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Research Article

Fixed Points and Stability in Nonlinear Equations with Variable Delays

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We consider two nonlinear scalar delay differential equations with variable delays and give some new conditions for the boundedness and stability by means of the contraction mapping principle. We obtain the differences of the two equations about the stability of the zero solution. Previous results are improved and generalized. An example is given to illustrate our theory.

1. Introduction

Fixed point theory has been used to deal with stability problems for several years. It has conquered many difficulties which Liapunov method cannot. While Liapunov's direct method usually requires pointwise conditions, fixed point theory needs average conditions.

In this paper, we consider the nonlinear delay differential equations

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)g(x(t - r_2(t))), \tag{1.1}$$

$$x'(t) = -a(t)f(x(t - r_1(t))) + b(t)g(x(t - r_2(t))), \tag{1.2}$$

where $r_1(t), r_2(t) : [0,\infty) \to [0,\infty)$, $r = \max\{r_1(0), r_2(0)\}$, $a, b : [0,\infty) \to R$, $f, g : R \to R$ are continuous functions. We assume the following:

(A1) $r_1(t)$ is differentiable,

(A2) the functions $t - r_1(t), t - r_2(t) : [0,\infty) \to [-r,\infty)$ is strictly increasing,

(A3) $t - r_1(t), t - r_2(t) \to \infty$ as $t \to \infty$.

Many authors have investigated the special cases of (1.1) and (1.2). Since Burton [1] used fixed point theory to investigate the stability of the zero solution of the equation $x'(t) = -a(t)x(t-r)$, many scholars continued his idea. For example, Zhang [2] has studied the equation

$$x'(t) = -a(t)x(t-r(t)), \tag{1.3}$$

Becker and Burton [3] have studied the equation

$$x'(t) = -a(t)f(x(t-r(t))), \tag{1.4}$$

Jin and Luo [4] have studied the equation

$$x'(t) = -a(t)x(t-r_1(t)) + b(t)x^{1/3}(t-r_2(t)). \tag{1.5}$$

Burton [5] and Zhang [6] have also studied similar problems. Their main results are the following.

Theorem 1.1 (Burton [1]). *Suppose that $r(t) = r$, a constant, and there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |a(s+r)|ds + \int_{0}^{t} |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)|du\, ds \le \alpha, \tag{1.6}$$

for all $t \ge 0$ and $\int_0^\infty a(s)ds = \infty$. Then, for every continuous initial function $\psi : [-r,0] \to R$, the solution $x(t) = x(t,0,\psi)$ of (1.3) is bounded and tends to zero as $t \to \infty$.

Theorem 1.2 (Zhang [2]). *Suppose that r is differentiable, the inverse function $h(t)$ of $t-r(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for $t \ge 0$*

(i)

$$\liminf_{t\to\infty} \int_0^t a(h(s)) > -\infty, \tag{1.7}$$

(ii)

$$\int_{t-r(t)}^{t} |a(h(s))|ds + \int_0^t e^{-\int_s^t a(h(u))du}|a(h(s))| \int_{s-r(s)}^{s} |a(h(v))|dv\, ds + \theta(s), \tag{1.8}$$

where $\theta(t) = \int_0^t e^{-\int_s^t a(h(u))du}|a(s)||r'(s)|ds$. Then, the zero solution of (1.3) is asymptotically stable if and only if

(iii)

$$\int_0^t a(h(s))ds \longrightarrow \infty, \quad as\ t \longrightarrow \infty. \tag{1.9}$$

Theorem 1.3 (Burton [7]). *Suppose that* $r(t) = r$*, a constant. Let* f *be odd, increasing on* $[0, L]$*, and satisfies a Lipschitz condition, and let* $x - f(x)$ *be nondecreasing on* $[0, L]$*. Suppose also that for each* $L_1 \in [0, L]$*, one has*

$$\begin{aligned} \left|L_1 - f(L_1)\right| \sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u+r)du} |a(s+r)| ds + f(L_1) \sup_{t\geq 0} \int_{t-r}^t |a(u+r)| du \\ + f(L_1) \sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u+r)du} |a(s+r)| \int_{s-r}^s |a(u+r)| du\, ds < L_1, \end{aligned} \tag{1.10}$$

and there exists $J > 0$ *such that*

$$-\int_0^t a(s+r) ds \leq J \quad \text{for } t \geq 0. \tag{1.11}$$

Then, the zero solution of (1.4) *is stable.*

Theorem 1.4 (Becker and Burton [3]). *Suppose* f *is odd, strictly increasing, and satisfies a Lipschitz condition on an interval* $[-l, l]$ *and that* $x - f(x)$ *is nondecreasing on* $[0, l]$*. If*

