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Research Article

A New General Iterative Method for a Finite Family of Nonexpansive Mappings in Hilbert Spaces

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We introduce a new general iterative method by using the K-mapping for finding a common fixed point of a finite family of nonexpansive mappings in the framework of Hilbert spaces. A strong convergence theorem of the purposed iterative method is established under some certain control conditions. Our results improve and extend the results announced by many others.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T provided that $Tx = x$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Recall that a self-mapping $f : C \to C$ is a contraction on C, if there exists a constant $\alpha \in (0, 1)$ such that $\|fx - fy\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

In 1953, Mann [1] introduced a well-known classical iteration to approximate a fixed point of a nonexpansive mapping. This iteration is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily, and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$. But Mann's iteration process has only weak convergence, even in a Hilbert space setting. In general for example, Reich [2] showed that if E is a uniformly convex Banach space and has a Frehet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to a point in $F(T)$. Therefore, many authors try to modify Mann's iteration process to have strong convergence.

In 2005, Kim and Xu [3] introduced the following iteration process:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n. \end{aligned} \tag{1.3}$$

They proved in a uniformly smooth Banach space that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

In 2008, Yao et al. [4] alsomodified Mann's iterative scheme 1.2 to get a strong convergence theorem.

Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings with $F := \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$. There are many authors introduced iterative method for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \bmod N)}$ with the mod function taking values in $\{1,2,\ldots,N\}$. Let u be a fixed element of H.

In 2003, Xu [5] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1-\epsilon_n A)T_{n+1}x_n + \epsilon_{n+1}u \tag{1.4}$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x\in F}\frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.5}$$

under suitable hypotheses on ϵ_n and under the additional hypothesis

$$F = F(T_1T_2\cdots T_N) = F(T_NT_1\cdots T_{N-1}) = \cdots = F(T_2T_3\cdots T_NT_1). \tag{1.6}$$

In 1999, Atsushiba and Takahashi [6] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I, \\ U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})I, \\ U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})I, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1}T_N - 1U_{n,N-2} + (1-\gamma_{n,N-1})I, \\ W_n = U_{n,N} &= \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})I, \end{aligned} \tag{1.7}$$

where $\{\gamma_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$.

In 2000, Takahashi and Shimoji [7] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \ldots, N$.

In 2007,Shang et al. [8] introduced a composite iteration scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen}, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \end{aligned} \tag{1.8}$$

where $f \in \prod_C$ is a contraction, and A is a linear bounded operator.

Note that the iterative scheme (1.8) is not well-defined, because $x_n (n \geq 1)$ may not lie in C, so $W_n x_n$ is not defined. However, if $C = H$, the iterative scheme (1.8) is well-defined and Theorem 2.1 [8] is obtained. In the case $C \neq H$, we have to modify the iterative scheme (1.8) in order to make it well-defined.

In 2009, Kangtunyakarn and Suantai [9] introduced a new mapping, called K-mapping, for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\gamma_{n,i}\}_i^N$ in $[0,1]$, the mapping $K_n : C \to C$ is defined as follows:

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1-\gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1-\gamma_{n,2}) U_{n,1}, \\ U_{n,3} &= \gamma_{n,3} T_3 U_{n,2} + (1-\gamma_{n,3}) U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_N - 1 U_{n,N-2} + (1-\gamma_{n,N-1}) U_{n,N-2}, \\ K_n = U_{n,N} &= \gamma_{n,N} T_N U_{n,N-1} + (1-\gamma_{n,N}) U_{n,N-1}. \end{aligned} \tag{1.9}$$

The mapping K_n is called the *K-mapping* generated by $T_1, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$.

In this paper, motivated by Kim and Xu [3], Marino and Xu [10], Xu [5], Yao et al. [4], andShang et al. [8], we introduce a composite iterative scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen}, \\ y_n &= \beta_n x_n + (1-\beta_n) K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n), \end{aligned} \tag{1.10}$$

where $f \in \prod_C$ is a contraction, and A is a bounded linear operator. We prove, under certain appropriate conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of the finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, which solves a variational inequaility problem.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad x, y \in H. \tag{1.11}$$

Lemma 1.2 (see [11]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n \tag{1.12}$$

for all integer $n \ge 0$, and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{1.13}$$

