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Research Article

On the Convergence for an Iterative Method for Quasivariational Inclusions

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We introduce an iterative algorithm for finding a common element of the set of solutions of quasivariational inclusion problems and of the set of fixed points of strict pseudocontractions in the framework Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by many others.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $S : H \to H$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S.

Recall the following definitions.

(1) The mapping S is said to be *contractive* with the coefficient $\alpha \in (0,1)$ if

$$\|Sx - Sy\| \le \alpha\|x - y\|, \quad \forall x, y \in H. \tag{1.1}$$

(2) The mapping S is said to be *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in H. \tag{1.2}$$

(3) The mapping S is said to be *strictly pseudocontractive* with the coefficient $k \in [0,1)$ if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H. \tag{1.3}$$

(4) The mapping S is said to be *pseudocontractive* if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H. \tag{1.4}$$

Clearly, the class of strict pseudocontractions falls into the one between classes of nonexpansive mappings and pseudocontractions. Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. See, for example, [1–6] and the references therein.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping S on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.5}$$

where A is a linear bounded and strongly positive operator and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Marino and Xu [2] studied the following iterative scheme:

$$x_0 \in H, \qquad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.6}$$

They proved that the sequence $\{x_n\}$ generated in the above iterative scheme converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(S), \tag{1.7}$$

which is the optimality condition for the minimization problem (1.5).

Next, let $B : H \to H$ be a nonlinear mapping. Recall the following definitions.

(1) The mapping B is said to be *monotone* if for each $x, y \in H$, we have

$$\langle Bx - By, x - y \rangle \geq 0. \tag{1.8}$$

(2) B is said to be *μ-strongly monotone* if

$$\langle Bx - By, x - y \rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in H. \tag{1.9}$$

(3) The mapping B is said to be *μ-inverse-strongly monotone* if there exists a constant $\mu > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \mu\|Bx - By\|^2, \quad \forall x, y \in H. \tag{1.10}$$

(4) The mapping B is said to be *relaxed δ-cocoercive* if there exists a constant $\delta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\delta) \|Bx - By\|^2, \quad \forall x, y \in H. \tag{1.11}$$

(5) The mapping B is said to be *relaxed (δ, r)-cocoercive* if there exist two constants $\delta, r > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\delta) \|Bx - By\|^2 + r\|x - y\|^2, \quad \forall x, y \in H. \tag{1.12}$$

(6) Recall also that a set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. The monotone mapping $M : H \to 2^H$ is *maximal* if the graph of $G(M)$ of T is not properly contained in the graph of any other monotone mapping.

The so-called quasi-variational inclusion problem is to find a $u \in H$ for a given element $f \in H$ such that

$$f \in Bu + Mu, \tag{1.13}$$

where $B : H \to H$ and $M : H \to 2^H$ are two nonlinear mappings. See, for example, [7–12]. A special case of the problem (1.13) is to find an element $u \in H$ such that

$$0 \in Bu + Mu. \tag{1.14}$$

In this paper, we use $VI(H, B, M)$ to denote the solution of the problem (1.14). A number of problems arising in structural analysis, mechanics, and economic can be studied in the framework of this class of variational inclusions.

Next, we consider two special cases of the problem (1.14).

(A) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential of ϕ, then the variational inclusion problem (1.14) is equivalent to finding $u \in H$ such that

$$\langle Bu, v - u \rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{1.15}$$

which is said to be the *mixed quasi-variational inequality*. See, for example, [7, 8] for more details.

(B) If ϕ is the indicator function of C, then the variational inclusion problem (1.14) is equivalent to the classical variational inequality problem, denoted by $VI(C, B)$, to find $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.16}$$

For finding a common element of the set of fixed points of a nonexpansive mapping and of the set of solutions to the variational inequality (1.16), Iiduka and Takahashi [13] proved the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let B be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,B) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) SP_C(x_n - \lambda_n B x_n) \tag{1.17}$$

for every $n = 1,2,\ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[a,b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \qquad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \tag{1.18}$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,B)} x$.

Recently, Zhang et al. [11] considered the problem (1.14). To be more precise, they proved the following theorem.