$$\sup_{t \geq t_1} \int_{t-r(t)}^t a(u) du < \frac{1}{2}, \tag{1.12}$$

where t_1 *is the unique solution of* $t - r(t) = 0$*, and if a continuous function* $\tilde{a} : [0, \infty) \to R$ *exists such that*

$$a(t) = \tilde{a}(t)\left(1 - r'(t)\right), \tag{1.13}$$

on $[0, \infty)$*, then the zero solution of* (1.5) *is stable at* $t = 0$*. Furthermore, if* f *is continuously differentiable on* $[-l, l]$ *with* $f'(0) \neq 0$ *and*

$$\int_0^t a(u) du \longrightarrow \infty \quad \text{as } t \longrightarrow \infty, \tag{1.14}$$

then the zero solution of (1.4) *is asymptotically stable.*

In the present paper, we adopt the contraction mapping principle to study the boundedness and stability of (1.1) and (1.2). That means we investigate how the stability property will be when (1.3) and (1.4) are added to the perturbed term $b(t)g(x(t - r_2(t)))$. We obtain their differences about the stability of the zero solution, and we also improve and generalize the special case $r_1(t) = r_1$. Finally, we give an example to illustrate our theory.

2. Main Results

From existence theory, we can conclude that for each continuous initial function $\psi : [-r, 0] \to R$ there is a continuous solution $x(t, 0, \psi)$ on an interval $[0, T)$ for some $T > 0$ and

$x(t,0,\psi)=\psi(t)$ on $[-r,0]$. Let $C(S_1,S_2)$ denote the set of all continuous functions $\phi: S_1 \to S_2$ and $\|\psi\| = \max\{|\psi(t)| : t \in [-r,0]\}$. Stability definitions can be found in [8].

Theorem 2.1. *Suppose that the following conditions are satisfied:*

(i) $g(0)=0$, *and there exists a constant* $L>0$ *so that if* $|x|,|y|\le L$, *then*

$$\left|g(x)-g(y)\right| \le \left|x-y\right|, \tag{2.1}$$

(ii) *there exists a constant* $\alpha\in(0,1)$ *and a continuous function* $h:[-r,\infty)\to R$ *such that*

$$\begin{aligned}\int_{t-r_1(t)}^{t}|h(s)|ds &+ \int_0^t e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|du\,ds\\ &+\int_0^t e^{-\int_s^t h(u)du}\left[\left|h(s-r_1(s))(1-r_1'(s))-a(s)\right|+|b(s)|\right]ds\le\alpha,\end{aligned} \tag{2.2}$$

(iii)

$$\liminf_{t\to\infty}\int_0^t h(s)ds > -\infty. \tag{2.3}$$

Then, the zero solution of (1.1) *is asymptotically stable if and only if*

(iv)

$$\int_0^t h(s)ds \longrightarrow \infty, \quad \text{as } t\longrightarrow\infty. \tag{2.4}$$

Proof. First, suppose that (iv) holds. We set

$$J=\sup_{t\ge 0}\left\{-\int_0^t h(s)ds\right\}. \tag{2.5}$$

Let $S=\{\phi \mid \phi\in C([-r,\infty),R), \|\phi\|=\sup_{t\ge -r}|\phi(t)|<\infty\}$, then S is a Banach space.

Multiply both sides of (1.1) by $e^{\int_0^t h(s)ds}$, and then integrate from 0 to t to obtain

$$\begin{aligned}x(t)={}&x_0e^{-\int_0^t h(s)ds}+\int_0^t e^{-\int_s^t h(u)du}h(s)x(s)ds\\ &-\int_0^t e^{-\int_s^t h(u)du}a(s)x(s-r_1(s))ds+\int_0^t e^{-\int_s^t h(u)du}b(s)g(x(s-r_2(s)))ds.\end{aligned} \tag{2.6}$$

