Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 240450, 11 pages
doi:10.1155/2010/240450

Research Article

Strong and Weak Convergence of the Modified Proximal Point Algorithms in Hilbert Space

Xinkuan Chai,[1] Bo Li,[2] and Yisheng Song[1]

[1] *College of Mathematics and Information Science, Henan Normal University, XinXiang 453007, China*
[2] *School of Mathematics and Statistics, AnYang Normal University, AnYang 455000, China*

Correspondence should be addressed to Yisheng Song, songyisheng123@yahoo.com.cn

Received 26 October 2009; Revised 25 November 2009; Accepted 10 December 2009

Academic Editor: Tomonari Suzuki

For a monotone operator T, we shall show weak convergence of Rockafellar's proximal point algorithm to some zero of T and strong convergence of the perturbed version of Rockafellar's to $P_Z u$ under some relaxed conditions, where P_Z is the metric projection from H onto $Z = T^{-1}0$. Moreover, our proof techniques are simpler than some existed results.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let I be on identity operator in H. We shall denote by $\mathbb{N}$ the set of all positive integers, by Z the set of all zeros of T, that is, $Z = T^{-1}0 = \{x \in D(T); 0 \in Tx\}$ and by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Let T be an operator with domain $D(T)$ and range $R(T)$ in H. Recall that T is said to be *monotone* if

$$\langle x - y, x' - y' \rangle \geq 0, \quad \forall x, y \in D(T),\ x' \in Tx,\ y' \in Ty. \tag{1.1}$$

A monotone operator T is said to be *maximal monotone* if T is monotone and $R(I + rT) = H$ for all $r > 0$.

In fact, theory of monotone operator is very important in nonlinear analysis and is connected with theory of differential equations. It is well known (see [1]) that many physically significant problems can be modeled by the initial-value problems of the form

$$x'(t) + Tx(t) = 0, \\ x(0) = x_0, \tag{1.2}$$

where T is a monotone operator in an appropriate space. Typical examples where such evolution equations occur can be found in the heat and wave equations or Schrodinger equations. On the other hand, a variety of problems, including convex programming and variational inequalities, can be formulated as finding a zero of monotone operators. Then the problem of finding a solution $x \in H$ with $0 \in Tx$ has been investigated by many researchers; see, for example, Bruck [2], Rockafellar [3], Brézis and Lions [4], Reich [5, 6], Nevanlinna and Reich [7], Bruck and Reich [8], Jung and Takahashi [9], Khang [10], Minty [11], Xu [12], and others. Some of them dealt with the weak convergence of (1.4) and others proved strong convergence theorems by imposing strong assumptions on T.

One popular method of solving $0 \in Tx$ is the proximal point algorithm of Rockafellar [3] which is recognized as a powerful and successful algorithm in finding a zero of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal point algorithm generates a sequence $\{x_k\}$ given by

$$x_{k+1} = J_{c_k}^T(x_k + e_k), \tag{1.3}$$

where $J_r^T = (I + rT)^{-1}$ for all $r > 0$ is the resolvent of T on the space H. Rockafellar's [3] proved the weak convergence of his algorithm (1.3) provided that the regularization sequence $\{c_k\}$ remains bounded away from zero and the error sequence $\{e_k\}$ satisfies the condition $\sum_{k=0}^{+\infty} \|e_k\| < \infty$. Güler's example [13] however shows that in an infinite-dimensional Hilbert space, Rochafellar's algorithm (1.3) has only weak convergence. Recently several authors proposed modifications of Rochafellar's proximal point algorithm (1.3) to have strong convergence. For examples, Solodov and Svaiter [14] and Kamimura and Takahashi [15] studied a modified proximal point algorithm by an additional projection at each step of iteration. Lehdili and Moudafi [16] obtained the convergence of the sequence $\{x_k\}$ generated by the algorithm

$$x_{k+1} = J_{\lambda_k}^{T_k} x_k, \quad k \geq 0, \tag{1.4}$$

where $T_k = \mu_k I + T$, $\mu_k > 0$, is viewed as a Tikhonov regularization of T. Using the technique of variational distance, Lehdili and Moudafi [16] were able to prove convergence theorems for the algorithm (1.4) and its perturbed version, under certain conditions imposed upon the sequences $\{\lambda_k\}$ and $\{\mu_k\}$. For a maximal monotone operator T, Xu [12] and Song and Yang [17] used the technique of nonexpansive mappings to get convergence theorems for $\{x_k\}$ defined by the perturbed version of the algorithm (1.4):