Theorem ZLC. *Let H be a real Hilbert space, $B : H \to H$ an α-inverse-strongly monotone mapping, $M : H \to 2^H$ a maximal monotone mapping, and $S : H \to H$ a nonexpansive mapping. Suppose that the set $F(S) \cap VI(H,B,M) \neq \emptyset$, where $VI(H,B,M)$ is the set of solutions of variational inclusion* (1.14)*. Suppose that $x_0 = x \in H$ and $\{x_n\}$ is the sequence defined by*

$$\begin{aligned} x_{n+1} &= \alpha_n x_0 + (1-\alpha_n) S y_n, \\ y_n &= J_{M,\lambda}(x_n - \lambda B x_n,) \quad n \geq 0, \end{aligned} \tag{1.19}$$

where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(H,B,M)} x_0$.

In this paper, motivated by the research work going on in this direction, see, for instance, [2, 3, 7–21], we introduce an iterative method for finding a common element of the set of fixed points of a strict pseudocontraction and of the set of solutions to the problem (1.14) with multivalued maximal monotone mapping and relaxed (δ, r)-cocoercive mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

In order to prove our main results, we need the following conceptions and lemmas.

Definition 1.1 (see [11]). Let $M : H \to 2^H$ be a multivalued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$, for all $u \in H$, is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Lemma 1.2 (see [4]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that* $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$*, where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.3 (see [22]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.4 (see [11]). $u \in H$ *is a solution of variational inclusion* (1.14) *if and only if* $u = J_{M,\lambda}(u - \lambda Bu,)$ *for all* $\lambda > 0$*, that is,*

$$VI(H,B,M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{1.20}$$

Lemma 1.5 (see [11]). *The resolvent operator* $J_{M,\lambda}$ *associated with* M *is single-valued and nonexpansive for all* $\lambda > 0$.

Lemma 1.6 (see [23]). *Let* C *be a closed convex subset of a strictly convex Banach space* E*. Let* S *and* T *be two nonexpansive mappings on* C*. Suppose that* $F(T) \cap F(S)$ *is nonempty. Then a mapping* R *on* C *defined by* $Rx = aSx + (1-a)Tx$*, where* $a \in (0,1)$*, for* $x \in C$ *is well defined and nonexpansive and* $F(R) = F(T) \cap F(S)$ *holds.*

Lemma 1.7 (see [24]). *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H*, and let* $S : C \to C$ *be a nonexpansive mapping. Then* $I - S$ *is demiclosed at zero.*

Lemma 1.8 (see [25]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : C \to C$ *a* k*-strict pseudocontraction. Define* $S : C \to H$ *by* $Sx = \alpha x + (1-\alpha)Tx$ *for each* $x \in C$*. Then, as* $\alpha \in [k,1)$*,* S *is nonexpansive such that* $F(S) = F(T)$.

2. Main Results

Theorem 2.1. *Let* H *be a real Hilbert space and* $M : H \to 2^H$ *a maximal monotone mapping. Let* $B : H \to H$ *be a relaxed* (δ, r)*-cocoercive and* ν*-Lipschitz continuous mapping, and* S *a* k*-strict pseudocontraction with a fixed point. Define a mapping* $S_k : H \to H$ *by* $S_k x = kx + (1-k)Sx$*. Let* f *be a contraction of* H *into itself with the contractive coefficient* $\alpha(0 < \alpha < 1)$*, and* A *a strongly positive linear bounded self-joint operator with the coefficient* $\overline{\gamma} > 0$*. Assume that* $0 < \gamma < \overline{\gamma}/\alpha$ *and* $\Omega = F(S) \cap VI(H,B,M) \neq \emptyset$*. Let* $x_1 \in H$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{gathered} y_n = J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left[(1-\beta_n)I - \alpha_n A\right]\left[\mu S_k x_n + (1-\mu)y_n\right], \quad \forall n \geq 1, \end{gathered} \tag{2.1}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$*. Assume that* $\lambda \in (0, 2(r - \delta\nu^2)/\nu^2)$*,* $r > \delta\nu^2$*. If the control consequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions:*

(C1) $0 < a \leq \beta_n \leq b < 1$, *for all* $n \geq 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ *converges strongly to* $z \in \Omega$, *which solves uniquely the following variational inequality:*

$$\langle (A-\gamma f)z, z-x^*\rangle \leq 0, \quad \forall x^* \in \Omega. \tag{2.2}$$

Equivalently, one has $P_\Omega(I-A+\gamma f)z = z$.