By performing an integration by parts, we have

$$\begin{aligned} x(t) = x_0 e^{-\int_0^t h(s)ds} &+ \int_0^t e^{-\int_s^t h(u)du} \left(\int_{s-r_1(s)}^{s} h(u)x(u)du \right)' ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} \left[h(s-r_1(s))\left(1-r_1'(s)\right) - a(s) \right] x(s-r_1(s))ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} b(s) g(x(s-r_2(s)))ds, \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} x(t) = x_0 e^{-\int_0^t h(s)ds} &- e^{-\int_0^t h(s)ds} \int_{-r_1(0)}^{0} h(s)x(s)ds + \int_{t-r_1(t)}^{t} h(s)x(s)ds \\ &- \int_0^t e^{-\int_s^t h(u)du} h(s) \int_{s-r_1(s)}^{s} h(u)x(u)du\, ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} \left[h(s-r_1(s))\left(1-r_1'(s)\right) - a(s) \right] x(s-r_1(s))ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} b(s) g(x(s-r_2(s)))ds. \end{aligned} \tag{2.8}$$

Let

$$M = \left\{ \phi \mid \phi \in S, \sup_{t \geq -r} \left|\phi(t)\right| \leq L,\ \phi(t) = \psi(t) \text{ for } t \in [-r, 0],\ \phi(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \right\}. \tag{2.9}$$

Then, M is a complete metric space with metric $\sup_{t \geq 0} |\phi(t) - \eta(t)|$ for $\phi, \eta \in M$. For all $\phi \in M$, define the mapping P

$$\begin{aligned} (P\phi)(t) &= \psi(t), \quad t \in [-r, 0], \\ (P\phi)(t) &= \psi(0) e^{-\int_0^t h(s)ds} - e^{-\int_0^t h(s)ds} \int_{-r_1(0)}^{0} h(s)\psi(s)ds + \int_{t-r_1(t)}^{t} h(s)\phi(s)ds \\ &\quad - \int_0^t e^{-\int_s^t h(u)du} h(s) \int_{s-r_1(s)}^{s} h(u)\phi(u)du\, ds \\ &\quad + \int_0^t e^{-\int_s^t h(u)du} \left[h(s-r_1(s))\left(1-r_1'(s)\right) - a(s) \right] \phi(s-r_1(s))ds \\ &\quad + \int_0^t e^{-\int_s^t h(u)du} b(s) g\left(\phi(s-r_2(s))\right)ds, \quad t \geq 0. \end{aligned} \tag{2.10}$$

By (i) and $g(0)=0$,

$$\begin{aligned}\left|(P\phi)(t)\right| &\leq \|\psi\|\left[1+\int_{-r_1(0)}^{0}|h(s)|ds\right]e^{-\int_0^t h(s)ds} \\ &\quad + L\left[\int_{t-r_1(t)}^{t}|h(s)|ds+\int_0^t e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|du\,ds\right. \\ &\quad\quad \left.+\int_0^t e^{-\int_s^t h(u)du}\left[\left|h(s-r_1(s))\left(1-r_1'(s)\right)-a(s)\right|+|b(s)|\right]ds\right] \\ &\leq \|\psi\|\left[1+\int_{-r_1(0)}^{0}|h(s)|ds\right]e^{J}+\alpha L.\end{aligned} \tag{2.11}$$

Thus, when $\|\psi\|\leq\delta=(1-\alpha)L/[1+\int_{-r_1(0)}^{0}|h(s)|ds]e^{J}$, $|(P\phi)(t)|\leq L$.

We now show that $(P\phi)(t)\to 0$ as $t\to\infty$. Since $\phi(t)\to 0$ and $t-r_1(t)\to\infty$ as $t\to\infty$, for each $\varepsilon>0$, there exists a $T_1>0$ such that $t>T_1$ implies $|\phi(t-r_1(t))|<\varepsilon$. Thus, for $t\geq T_1$,

$$|I_1|=\left|\int_{t-r_1(t)}^{t}h(s)\phi(s)ds\right|\leq\varepsilon\int_{t-r_1(t)}^{t}|h(s)|ds\leq\alpha\varepsilon. \tag{2.12}$$

Hence, $I_1\to 0$ as $t\to\infty$. And

$$\begin{aligned}|I_2| &= \left|\int_0^t e^{-\int_s^t h(u)du}h(s)\int_{s-r_1(s)}^{s}h(u)\phi(u)du\,ds\right| \\ &\leq \int_0^{T_1} e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|\left|\phi(u)\right|du\,ds \\ &\quad +\int_{T_1}^{t} e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|\left|\phi(u)\right|du\,ds \\ &\leq L\int_0^{T_1} e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|du\,ds+\varepsilon\int_{T_1}^{t} e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|du\,ds,\end{aligned} \tag{2.13}$$

By (ii) and (iv), there exists $T_2>T_1$ such that $t\geq T_2$ implies

$$L\int_0^{T_1} e^{-\int_s^t h(u)du}|h(s)|\int_{s-r_1(s)}^{s}|h(u)|du\,ds<\varepsilon. \tag{2.14}$$

Apply (ii) to obtain $|I_2|<\varepsilon+2\varepsilon<2\varepsilon$. Thus, $I_2\to 0$ as $t\to\infty$. Similarly, we can show that the rest term in (2.10) approaches zero as $t\to\infty$. This yields $(P\phi)(t)\to 0$ as $t\to\infty$, and hence $P\phi\in M$.