$$x_{k+1} = J_{r_k}^T(t_k u + (1 - t_k)x_k). \tag{1.5}$$

In this paper, under more relaxed conditions on the sequences $\{r_k\}$ and $\{t_k\}$, we shall show that the sequence $\{x_k\}$ generated by (1.5) converges strongly to $P_Z u \in T^{-1}0$ (where P_Z is the metric projection from H onto Z) and the sequence $\{x_k\}$ generated by (1.3) weakly converges to some $x^* \in T^{-1}0$. Moreover, our proof techniques are simpler than those of Lehdili and Moudafi [16], Xu [12], and Song and Yang [17].

2. Preliminaries and Basic Results

Let T be a monotone operator with $Z \neq \emptyset$. We use J_r^T and A_r to denote the resolvent and Yosida's approximation of T, respectively. Namely,

$$J_r^T = (I + rT)^{-1}, \qquad A_r = \frac{I - J_r^T}{r}, \quad r > 0. \tag{2.1}$$

For J_r^T and A_r, the following is well known. For more details, see [18, Pages 369–400] or [3, 19].

(i) $A_r x \in T J_r^T x$ for all $x \in R(I + rT)$;

(ii) $\|A_r x\| \leq |Tx| = \inf\{\|y\|; y \in Tx\}$ for all $x \in D(T) \cap R(I + rT)$;

(iii) $J_r^T : R(I + rT) \to D(I + rT) = D(T)$ is a single-valued nonexpansive mapping for each $r > 0$ (i.e., $\|J_r^T x - J_r^T y\| \leq \|x - y\|$ for all $x, y \in R(I + rT)$);

(iv) $Z = T^{-1}0 = F(J_r^T) = \{x \in D(J_r); J_r^T x = x\}$ is closed and convex;

(v) (The Resolvent Identity) For $r > 0$ and $t > 0$ and $x \in E$,

$$J_r^T x = J_t^T \left(\frac{t}{r} x + \left(1 - \frac{t}{r} \right) J_r^T x \right). \tag{2.2}$$

In the rest of this paper, it is always assumed that Z is nonempty so that the metric projection P_Z from H onto Z is well defined. It is known that P_Z is nonexpansive and characterized by the inequality: given $x \in H$ and $v \in Z$; then $v = P_Z x$ if and only if

$$\langle x - v, y - v \rangle \leq 0, \quad \forall y \in Z. \tag{2.3}$$

In order to facilitate our investigation in the next section we list a useful lemma.

Lemma 2.1 (see Xu [20, Lemma 2.5]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{k+1} \leq (1 - \lambda_k) a_k + \lambda_k \beta_k + \sigma_k, \quad \forall k \geq 0, \tag{2.4}$$

where $\{\lambda_k\}$, $\{\beta_k\}$, and $\{\sigma_k\}$ satisfy the conditions (i) $\sum_{k=0}^{\infty} \lambda_k = \infty$; (ii) either $\limsup_{k\to\infty} \beta_k \leq 0$ or $\sum_{k=0}^{\infty} |\lambda_k \beta_k| < \infty$; (iii) $\sigma_k \geq 0$ for all k and $\sum_{k=0}^{\infty} \sigma_k < \infty$. Then $\{a_k\}$ converges to zero as $k \to \infty$.

3. Strongly Convergence Theorems

Let T be a monotone operator on a Hilbert space H. Then J_r^T is a single-valued nonexpansive mapping from $R(I + rT)$ to $D(I + rT) = D(T) \cap D(I) = D(T)$. When K is a nonempty closed convex subset of H such that $\overline{D(T)} \subset K \subset R(I + rT)$ for all $r > 0$ (here $\overline{D(T)}$ is closure of $D(T)$), then we have $t_k u + (1 - t_k)x_k \in K \subset R(I + r_k T)$ for $u, x_k \in K$ and all $k \in \mathbb{N}$, and hence the following iteration is well defined

$$x_{k+1} = J_{r_k}^T (t_k u + (1 - t_k)x_k). \tag{3.1}$$

Next we will show strong convergence of $\{x_k\}$ defined by (3.1) to find a zero of T. For reaching this objective, we always assume $Z = T^{-1}0 \neq \emptyset$ in the sequel.