Then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 1.3 (see [5]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\gamma_n)a_n + \delta_n$ $n \ge 0$, where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 1.4 (see [10]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 1.5 (see [10]). *Let H be a Hilbert space. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1-tA)Tx$. Then x_t converges strongly as $t \to 0$ to a fixed point $\overline{x}$ of T, which solves the variational inequality*

$$\langle (A - \gamma f)\overline{x}, z - \overline{x}\rangle \ge 0, \quad z \in F(T). \tag{1.14}$$

Lemma 1.6 (see [1]). *Demiclosedness principle. Assume that T is nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here, I is identity mapping of H.*

Lemma 1.7 (see [9]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \le 1$. Let K be the K-mapping of C into itself generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

By using the same argument as in [9, Lemma 2.10], we obtain the following lemma.

Lemma 1.8. *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1,2,\ldots,N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$, and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every bounded sequence $x_n \in C$, one has $\lim_{n\to\infty}\|K_n x_n - K x_n\| = 0$.*

Let H be real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, C a nonempty closed convex subset of H. Recall that the metric (nearest point) projection P_C from a real Hilbert space H to a closed convex subset C of H is defined as follows. Given that $x \in H$, $P_C x$ is the only point in C with the property $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. Below Lemma 1.9 can be found in any standard functional analysis book.

Lemma 1.9. *Let C be a closed convex subset of a real Hilbert space H. Given that $x \in H$ and $y \in C$ then*

(i) *$y = P_C x$ if and only if the inequality $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$,*

(ii) *P_C is nonexpansive,*

(iii) *$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in H$,*

(iv) *$\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x \in H$ and $y \in C$.*

2. Main Result

In this section, we prove strong convergence of the sequences $\{x_n\}$ defined by the iteration scheme (1.10).

Theorem 2.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$, and let $f \in \prod_C$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $0 < \gamma < \overline{\gamma}/\alpha$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, *for all* $i = 1,2,\ldots,N$ *and* $\{\gamma_{n,i}\}_{i=1}^N \subset [a,b]$, *where* $0 < a \leq b < 1$;

(C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by (1.10), then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0, \quad p \in F. \tag{2.1}$$

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a point $u \in F$, and notice that

$$\|y_n - u\| \leq \beta_n\|x_n - u\| + (1-\beta_n)\|K_n x_n - u\| \leq \|x_n - u\|. \tag{2.2}$$

Since $\alpha_n \to 0$, we may assume that $\alpha_n \leq \|A^{-1}\|$ for all n. By Lemma 1.4, we have $\|I - \alpha_n A\| \leq 1 - \alpha_n\overline{\gamma}$ for all n.

It follows that

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|P_C(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n) - P_C(u)\right\| \\
&\leq \left\|\alpha_n(\gamma f(x_n) - Au) + (I-\alpha_n A)(y_n - u)\right\| \\
&\leq \alpha_n\left\|\gamma f(x_n) - Au\right\| + (1-\alpha_n\overline{\gamma})\left\|y_n - u\right\| \\
&\leq \alpha_n\left\|\gamma f(x_n) - \gamma f(u)\right\| + \alpha_n\left\|\gamma f(u) - Au\right\| + (1-\alpha_n\overline{\gamma})\left\|y_n - u\right\| \\
&\leq \alpha\gamma\alpha_n\|x_n - u\| + \alpha_n\left\|\gamma f(u) - Au\right\| + (1-\alpha_n\overline{\gamma})\|x_n - u\| \\
&= (1-(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n - u\| + \alpha_n\left\|\gamma f(u) - Au\right\| \\
&= (1-(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n - u\| + (\overline{\gamma}-\gamma\alpha)\alpha_n\frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma}-\gamma\alpha} \\
&\leq \max\left\{\|x_n - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma}-\gamma\alpha}\right\}.
\end{aligned} \tag{2.3}$$