Also, by (ii), P is a contraction mapping with contraction constant α. By the contraction mapping principle, P has a unique fixed point x in M which is a solution of (1.1) with $x(s) = \psi(s)$ on $[-r, 0]$ and $x(t) \to 0$ as $t \to \infty$.

In order to prove stability at $t = 0$, let $\varepsilon > 0$ be given. Then, choose $m > 0$ so that $m < \min\{\varepsilon, L\}$. Replacing L with m in M, we see there is a $\delta > 0$ such that $\|\psi\| < \delta$ implies that the unique continuous solution x agreeing with ψ on $[-r, 0]$ satisfies $|x(t)| \le m < \varepsilon$ for all $t \ge -r$. This shows that the zero solution of (1.1) is asymptotically stable if (iv) holds.

Conversely, suppose (iv) fails. Then, by (iii), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} h(s)ds = l$ for some $l \in R$. We may choose a positive constant N satisfying

$$-N \le \int_0^{t_n} h(s)ds \le N, \tag{2.15}$$

for all $n \ge 1$. To simplify the expression, we define

$$\omega(s) = \left|h(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right| + |b(s)| + h(s)\int_{s-r_1(s)}^{s} |h(u)|du, \tag{2.16}$$

for all $s \ge 0$. By (ii), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} h(u)du}\omega(s)ds \le \alpha. \tag{2.17}$$

This yields

$$\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds \le \alpha e^{\int_0^{t_n} h(u)du} \le \alpha e^N. \tag{2.18}$$

The sequence $\{\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds = \gamma, \tag{2.19}$$

for some $\gamma \in R^+$ and choose a positive integer $\overline{k}$ so large that

$$\int_{t_{\overline{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds < \frac{\delta_0}{4N}, \tag{2.20}$$

for all $n \ge \overline{k}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^N + \alpha < 1$.

By (iii), J in (2.5) is well defined. We now consider the solution $x(t) = x(t, t_{\overline{k}}, \varphi)$ of (1.1) with $\varphi(t_{\overline{k}}) = \delta_0$ and $|\varphi(s)| \le \delta_0$ for $s \le t_{\overline{k}}$. We may choose φ so that $|x(t)| \le L$ for $t \ge t_{\overline{k}}$ and

$$\varphi(t_{\overline{k}}) - \int_{t_{\overline{k}}-r_1(t_{\overline{k}})}^{t_{\overline{k}}} h(s)\varphi(s)ds \ge \frac{1}{2}\delta_0. \tag{2.21}$$

It follows from (2.10) with $x(t) = (Px)(t)$ that for $n \ge t_{\overline{k}}$,

$$\begin{aligned}
\left| x(t_n) - \int_{t_n-r_1(t_n)}^{t_n} h(s)x(s)ds \right| &\ge \frac{1}{2}\delta_0 e^{-\int_{t_{\overline{k}}}^{t_n} h(u)du} - \int_{t_{\overline{k}}}^{t_n} e^{-\int_s^{t_n} h(u)du}\omega(s)ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_{\overline{k}}}^{t_n} h(u)du} - e^{-\int_0^{t_n} h(u)du}\int_{t_{\overline{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds \\
&= e^{-\int_{t_{\overline{k}}}^{t_n} h(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_{\overline{k}}} h(u)du}\int_{t_{\overline{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right) \\
&\ge e^{-\int_{t_{\overline{k}}}^{t_n} h(u)du}\left(\frac{1}{2}\delta_0 - N\int_{t_{\overline{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right) \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_{\overline{k}}}^{t_n} h(u)du} \ge \frac{1}{4}\delta_0 e^{-2N} > 0.
\end{aligned} \tag{2.22}$$

On the other hand, if the solution of (1.1) $x(t) = x(t, t_{\overline{k}}, \varphi) \to 0$ as $t \to \infty$, since $t_n - r_1(t_n) \to \infty$ as $n \to \infty$ and (ii) holds, we have

$$x(t_n) - \int_{t_n-r_1(t_n)}^{t_n} h(s)x(s)ds \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.23}$$

which contradicts (2.22). Hence, condition (iv) is necessary for the asymptotically stability of the zero solution of (1.1). The proof is complete. □

When $r_1(t) = r_1$, a constant, $h(t) = a(t + r_1)$, we can get the following.