Theorem 3.1. *Let T be a monotone operator on a Hilbert space H with $Z = T^{-1}0 \neq \emptyset$. Assume that K is a nonempty closed convex subset of H such that $\overline{D(T)} \subset K \subset R(I + rT)$ for all $r > 0$ and for an anchor point $u \in K$ and an initial value $x_0 \in K$, $\{x_k\}$ is iteratively defined by* (3.1)*. If $\{t_k\} \subset (0,1)$ and $\{r_k\} \subset (0, +\infty)$ satisfy*

(i) $\lim_{k\to\infty} t_k = 0$;

(ii) $\sum_{k=0}^{+\infty} t_k = \infty$;

(iii) $\lim_{k\to\infty} r_k = \infty$,

then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.

Proof. The proof consists of the following steps:

Step 1. The sequence $\{x_k\}$ is bounded. Let $y_k = t_k u + (1 - t_k)x_k$, then $x_{k+1} = J_{r_k}^T y_k$ and for some $z \in T^{-1}0 = F(J_r^T)$, we have

$$\begin{aligned} \|x_{k+1} - z\| &= \left\| J_{r_k}^T y_k - z \right\| \le \|y_k - z\| = \|t_k u + (1 - t_k)x_k - z\| \\ &\le t_k \|u - z\| + (1 - t_k)\|x_k - z\| \\ &\le \max\{\|x_k - z\|, \|u - z\|\} \\ &\;\vdots \\ &\le \max\{\|x_0 - z\|, \|u - z\|\}. \end{aligned} \tag{3.2}$$

So, the sequences $\{x_k\}$, $\{y_k\}$, and $\{J_{r_k}^T y_k\}$ are bounded.

Step 2. $\lim_{k\to\infty}\|x_k - J_r^T x_k\| = 0$ for each $r > 0$. Since

$$\begin{aligned} \left\| x_{k+1} - J_r^T x_{k+1} \right\| &= \left\| J_{r_k}^T y_k - J_r^T J_{r_k}^T y_k \right\| = \left\| \left(I - J_r^T\right) J_{r_k}^T y_k \right\| \\ &= r \left\| A_r J_{r_k}^T y_k \right\| \le r \left| T J_{r_k}^T y_k \right| \le r \left\| A_{r_k} y_k \right\| \\ &= r \frac{\left\| y_k - J_{r_k}^T y_k \right\|}{r_k} \longrightarrow 0 \quad (k \longrightarrow \infty), \end{aligned} \tag{3.3}$$

we have

$$\lim_{k\to\infty}\left\|x_k - J_r^T x_k\right\| = 0. \tag{3.4}$$

Step 3. $\limsup_{k\to\infty}\langle u - P_Z u, x_k - P_Z u\rangle \le 0$. Indeed, we can take a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\langle u - P_S u, x_k - P_S u\rangle = \lim_{i\to\infty}\langle u - P_S u, x_{k_i} - P_S u\rangle. \tag{3.5}$$

We may assume that $x_{k_i} \rightharpoonup x^*$ by the reflexivity of H and the boundedness of $\{x_k\}$. Then $x^* \in Z = T^{-1}0 = F(J_r^T)$. In fact, since

$$\begin{aligned}
\left\|x_{k_i} - J_r^T x^*\right\|^2 &= \left\|x_{k_i} - x^* + x^* - J_r^T x^*\right\|^2 \\
&= \|x_{k_i} - x^*\|^2 + 2\left\langle x_{k_i} - x^*, x^* - J_r^T x^*\right\rangle + \left\|x^* - J_r^T x^*\right\|^2, \\
\left\|x_{k_i} - J_r^T x^*\right\| &= \left\|x_{k_i} - J_r^T x_{k_i} + J_r^T x_{k_i} - J_r^T x^*\right\| \\
&\le \left\|x_{k_i} - J_r^T x_{k_i}\right\| + \left\|J_r^T x_{k_i} - J_r^T x^*\right\| \\
&\le \left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|,
\end{aligned} \tag{3.6}$$

then, for some constant $L > 0$, we have

$$\begin{aligned}
&\|x_{k_i} - x^*\|^2 + 2\left\langle x_{k_i} - x^*, x^* - J_r^T x^*\right\rangle + \left\|x^* - J_r^T x^*\right\|^2 \\
&\quad = \left\|x_{k_i} - J_r^T x^*\right\|^2 \le \left(\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|\right)^2 \\
&\quad = \left(\left\|x_{k_i} - J_r^T x_{k_i}\right\| + 2\|x_{k_i} - x^*\|\right)\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|^2 \le L\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|^2.
\end{aligned} \tag{3.7}$$