By simple inductions, we have

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma}-\gamma\alpha}\right\}, \quad n \geq 0. \tag{2.4}$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{f(x_n)\}$. Since K_n is nonexpansive and $y_n = \beta_n x_n + (1-\beta_n)K_n x_n$, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \left\|(\beta_{n+1}x_{n+1} + (1-\beta_{n+1})K_{n+1}x_{n+1}) - (\beta_n x_n + (1-\beta_n)K_n x_n)\right\| \\
&= \left\|\beta_{n+1}x_{n+1} - \beta_{n+1}x_n + \beta_{n+1}x_n - \beta_n x_n + (1-\beta_{n+1})(K_{n+1}x_{n+1} - K_{n+1}x_n)\right. \\
&\quad \left. +(1-\beta_{n+1})(K_{n+1}x_n - K_n x_n) + (1-\beta_{n+1})K_n x_n - (1-\beta_n)K_n x_n\right\| \\
&\leq \beta_{n+1}\|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| + (1-\beta_{n+1})\|K_{n+1}x_{n+1} - K_{n+1}x_n\| \\
&\quad + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\| + \left|\beta_n - \beta_{n+1}\right|\|K_n x_n\| \\
&\leq \beta_{n+1}\|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| + (1-\beta_{n+1})\|x_{n+1} - x_n\| \\
&\quad + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\| + \left|\beta_n - \beta_{n+1}\right|\|K_n x_n\| \\
&= \|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| \ + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\|\left|\beta_n - \beta_{n+1}\right|\|K_n x_n\|.
\end{aligned} \tag{2.5}$$

By using the inequalities (2.6) and (2.11) of [9, Lemma 2.11], we can conclude that

$$\|K_n x_{n-1} - K_{n-1}x_{n-1}\| \le M\sum_{j=1}^{N}\left|\gamma_{n,j} - \gamma_{n-1,j}\right|, \tag{2.6}$$

where $M = \sup\{\sum_{j=2}^{N}(\|T_jU_{n,j-1}x_n\| + \|U_{n,j-1}x_n\|) + \|T_1x_n\| + \|x_n\|\}$.

By (2.5) and (2.6), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|\left(P_C\left(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n\right)\right) - \left(P_C\left(\alpha_{n-1}\gamma f(x_{n-1}) + (I-\alpha_{n-1}A)y_{n-1}\right)\right)\right\| \\
&\le \left\|(I-\alpha_n A)\left(y_n - y_{n-1}\right) - (\alpha_n - \alpha_{n-1})Ay_{n-1}\right. \\
&\quad \left. +\gamma\alpha_n\left(f(x_n) - f(x_{n-1})\right) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1})\right\| \\
&\le (1-\alpha_n\overline{\gamma})\left\|y_n - y_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\| \\
&\quad + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\le (1-\alpha_n\overline{\gamma})\left[\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|\|x_{n-1}\|\right. \\
&\qquad\qquad \left. +\left|1-\beta_n\right|\|K_nx_{n-1} - K_{n-1}x_{n-1}\| + \left|\beta_{n-1} - \beta_n\right|\|K_{n-1}x_{n-1}\|\right] \\
&\quad + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\| + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\le (1-\alpha_n\overline{\gamma})\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|\|x_{n-1}\| \\
&\quad + \left|1-\beta_n\right|\|K_nx_{n-1} - K_{n-1}x_{n-1}\| + \left|\beta_{n-1} - \beta_n\right|\|K_{n-1}x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&= \left(1 - (\overline{\gamma} - \gamma\alpha)\alpha_n\right)\|x_n - x_{n-1}\| + L\left|\beta_{n-1} - \beta_n\right| + M'|\alpha_n - \alpha_{n-1}| \\
&\quad + \left|1-\beta_n\right|M\sum_{j=1}^{N}\left|\gamma_{n,j} - \gamma_{n-1,j}\right|,
\end{aligned} \tag{2.7}$$

where $L = \sup\{\|x_{n-1}\|+\|K_{n-1}x_{n-1}\| : n \in \mathbb{N}\}$, $M' = \max\{\|Ay_{n-1}\|+\gamma\|f(x_{n-1})\|\}$. Since $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n-1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty}|\gamma_{n,j} - \gamma_{n-1,j}| < \infty$, for all $j = 1, 2, \ldots, N$, by Lemma 1.3, we obtain $\|x_{n+1} - x_n\| \to 0$. It follows that

$$\begin{aligned}
\left\|x_{n+1} - y_n\right\| &= \left\|P_C\left(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n\right) - P_C\left(y_n\right)\right\| \\
&\le \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n - y_n\right\| \\
&= \alpha_n\left\|\gamma f(x_n) + Ay_n\right\|.
\end{aligned} \tag{2.8}$$

Since $\alpha_n \to 0$ and $\{f(x_n)\}$, $\{Ay_n\}$ are bounded, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Since

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \tag{2.9}$$

it implies that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

On the other hand, we have

$$\|K_n x_n - x_n\| \le \|x_n - y_n\| + \|y_n - K_n x_n\| = \|x_n - y_n\| + \beta_n \|x_n - K_n x_n\|, \tag{2.10}$$

which implies that $(1 - \beta_n)\|K_n x_n - x_n\| \le \|x_n - y_n\|$.