Corollary 2.2. *Suppose that the following conditions are satisfied:*

(i) $g(0) = 0$, *and there exists a constant* $L > 0$ *so that if* $|x|, |y| \le L$, *then*

$$|g(x) - g(y)| \le |x - y|, \tag{2.24}$$

(ii) *there exists a constant* $\alpha \in (0,1)$ *such that for all* $t \geq 0$, *one has*

$$\int_{t-r_1}^{t} |a(s+r_1)|ds + \int_0^t e^{-\int_s^t a(u+r_1)du} |a(s+r_1)| \int_{s-r_1}^{s} |a(u+r_1)|du\,ds + \int_0^t e^{-\int_s^t a(u+r_1)du} |b(s)|ds \leq \alpha, \tag{2.25}$$

(iii)

$$\liminf_{t\to\infty} \int_0^t a(s+r_1)ds > -\infty. \tag{2.26}$$

Then, the zero solution of (1.1) *is asymptotically stable if and only if*

(iv)

$$\int_0^t a(s+r_1)ds \longrightarrow \infty, \quad \text{as } t \longrightarrow \infty. \tag{2.27}$$

Remark 2.3. We can also obtain the result that $x(t)$ is bounded by L on $[-r,\infty)$. Our results generalize Theorems 1.1 and 1.2.

Theorem 2.4. *Suppose that a continuous function* $\widetilde{a} : [0,\infty) \to R$ *exists such that* $a(t) = \widetilde{a}(t)(1 - r_1'(t))$ *and that the inverse function* $h(t)$ *of* $t - r_1(t)$ *exists. Suppose also that the following conditions are satisfied:*

(i) *there exists a constant* $J > 0$ *such that* $\sup_{t\geq 0}\{-\int_0^t \widetilde{a}(h(s))ds\} < J$,

(ii) *there exists a constant* $L > 0$ *such that* $f(x), x - f(x), g(x)$ *satisfy a Lipschitz condition with constant* $K > 0$ *on an interval* $[-L, L]$,

(iii) f *and* g *are odd, increasing on* $[0, L]$. $x - f(x)$ *is nondecreasing on* $[0, L]$,

(iv) *for each* $L_1 \in (0, L]$, *one has*

$$\left|L_1 - f(L_1)\right| \sup_{t\geq 0} \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} |\widetilde{a}(h(s))|ds + f(L_1) \sup_{t\geq 0} \int_{t-r_1(t)}^{t} |\widetilde{a}(h(s))|ds + g(L_1) \sup_{t\geq 0} \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} |b(s)|ds < L_1. \tag{2.28}$$

Then, the zero solution of (1.2) *is stable.*

Proof. By (iv), there exists $\alpha \in (0,1)$ such that

$$\begin{aligned} &\left|L - f(L)\right| \sup_{t\geq 0} \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} |\widetilde{a}(h(s))| ds + f(L) \sup_{t\geq 0} \int_{t-r_1(t)}^t |\widetilde{a}(h(s))| ds \\ &\quad + g(L) \sup_{t\geq 0} \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} |b(s)| ds \leq \alpha L. \end{aligned} \tag{2.29}$$

Let S be the space of all continuous functions $\phi : [-r,\infty) \to R$ such that

$$\left|\phi\right|_K := \sup\left\{ e^{-(dK+2)\int_0^t (|\widetilde{a}(h(s))|+|b(s)|)ds} \left|\phi(t)\right| \;:\; t \in [-r,\infty) \right\} < \infty, \tag{2.30}$$

where $d > 3$ is a constant. Then, $(S, |\cdot|_K)$ is a Banach space, which can be verified with Cauchy's criterion for uniform convergence.