Thus,

$$2\langle x_{k_i} - x^*, x^* - J_r^T x^*\rangle + \left\|x^* - J_r^T x^*\right\|^2 \le L\left\|x_{k_i} - J_r^T x_{k_i}\right\|. \tag{3.8}$$

Take $i \to \infty$ on two sides of the above equation by means of (3.4), we must have $\|x^* - J_r^T x^*\|^2 = 0$. So, $x^* \in Z$. Hence, noting the projection inequality (2.3), we obtain

$$\limsup_{k\to\infty}\langle u - P_Z u, x_k - P_Z u\rangle = \lim_{i\to\infty}\langle u - P_Z u, x_{k_i} - P_Z u\rangle = \langle u - P_Z u, x^* - P_Z u\rangle \le 0. \tag{3.9}$$

Step 4. $x_k \to P_Z u$. Indeed,

$$\begin{aligned}\|x_{k+1} - P_Z u\|^2 &= \left\| J_{r_k}^T (t_k u + (1-t_k)x_k) - P_Z u \right\|^2 \\ &= \left\| J_{r_k}^T y_k - P_Z u \right\|^2 \le \left\| y_k - P_Z u \right\|^2 \\ &\le \|t_k(u - P_Z u) + (1-t_k)(x_k - P_Z u)\|^2 \\ &\le (1-t_k)^2 \|x_k - P_Z u\|^2 + t_k^2 \|u - P_Z u\|^2 + 2t_k(1-t_k)\langle u - P_Z u, x_k - P_Z u\rangle. \end{aligned} \tag{3.10}$$

Therefore,

$$\|x_{k+1} - P_Z u\|^2 \le (1-t_k)\|x_k - P_Z u\|^2 + t_k \beta_k, \tag{3.11}$$

where $\beta_k = t_k \|u - P_Z u\|^2 + 2(1-t_k)\langle u - P_Z u, x_k - P_Z u\rangle$. So, an application of Lemma 2.1 onto (3.11) yields the desired result. □

Theorem 3.2. *Let $T, H, Z, K, \{x_k\}, \{t_k\}$ be as Theorem 3.1, the condition (iii) $\lim_{k\to\infty} r_k = \infty$ is replaced by the following condition:*

$$\sum_{k=0}^{+\infty} |t_{k+1} - t_k| < \infty; \qquad 0 < \liminf_{k\to\infty} r_k, \qquad \sum_{k=0}^{\infty} \left| 1 - \frac{r_k}{r_{k+1}} \right| < +\infty. \tag{3.12}$$

Then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.

Proof. From the proof of Theorem 3.1, we can observe that Steps 1, 3 and 4 still hold. So we only need show to Step 2: $\lim_{k\to\infty}\|x_k - J_r^T x_k\| = 0$ for each $r > 0$.

We first estimate $\|x_{k+1} - x_k\|$. From the resolvent identity (2.2), we have

$$J_{r_k}^T y_k = J_{r_{k-1}}^T \left(\frac{r_{k-1}}{r_k} y_k + \left(1 - \frac{r_{k-1}}{r_k}\right) J_{r_k}^T y_k \right). \tag{3.13}$$