From condition (C3) and $\|x_n - y_n\| \to 0$ as $n \to \infty$, we obtain

$$\|K_n x_n - x_n\| \to 0. \tag{2.11}$$

By (C4), we have $\lim_{n\to\infty} \gamma_{n,i} = \gamma_i \in [a,b]$ for all $i = 1,2,\dots,N$. Let K be the K-mapping generated by $T_1,\dots,T_N$ and $\gamma_1,\dots,\gamma_N$. Next, we show that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \le 0, \tag{2.12}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Kx$. Thus, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Kx_t$. By Lemma 1.5 and Lemma 1.7, we have $q \in F$ and $\langle \gamma f(q) - Aq,\ p - q\rangle \ge 0$ for all $p \in F$. It follows by (2.11) and Lemma 1.8 that $\|Kx_n - x_n\| \to 0$. Thus, we have $\|x_t - x_n\| = \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 1.1 that for $0 < t < \|A\|^{-1}$,

$$\begin{aligned}
\|x_t - x_n\|^2 &= \left\|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\right\|^2 \\
&\le (1 - \overline{\gamma} t)^2 \|Kx_t - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n\rangle \\
&\le (1 - \overline{\gamma} t)^2 \left(\|Kx_t - Kx_n\|^2 + 2\|Kx_t - K_n x_n\|\|Kx_n - x_n\| + \|Kx_n - x_n\|^2\right) \\
&\quad + 2t(\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + \langle Ax_t - Ax_n, x_t - x_n\rangle) \\
&\le \left(1 - 2\overline{\gamma} t + (\overline{\gamma} t)^2\right) \|x_t - x_n\|^2 + f_n(t) + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle \\
&\quad + 2t\langle Ax_t - Ax_n, x_t - x_n\rangle,
\end{aligned} \tag{2.13}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Kx_n\|)\|x_n - Kx_n\| \longrightarrow 0, \quad \text{as } n \to 0. \tag{2.14}$$

It follows that

$$\begin{aligned}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle &\le \left(\frac{-2\overline{\gamma}t+(\overline{\gamma}t)^2}{2t}\right)\|x_t-x_n\|^2+\frac{1}{2t}f_n(t)+\langle Ax_t-Ax_n,x_t-x_n\rangle\\ &\le \left(\frac{-2+\overline{\gamma}t}{2}\right)\overline{\gamma}\|x_t-x_n\|^2+\frac{1}{2t}f_n(t)+\langle Ax_t-Ax_n,x_t-x_n\rangle\\ &\le \left(-1+\frac{\overline{\gamma}t}{2}\right)\langle Ax_t-Ax_n,x_t-x_n\rangle+\frac{1}{2t}f_n(t)+\langle Ax_t-Ax_n,x_t-x_n\rangle\\ &\le \frac{\overline{\gamma}t}{2}\langle Ax_t-Ax_n,x_t-x_n\rangle+\frac{1}{2t}f_n(t).\end{aligned}\tag{2.15}$$

Letting $n\to\infty$ in (2.15) and (2.14), we get

$$\limsup_{n\to\infty}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle\le\frac{t}{2}M_0,\tag{2.16}$$

where $M_0>0$ is a constant such that $M_0\ge\overline{\gamma}\langle Ax_t-Ax_n,x_t-x_n\rangle$ for all $t\in(0,1)$ and $n\ge 1$. Taking $t\to 0$ in (2.16), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle\le 0.\tag{2.17}$$