The equation (1.2) can be transformed as

$$\begin{aligned} x'(t) &= -\widetilde{a}(h(t)) f(x(t)) + \frac{d}{dt} \int_{t-r_1(t)}^t \widetilde{a}(h(s)) f(x(s)) ds + b(t) g(x(t - r_2(t))) \\ &= -\widetilde{a}(h(t)) x(t) + \widetilde{a}(h(t)) \left[x(t) - f(x(t))\right] \\ &\quad + \frac{d}{dt} \int_{t-r_1(t)}^t \widetilde{a}(h(s)) f(x(s)) ds + b(t) g(x(t - r_2(t))). \end{aligned} \tag{2.31}$$

By the variation of parameters formula, we have

$$\begin{aligned} x(t) &= x_0 e^{-\int_0^t \widetilde{a}(h(s))ds} - e^{-\int_0^t \widetilde{a}(h(s))ds} \int_{-r_1(0)}^0 \widetilde{a}(h(s)) f(x(s)) ds + \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} \left[x(s) - f(x(s))\right] ds \\ &\quad + \int_{t-r_1(t)}^t \widetilde{a}(h(s)) f(x(s)) ds + \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du} b(s) g(x(s - r_2(s))) ds. \end{aligned} \tag{2.32}$$

Let

$$M = \left\{ \phi \mid \phi \in S,\ \sup_{t\geq -r} \left|\phi(t)\right| \leq L,\ \phi(t) = \psi(t),\ t \in [-r,0] \right\}, \tag{2.33}$$

then M is a complete metric space with metric $|\phi-\eta|_K$ for $\phi,\eta\in M$. For all $\phi\in M$, define the mapping P

$$\begin{aligned}
(P\phi)(t) &= \psi(t), \quad t\in[-r,0],\\
(P\phi)(t) &= \psi(0)e^{-\int_0^t \tilde{a}(h(s))ds} - e^{-\int_0^t \tilde{a}(h(s))ds}\int_{-r_1(0)}^{0}\tilde{a}(h(s))f(\psi(s))ds\\
&\quad+\int_0^t e^{-\int_s^t \tilde{a}(h(u))du}\left[\phi(s)-f(\phi(s))\right]ds\\
&\quad+\int_{t-r_1(t)}^{t}\tilde{a}(h(s))f(\phi(s))ds+\int_0^t e^{-\int_s^t \tilde{a}(h(u))du}b(s)g(\phi(s-r_2(s)))ds.
\end{aligned}\tag{2.34}$$

By (i), (iii), and (2.29), we have

$$\begin{aligned}
\left|(P\phi)(t)\right| &\le \|\psi\|e^J+e^J\|f(\psi)\|\int_{-r_1(0)}^{0}|\tilde{a}(h(s))|ds+\left|L-f(L)\right|\sup_{t\ge0}\int_0^t e^{-\int_{s-r_1(s)}^{t}\tilde{a}(h(u))du}\tilde{a}(h(s))ds\\
&\quad+f(L)\sup_{t\ge0}\int_{t-r_1(t)}^{t}|\tilde{a}(h(s))|ds+g(L)\sup_{t\ge0}\int_0^t e^{\int_s^t \tilde{a}(h(u))du}|b(s)|ds\\
&\le \|\psi\|e^J+e^J\|f(\psi)\|\int_{-r_1(0)}^{0}|\tilde{a}(h(s))|ds+\alpha L.
\end{aligned}\tag{2.35}$$

Thus, there exists $\delta\in(0,L)$ such that $e^J[1+K\delta\int_{-r_1(0)}^{0}|\tilde{a}(h(s))|ds]<(1-\alpha)L$ and $|(P\phi)(t)|\le L$. Hence, $P\phi\in M$.

We now show that P is a contraction mapping in M. For all $\phi,\eta\in M$,

$$\begin{aligned}
\left|(P\phi)(t)-(P\eta)(t)\right| &\le \int_0^t e^{-\int_s^t \tilde{a}(h(u))du}|\tilde{a}(h(s))|\left|\phi(s)-f(\phi(s))-\eta(s)+f(\eta(s))\right|ds\\
&\quad+\int_{t-r_1(t)}^{t}|\tilde{a}(h(s))|\left|f(\phi(s))-f(\eta(s))\right|ds\\
&\quad+\int_0^t e^{-\int_s^t \tilde{a}(h(u))du}|b(s)|\left|g(\phi(s-r_2(s)))-g(\eta(s-r_2(s)))\right|ds.
\end{aligned}\tag{2.36}$$