Therefore, for a constant $M > 0$ with $M \ge \max\{\|u\|, \|x_k\|, \|J_{r_k}^T y_k\|, \|y_k\|\}$,

$$\begin{aligned}\|x_{k+1} - x_k\| &= \left\| J_{r_k}^T y_k - J_{r_{k-1}}^T y_{k-1} \right\| \le \left\| \frac{r_{k-1}}{r_k} y_k + \left(1 - \frac{r_{k-1}}{r_k}\right) J_{r_k}^T y_k - y_{k-1} \right\| \\ &\le \left\| \frac{r_{k-1}}{r_k}(y_k - y_{k-1}) + \left(1 - \frac{r_{k-1}}{r_k}\right)\left(J_{r_k}^T y_k - y_{k-1}\right) \right\| \end{aligned}$$

$$
\begin{aligned}
&\le \left\|y_k - y_{k-1}\right\| + \left|1 - \frac{r_{k-1}}{r_k}\right| \left\|J_{r_k}^T y_k - y_k\right\| \\
&\le |t_k - t_{k-1}|(\|u\| + \|x_{k-1}\|) + (1 - t_k)\|x_k - x_{k-1}\| + 2M\left|1 - \frac{r_{k-1}}{r_k}\right| \\
&\le (1 - t_k)\|x_k - x_{k-1}\| + 2M\left(|t_k - t_{k-1}| + \left|1 - \frac{r_{k-1}}{r_k}\right|\right).
\end{aligned}
\tag{3.14}
$$

It follows from Lemma 2.1 that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.15}$$

As $\|y_k - J_{r_k}^T y_k\| = \|y_k - x_{k+1}\| \le t_k\|u - x_{k+1}\| + (1 - t_k)\|x_k - x_{k+1}\|$, then

$$\lim_{k\to\infty} \left\|y_k - J_{r_k}^T y_k\right\| = 0. \tag{3.16}$$

Since $0 < \liminf_{k\to\infty} r_k$, then there exists $\varepsilon > 0$ and a positive integer $N > 0$ such that for all $k > N$, $r_k \ge \varepsilon$. Thus for each $r > 0$, we also have

$$
\begin{aligned}
\left\|x_{k+1} - J_r^T x_{k+1}\right\| &= \left\|J_{r_k}^T y_k - J_r^T J_{r_k}^T y_k\right\| = \left\|\left(I - J_r^T\right) J_{r_k}^T y_k\right\| \\
&= r\left\|A_r J_{r_k}^T y_k\right\| \le r\left|T J_{r_k}^T y_k\right| \le r\left\|A_{r_k} y_k\right\| \\
&= r\frac{\left\|y_k - J_{r_k}^T y_k\right\|}{r_k} \le \frac{r}{\varepsilon}\left\|y_k - J_{r_k}^T y_k\right\| \longrightarrow 0 \quad (k \longrightarrow \infty);
\end{aligned}
\tag{3.17}
$$

we have $\lim_{k\to\infty}\|x_k - J_r^T x_k\| = 0$. □

Corollary 3.3. *Let $H, \{t_k\}, \{r_k\}, Z$ be as Theorem 3.1 or 3.2. Suppose that T is a maximal monotone operator on H and for $x_0, u \in H$, $\{x_k\}$ is defined by* (3.1)*. Then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.*

Proof. Since T is a maximal monotone, then T is monotone and satisfies the condition $\overline{D(T)} \subset H = R(I + rT)$ for all $r > 0$. Putting $K = H$, the desired result is reached. □

Corollary 3.4. *Let $H, \{t_k\}, \{r_k\}, Z$ be as Theorem 3.1 or 3.2. Suppose that T is a monotone operator on H satisfying the condition $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and for $x_0, u \in \overline{D(T)}$, $\{x_k\}$ is defined by* (3.1)*. If $D(T)$ is convex, then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.*

Proof. Taking $K = \overline{D(T)}$, following Theorem 3.1 or 3.2, we easily obtain the desired result. □

4. Weakly Convergence Theorems

For a monotone operator T, if $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and $x_0 \in \overline{D(T)}$, then the iteration $x_{k+1} = J_{r_k}^T x_k$ $(k \in \mathbb{N})$ is well defined. Next we will show weak convergence of $\{x_k\}$ under some assumptions.

Theorem 4.1. *Let T be a monotone operator on a Hilbert space H with $Z = T^{-1}0 \neq \emptyset$. Assume that $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and for an initial value $x_0 \in \overline{D(T)}$, iteratively define*

$$x_{k+1} = J_{r_k}^T x_k. \tag{4.1}$$

If $\{r_k\} \subset (0, +\infty)$ satisfies

$$\lim_{k \to \infty} r_k = \infty, \tag{4.2}$$

then the sequence $\{x_k\}$ converges weakly to some $x^* \in Z$.

