupled fixed point in Ω. Moreover, $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to the coupled fixed point of F.*

Proof. For any $(x,y),(u,v)\in X^2$, by (3.45) and Theorem 2.1, we obtain

$$\begin{aligned}d_p(F(x,y),F(u,v)) &\le \frac{1}{2}\varphi(d_p(x,u)+d_p(y,v))[d_p(x,u)+d_p(y,v)]\\ &= \frac{1}{2}\varphi(\sigma_p((x,y),(u,v)))\sigma_p((x,y),(u,v))\\ &< \frac{1}{2}\sigma_p((x,y),(u,v)).\end{aligned} \tag{3.46}$$

From (3.46), we know that F is d_p-continuous on (X^2,σ_p). Following the same argument as in the proof of Theorem 3.5, we can prove that conclusions (a) and (b) hold and there exists $(\widehat{x},\widehat{y})\in\Omega$, such that $\{(x_n,y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to $(\widehat{x},\widehat{y})$ and $(\widehat{x},\widehat{y})$ is a coupled fixed point of F. To complete the proof, it suffices to show the uniqueness of the coupled fixed point of F. On the contrary, suppose that there exists $(\widehat{u},\widehat{v})\in X\times X$, such that $\widehat{u}=F(\widehat{u},\widehat{v})$ and $\widehat{v}=F(\widehat{v},\widehat{u})$. By (3.46), we have

$$\begin{aligned}d_p(\widehat{x},\widehat{u}) &= d_p(F(\widehat{x},\widehat{y}),F(\widehat{u},\widehat{v})) < \frac{1}{2}[d_p(\widehat{x},\widehat{u})+d_p(\widehat{y},\widehat{v})],\\ d_p(\widehat{y},\widehat{v}) &= d_p(F(\widehat{y},\widehat{x}),F(\widehat{v},\widehat{u})) < \frac{1}{2}[d_p(\widehat{x},\widehat{u})+d_p(\widehat{y},\widehat{v})].\end{aligned} \tag{3.47}$$

So, it follows from (3.47) that

$$d_p(\widehat{x},\widehat{u})+d_p(\widehat{y},\widehat{v}) < d_p(\widehat{x},\widehat{u})+d_p(\widehat{y},\widehat{v}), \tag{3.48}$$

which leads to a contradiction. The proof is completed. □

The following results are immediate from Theorem 3.9.

Theorem 3.10. *Let (X,p) be a TVS-cone complete metric space, $F: X^2 \to X$ a map, and $d_p := \xi_e\circ p$. Assume that there exists a $\mathcal{MT}$-function $\varphi:[0,\infty)\to[0,1)$ such that for any $(x,y),(u,v)\in X^2$,*

$$p(F(x,y),F(u,v)) \precsim_K \frac{1}{2}\varphi(d_p(x,u)+d_p(y,v))\rho((x,y),(u,v)). \tag{3.49}$$

Let $x_0,y_0\in X$. Define the iterative sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ in X^2 by $x_n=F(x_{n-1},y_{n-1})$ and $y_n=F(y_{n-1},x_{n-1})$ for $n\in\mathbb{N}$. Then the following statements hold.

(a) *There exists a nonempty subset $\mathfrak{D}$ of X, such that $(\mathfrak{D},d_p)$ is a complete metric space.*

(b) *There exists a nonempty subset Ω of X^2, such that (Ω,σ_p) is a complete metric space.*

(c) *F has a unique coupled fixed point in Ω. Moreover, $\{(x_n,y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to the coupled fixed point of F.*

Theorem 3.11. *Let (X, p) be a TVS-cone complete metric space and $F : X^2 \to X$ a map. Assume that there exists a nonnegative number $\gamma < 1$ such that for any $(x, y), (u, v) \in X^2$,*

$$p(F(x, y), F(u, v)) \precsim_K \frac{\gamma}{2}\rho((x, y), (u, v)). \tag{3.50}$$

Let $x_0, y_0 \in X$. Define the iterative sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ in X^2 by $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$ for $n \in \mathbb{N}$. Then the following statements hold.

(a) *There exists a nonempty subset $\mathfrak{D}$ of X, such that $(\mathfrak{D}, d_p)$ is a complete metric space.*

(b) *There exists a nonempty subset Ω of X^2, such that (Ω, σ_p) is a complete metric space.*

(c) *F has a unique coupled fixed point in Ω. Moreover, $\{(x_n, y_n)\}_{n\in\mathbb{N}\cup\{0\}}$ TVS-cone converges to the coupled fixed point of F.*

Remark 3.12. (a) Theorems 3.9 and 3.10 all generalize and improve [4, Theorem 2.12].

(b) Theorems 3.9–3.11 all generalize some results in [2, 9, 11].

Acknowledgment

This research was supported by the National Science Council of the Republic of China.

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