Since

$$
\begin{aligned}
&e^{-(dK+2)\int_0^t(|\widetilde{a}(h(s))|+|b(s)|)ds}\int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|\widetilde{a}(h(s))|\left|\phi(s)-f(\phi(s))-\eta(s)+f(\eta(s))\right|ds \\
&\quad\le \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|\widetilde{a}(h(s))|K\left|\phi(s)-\eta(s)\right|e^{-(dK+2)\int_0^s(|\widetilde{a}(h(u))|+|b(u)|)du} \\
&\quad\quad\times e^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds \\
&\quad\le \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|\widetilde{a}(h(s))|Ke^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds\left|\phi-\eta\right|_K \\
&\quad\le \frac{1}{d}\left|\phi-\eta\right|_K, \\
&e^{-(dK+2)\int_0^t(|\widetilde{a}(h(s))|+|b(s)|)ds}\int_{t-r_1(t)}^t |\widetilde{a}(h(s))|\left|f(\phi(s))-f(\eta(s))\right|ds \\
&\quad\le \int_{t-r_1(t)}^t |\widetilde{a}(h(s))|K\left|\phi(s)-\eta(s)\right|e^{-(dK+2)\int_0^s(|\widetilde{a}(h(u))|+|b(u)|)du}e^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds \\
&\quad\le \int_{t-r_1(t)}^t |\widetilde{a}(h(s))|Ke^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds\left|\phi-\eta\right|_K \\
&\quad\le \frac{1}{d}\left|\phi-\eta\right|_K, \\
&e^{-(dK+2)\int_0^t(|\widetilde{a}(h(s))|+|b(s)|)ds}\int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|b(s)|\left|g(\phi(s-r_2(s)))-g(\eta(s-r_2(s)))\right|ds \\
&\quad\le \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|b(s)|K\left|\phi(s-r_2(s))-\eta(s-r_2(s))\right|e^{-(dK+2)\int_0^{s-r_2(s)}(|\widetilde{a}(h(u))|+|b(u)|)du} \\
&e^{-(dK+2)\int_{s-r_2(s)}^s(|\widetilde{a}(h(u))|+|b(u)|)du}e^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds \\
&\quad\le \int_0^t e^{-\int_s^t \widetilde{a}(h(u))du}|b(s)|Ke^{-(dK+2)\int_s^t(|\widetilde{a}(h(u))|+|b(u)|)du}ds\left|\phi-\eta\right|_K \\
&\quad\le \frac{1}{d}\left|\phi-\eta\right|_K,
\end{aligned}
\tag{2.37}
$$

we have $e^{-(dK+2)\int_0^t(|\widetilde{a}(h(s))|+|b(s)|)ds}|(P\phi)(t)-(P\eta)(t)| \le (3/d)|\phi-\eta|_K$. That means $|P\phi-P\eta| \le (3/d)|\phi-\eta|_K$. Hence, P is a contraction mapping in M with constant $3/d$. By the contraction mapping principle, P has a unique fixed point x in M, which is a solution of (1.2) with $x(s)=\psi(s)$ on $[-r,0]$ and $\sup_{t\ge -r}|x(t)|\le L$.

In order to prove stability at $t=0$, let $\varepsilon>0$ be given. Then, choose $m>0$ so that $m<\min\{\varepsilon,L\}$. Replacing L with m in M, we see there is a $\delta>0$ such that $\|\psi\|<\delta$ implies that the unique continuous solution x agreeing with ψ on $[-r,0]$ satisfies $|x(t)|\le m<\varepsilon$ for all $t\ge -r$. This shows that the zero solution of (1.2) is stable. That completes the proof. □

When $r_1(t) = r_1$, a constant, we have the following.

Corollary 2.5. *Suppose that the following conditions are satisfied:*

(i) *there exists a constant* $J > 0$ *such that* $\sup_{t\geq 0}\{-\int_0^t a(s+r_1)ds\} < J$,

(ii) *there exists a constant* $L > 0$ *such that* $f(x), x - f(x), g(x)$ *satisfy a Lipschitz condition with constant* $K > 0$ *on an interval* $[-L, L]$,

(iii) f *and* g *are odd, increasing on* $[0, L]$. $x - f(x)$ *is nondecreasing on* $[0, L]$,

(iv) *for each* $L_1 \in (0, L]$, *one has*

$$\begin{aligned}&\left|L_1 - f(L_1)\right|\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u+r_1)du}|a(s+r_1)|ds + f(L_1)\sup_{t\geq 0}\int_{t-r_1}^t |a(u+r_1)|du\\&\quad + g(L_1)\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u+r_1)du}|b(s)|ds < L_1.\end{aligned} \tag{2.38}$$

Then, the zero solution of the equation

$$x'(t) = -a(t)f(x(t-r_1)) + b(t)g(x(t-r_2(t))) \tag{2.39}$$

is stable.