Proof. Take $z \in Z = T^{-1}0 = F(J_r^T)$, we have

$$\|x_{k+1} - z\| = \left\| J_{r_k}^T x_k - z \right\| \leq \|x_k - z\|. \tag{4.3}$$

Therefore, $\{\|x_k - z\|\}$ is nonincreasing and bounded below, and hence the limit $\lim_{k \to \infty} \|x_k - z\|$ exists for each $z \in Z$. Further, $\{x_k\}$ is bounded. So we have

$$\begin{aligned} \left\| x_{k+1} - J_r^T x_{k+1} \right\| &= \left\| J_{r_k}^T x_k - J_r^T J_{r_k}^T x_k \right\| = \left\| \left(I - J_r^T \right) J_{r_k}^T x_k \right\| \\ &= r \left\| A_r J_{r_k}^T x_k \right\| \leq r \left| T J_{r_k}^T x_k \right| \leq r \|A_{r_k} x_k\| \\ &= r \frac{\left\| x_k - J_{r_k}^T x_k \right\|}{r_k} = \frac{r \|x_k - x_{k+1}\|}{r_k} \longrightarrow 0 \quad (k \longrightarrow \infty). \end{aligned} \tag{4.4}$$

Hence,

$$\lim_{k \to \infty} \left\| x_k - J_r^T x_k \right\| = 0. \tag{4.5}$$

As $\{x_k\}$ is weakly sequentially compact by the reflexivity of H, and hence we may assume that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup x^*$. Using the proof technique of Step 3 in Theorem 3.1, we must have that $x^* \in Z = T^{-1}0$.

Now we prove that $\{x_n\}$ converges weakly to x^*. Supposed that there exists another subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which weakly converges to some $y \in K$. We also have $y \in Z = T^{-1}0$. Because $\lim_{k \to \infty} \|x_k - z\|$ exists for each $z \in Z = T^{-1}0$ and

$$\begin{aligned} \left\| x_{k_j} - y \right\|^2 &= \left\| x_{k_j} - x^* \right\|^2 + 2 \left\langle x_{k_j} - x^*, x^* - y \right\rangle + \|x^* - y\|^2, \\ \|x_{k_i} - x^*\|^2 &= \left\| x_{k_i} - y \right\|^2 + 2 \langle x_{k_i} - y, y - x^* \rangle + \|y - x^*\|^2, \end{aligned} \tag{4.6}$$

thus,

$$\begin{aligned} \lim_{k\to\infty}\|x_k-y\|^2 &= \limsup_{j\to\infty}\left\|x_{k_j}-y\right\|^2 \\ &= \limsup_{j\to\infty}\left(\left\|x_{k_j}-x^*\right\|^2+2\left\langle x_{k_j}-x^*,x^*-y\right\rangle+\|x^*-y\|^2\right) \\ &\le \lim_{k\to\infty}\|x_k-x^*\|^2-\|x^*-y\|^2. \end{aligned} \tag{4.7}$$

Similarly, we also have

$$\lim_{k\to\infty}\|x_k-x^*\|^2 \le \lim_{k\to\infty}\|x_k-y\|^2-\|x^*-y\|^2. \tag{4.8}$$

Adding up the above two equations, we must have $-\|x^*-y\|^2 \ge 0$. So, $x^*=y$.

In a summary, we have proved that the set $\{x_k\}$ is weakly sequentially compact and each cluster point in the weak topology equals to $x^*\in Z$. Hence, $\{x_k\}$ converges weakly to $x^*\in T^{-1}0$. The proof is complete. □

Theorem 4.2. *Let T be a maximal monotone operator on a Hilbert space H with $Z=T^{-1}0\neq\emptyset$. For an initial value $x_0\in H$, iteratively define*

$$x_{k+1}=J_{r_k}^T(x_k+e_k). \tag{4.9}$$

If $\{r_k\}\subset(0,+\infty)$ and $e_k\in H$ satisfy

$$\lim_{k\to\infty}r_k=\infty,\quad \sum_{k=0}^{+\infty}\|e_k\|<+\infty, \tag{4.10}$$

then the sequence $\{x_k\}$ converges weakly to some $x^*\in Z$.