On the other hand, one has

$$\begin{aligned}\langle\gamma f(q)-Aq,x_n-q\rangle &= \langle\gamma f(q)-Aq,x_n-q\rangle-\langle\gamma f(q)-Aq,x_n-x_t\rangle\\ &\quad+\langle\gamma f(q)-Aq,x_n-x_t\rangle-\langle\gamma f(q)-Ax_t,x_n-x_t\rangle\\ &\quad+\langle\gamma f(q)-Ax_t,x_n-x_t\rangle-\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\\ &\quad+\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle.\\ &=\langle\gamma f(q)-Aq,x_t-q\rangle+\langle Ax_t-Aq,x_n-x_t\rangle\\ &\quad+\langle\gamma f(q)-\gamma f(x_t),x_n-x_t\rangle+\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\\ &\le\|\gamma f(q)-Aq\|\|x_t-q\|+(\|A\|\|x_t-q\|+\gamma\alpha\|x_t-q\|)\|x_n-x_t\|\\ &\quad+\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\\ &=\|\gamma f(q)-Aq\|\|x_t-q\|+(\|A\|+\gamma\alpha)\|x_t-q\|\|x_n-x_t\|\\ &\quad+\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle.\end{aligned}\tag{2.18}$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle &\le\|\gamma f(q)-Aq\|\|x_t-q\|+(\|A\|+\gamma\alpha)\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &\quad+\limsup_{n\to\infty}\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle.\end{aligned}\tag{2.19}$$

Therefore, from (2.17) and $\lim_{t\to 0}\|x_t - q\| = 0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle &\le \limsup_{t\to 0}\left(\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle\right)\\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle \le 0.\end{aligned} \tag{2.20}$$

Hence (2.12) holds. Finally, we prove that $x_n \to q$. By using (2.2) and together with the Schwarz inequality, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(q)\|^2\\
&\le \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(y_n - q)\|^2\\
&= \|(I - \alpha_n A)(y_n - q)\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle (I - \alpha_n A)(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|y_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle y_n - q, \gamma f(x_n) - Aq\rangle - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle y_n - q, \gamma f(x_n) - \gamma f(q)\rangle + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\|y_n - q\|\|\gamma f(x_n) - \gamma f(q)\| + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\gamma\alpha\alpha_n\|y_n - q\|\|x_n - q\| + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\gamma\alpha\alpha_n\|x_n - q\|^2 + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le \left((1 - \alpha_n\overline{\gamma})^2 + 2\gamma\alpha\alpha_n\right)\|x_n - q\|^2 + 2\alpha_n\langle y_n - q, \gamma f(x_n) - Aq\rangle\\
&\quad + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 + 2\alpha_n^2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\|\\
&= (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2\\
&\quad + \alpha_n\Big(2\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\qquad + \alpha_n\left(\|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\| + \overline{\gamma}^2\|x_n - q\|^2\right)\Big).\end{aligned} \tag{2.21}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{y_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\| + \overline{\gamma}^2\|x_n - q\|^2 \tag{2.22}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2 + \alpha_n\beta_n, \tag{2.23}$$

where $\beta_n = 2\langle y_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty}\langle(\gamma f - A)q, y_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\beta_n \leq 0$. By applying Lemma 1.3 to (2.23), we can conclude that $x_n \to q$. This completes the proof. □

If $A = I$ and $\gamma = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let H be a Hilbert space, C a closed convex nonempty subset of H, and let $f \in \prod_c$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, *for all* $i = 1,2,\ldots,N$ *and* $\{\gamma_{n,i}\}_{i=1}^N \subset [a,b]$, *where* $0 < a \leq b < 1$;

(C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)K_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \end{aligned} \tag{2.24}$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle(f - I)q, p - q\rangle \leq 0, \quad p \in F. \tag{2.25}$$

If $N = 1$, $A = I$, $\gamma = 1$, and $f \equiv u \in C$ is a constant in Theorem 2.1, we get the results of Kim and Xu [3].

Corollary 2.3. *Let H be a Hilbert space, C a closed convex nonempty subset of H, and let $f \in \prod_c$. Let T be a nonexpansive mapping of C into itself. $F(T) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;

(C5) $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$.

If $\{x_n\}_{n=1}^{\infty}$ *is the composite process defined by*

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (I-\alpha_n)y_n, \end{aligned} \tag{2.26}$$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $q \in F$*, which also solves the following variational inequality:*

$$\langle u-q, p-q\rangle \leq 0, \quad p \in F. \tag{2.27}$$

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