Corollary 2.6. *Suppose that the following conditions are satisfied:*

(i) *there exists a constant* $J > 0$ *such that* $\sup_{t\geq 0}\{-\int_0^t a(s)ds\} < J$,

(ii) *there exists a constant* $L > 0$ *such that* $f(x)$, $x - f(x)$, $g(x)$ *satisfy a Lipschitz condition with constant* $K > 0$ *on an interval* $[-L, L]$,

(iii) f *and* g *are odd, increasing on* $[0, L]$. $x - f(x)$ *is nondecreasing on* $[0, L]$,

(iv) *for each* $L_1 \in (0, L]$, *one has*

$$\left|L_1 - f(L_1)\right|\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u)du}|a(s)|ds + g(L_1)\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds < L_1. \tag{2.40}$$

Then, the zero solution of

$$x'(t) = -a(t)f(x(t)) + b(t)g(x(t-r(t))) \tag{2.41}$$

is stable.

Remark 2.7. The zero solution of (1.2) is not as asymptotically stable as that of (1.1). The key is that M is not complete under the weighted metric when added the condition to M that $\phi(t) \to 0$ as $t \to \infty$.

Remark 2.8. Theorem 2.4 makes use of the techniques of Theorems 1.3 and 1.4.

3. An Example

We use an example to illustrate our theory. Consider the following differential equation:

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)g(x(t - r_2(t))). \tag{3.1}$$

where $r_1(t) = 0.281t$, $r_2 \in C(R^+, R)$, $g(x) = x^3$, $a(t) = 1/(0.719t + 1)$, and $b(t) = \mu \sin t/(t + 1)$, $\mu > 0$. This equation comes from [4].

Choosing $h(t) = 1.2/(t + 1)$, we have

$$\begin{aligned}
&\int_{t-r_1(t)}^{t} |h(s)|ds = \int_{0.719t}^{t} \frac{1.2}{s+1} ds = 1.2 \ \ln \frac{t+1}{0.719t+1} < 0.396, \\
&\int_0^t e^{-\int_s^t h(u)du} \big| h(s - r_1(s))\big(1 - r_1'(s)\big) - a(s) \big| ds \\
&\textstyle\int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-r_1(s)}^{s} |h(u)| du\, ds < 0.396, \\
&= \textstyle\int_0^t e^{-\int_s^t (1.2/(u+1))du} \dfrac{1 - 1.2 \times 0.719}{0.719s + 1} ds \\
&< \dfrac{1 - 1.2 \times -0.719}{0.719s + 1} \textstyle\int_0^t e^{-\int_s^t (1.2/(u+1))du} \dfrac{1.2}{s+1} ds < 0.1592, \\
&\int_0^t e^{-\int_s^t h(u)du} |b(s)| ds \le \frac{\mu}{1.2}.
\end{aligned} \tag{3.2}$$

Let $\alpha := 0.396 + 0.396 + 0.1592 + \mu/1.2$, when μ is sufficiently small, $\alpha < 1$. Then, the condition (ii) of Theorem 2.1 is satisfied.

Let $L = \sqrt{3}/3$, then the condition (i) of Theorem 2.1 is satisfied.

And $\int_0^t h(s)ds = \int_0^t (1.2/(s+1))ds = 1.2 \ \ln(t + 1)$, then the condition (iii) and (iv) of Theorem 2.1 are satisfied.

According to Theorem 2.1, the zero solution of (3.1) is asymptotically stable.

References

[1] T. A. Burton, "Stability by fixed point theory or Liapunov theory: a comparison," *Fixed Point Theory*, vol. 4, no. 1, pp. 15–32, 2003.

[2] B. Zhang, "Fixed points and stability in differential equations with variable delays," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 63, no. 5–7, pp. e233–e242, 2005.

[3] L. C. Becker and T. A. Burton, "Stability, fixed points and inverses of delays," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 136, no. 2, pp. 245–275, 2006.

[4] C. Jin and J. Luo, "Stability in functional differential equations established using fixed point theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 11, pp. 3307–3315, 2008.

[5] T. A. Burton, "Stability and fixed points: addition of terms," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 459–477, 2004.

[6] B. Zhang, "Contraction mapping and stability in a delay-differential equation," in *Dynamic Systems and Applications*, vol. 4, pp. 183–190, Dynamic, Atlanta, Ga, USA, 2004.

[7] T. A. Burton, "Stability by fixed point methods for highly nonlinear delay equations," *Fixed Point Theory*, vol. 5, no. 1, pp. 3–20, 2004.

[8] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, NY, USA, 1993.