Proof. Take $z\in Z=T^{-1}0=F(J_r^T)$ and $y_k=x_k+e_k$, we have

$$\|x_{k+1}-z\|=\left\|J_{r_k}^Ty_k-z\right\|\le\|x_k-z\|+\|e_k\|. \tag{4.11}$$

It follows from Liu [21, Lemma 2] that the limit $\lim_{k\to\infty}\|x_k-z\|$ exists for each $z\in Z$ and hence both $\{x_k\}$ and $\{y_k\}$ are bounded. So we have

$$\begin{aligned} \left\|x_{k+1}-J_r^Tx_{k+1}\right\| &= \left\|J_{r_k}^Ty_k-J_r^TJ_{r_k}^Ty_k\right\|=\left\|\left(I-J_r^T\right)J_{r_k}^Ty_k\right\| \\ &= r\left\|A_rJ_{r_k}^Ty_k\right\|\le r\left|TJ_{r_k}^Ty_k\right|\le r\|A_{r_k}y_k\| \\ &= r\frac{\left\|y_k-J_{r_k}^Ty_k\right\|}{r_k}=\frac{r\|y_k-x_{k+1}\|}{r_k}\longrightarrow 0\quad (k\longrightarrow\infty). \end{aligned} \tag{4.12}$$

Hence,

$$\lim_{k\to\infty}\left\|x_k - J_r^T x_k\right\| = 0. \tag{4.13}$$

The remainder of the proof is the same as Theorem 4.1; we omit it. □

Acknowledgments

The authors are grateful to the anonymous referee for his/her valuable suggestions which helps to improve this manuscript. This work is supported by Youth Science Foundation of Henan Normal University(2008qk02) and by Natural Science Research Projects (Basic Research Project) of Education Department of Henan Province (2009B110011, 2009B110001).

References

[1] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, Part II: Monotone Operators*, Springer, Berlin, Germany, 1985.
[2] R. E. Bruck Jr., "A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 48, pp. 114–126, 1974.
[3] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.
[4] H. Brézis and P.-L. Lions, "Produits infinis de résolvantes," *Israel Journal of Mathematics*, vol. 29, no. 4, pp. 329–345, 1978.
[5] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.
[6] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.
[7] O. Nevanlinna and S. Reich, "Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces," *Israel Journal of Mathematics*, vol. 32, no. 1, pp. 44–58, 1979.
[8] R. E. Bruck and S. Reich, "A general convergence principle in nonlinear functional analysis," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 939–950, 1980.
[9] J. S. Jung and W. Takahashi, "Dual convergence theorems for the infinite products of resolvents in Banach spaces," *Kodai Mathematical Journal*, vol. 14, no. 3, pp. 358–365, 1991.
[10] D. B. Khang, "On a class of accretive operators," *Analysis*, vol. 10, no. 1, pp. 1–16, 1990.
[11] G. J. Minty, "On the monotonicity of the gradient of a convex function," *Pacific Journal of Mathematics*, vol. 14, pp. 243–247, 1964.
[12] H.-K. Xu, "A regularization method for the proximal point algorithm," *Journal of Global Optimization*, vol. 36, no. 1, pp. 115–125, 2006.
[13] O. Güler, "On the convergence of the proximal point algorithm for convex minimization," *SIAM Journal on Control and Optimization*, vol. 29, no. 2, pp. 403–419, 1991.
[14] M. V. Solodov and B. F. Svaiter, "Forcing strong convergence of proximal point iterations in a Hilbert space," *Mathematical Programming. Series A*, vol. 87, no. 1, pp. 189–202, 2000.
[15] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.
[16] N. Lehdili and A. Moudafi, "Combining the proximal algorithm and Tikhonov regularization," *Optimization*, vol. 37, no. 3, pp. 239–252, 1996.
[17] Y. Song and C. Yang, "A note on a paper "A regularization method for the proximal point algorithm"," *Journal of Global Optimization*, vol. 43, no. 1, pp. 171–174, 2009.
[18] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons, New York, NY, USA, 1984.
[19] W. Takahashi, *Nonlinear Functional Analysis—Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.

[20] H.-K. Xu, "Strong convergence of an iterative method for nonexpansive and accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 631–643, 2006.

[21] Q. Liu, "Iterative sequences for asymptotically quasi-nonexpansive mappings with error member," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 18–24, 2001.