Proof. The uniqueness of the solution of the variational inequality (2.2) is a consequence of the strong monotonicity of $A-\gamma f$. Suppose that $z_1 \in \Omega$ and $z_2 \in \Omega$ both are solutions to (2.2); then $\langle (A-\gamma f)z_1, z_1-z_2\rangle \leq 0$ and $\langle (A-\gamma f)z_2, z_2-z_1\rangle \leq 0$. Adding up the two inequalities, we see that

$$\langle (A-\gamma f)z_1 - (A-\gamma f)z_2, z_1-z_2\rangle \leq 0. \tag{2.3}$$

The strong monotonicity of $A-\gamma f$ (see [2, Lemma 2.3]) implies that $z_1 = z_2$ and the uniqueness is proved. Below we use z to denote the unique solution of (2.2).

Next, we show that the mapping $I-\lambda B$ is nonexpansive. Indeed, for all $x, y \in H$, one see from the condition $\lambda \in (0, 2(r-\gamma\mu^2)/\mu^2)$ that

$$\begin{aligned}
&\left\|(I-\lambda B)x-(I-\lambda B)y\right\|^2 \\
&\quad = \left\|(x-y)-\lambda(Bx-By)\right\|^2 \\
&\quad = \left\|x-y\right\|^2 - 2\lambda\langle Bx-By, x-y\rangle + \lambda^2\left\|Bx-By\right\|^2 \\
&\quad \leq \left\|x-y\right\|^2 - 2\lambda\left[(-\delta)\left\|Bx-By\right\|^2 + r\left\|x-y\right\|^2\right] + \lambda^2\nu^2\left\|x-y\right\|^2 \\
&\quad = \left(1+\lambda^2\nu^2-2\lambda r+2\lambda\delta\nu^2\right)\left\|x-y\right\|^2 \\
&\quad \leq \left\|x-y\right\|^2,
\end{aligned} \tag{2.4}$$

which implies that the mapping $I-\lambda B$ is nonexpansive. Taking $x^* \in \Omega$, we have $x^* = J_{M,\lambda}(x^* - \lambda Bx^*)$. It follows from Lemma 1.5 that

$$\left\|y_n - x^*\right\| = \|J_{M,\lambda}(x_n-\lambda Bx_n) - J_{M,\lambda}(x^*-\lambda Bx^*)\| \leq \|x_n - x^*\|. \tag{2.5}$$

Note that from the conditions (C1) and (C2), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \geq 1$. Since A is a strongly positive linear bounded self-adjoint operator, we have $\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$. Now for $x \in H$ with $\|x\| = 1$, we see that

$$\langle ((1-\beta_n)I-\alpha_n A)x, x\rangle = 1-\beta_n-\alpha_n\langle Ax, x\rangle \geq 1-\beta_n-\alpha_n\|A\| \geq 0; \tag{2.6}$$

that is, $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\left\|(1-\beta_n)I-\alpha_n A\right\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)x,x\rangle : x\in C, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in C, \|x\|=1\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned} \tag{2.7}$$

Set $t_n=\mu S_k x_n+(1-\mu)y_n$. From Lemma 1.8, we see that S_k is nonexpansive. It follows from (2.5) that

$$\|t_n-x^*\|\le\mu\|S_k x_n-x^*\|+(1-\mu)\left\|y_n-x^*\right\|\le\|x_n-x^*\|. \tag{2.8}$$

From (2.7) and (2.8), we arrive at

$$\begin{aligned}&\|x_{n+1}-x^*\|\\ &\quad=\left\|\alpha_n\gamma f(x_n)+\beta_n x_n+\left[(1-\beta_n)I-\alpha_n A\right]t_n-x^*\right\|\\ &\quad\le\alpha_n\left\|\gamma f(x_n)-Ax^*\right\|+\beta_n\|x_n-x^*\|+\left\|(1-\beta_n)I-\alpha_n A\right\|\|t_n-x^*\|\\ &\quad\le\alpha_n\left\|\gamma f(x_n)-Ax^*\right\|+\beta_n\|x_n-x^*\|+(1-\beta_n-\alpha_n\overline{\gamma})\|t_n-x^*\|\\ &\quad\le\alpha_n\left\|\gamma f(x_n)-\gamma f(x^*)\right\|+\alpha_n\left\|\gamma f(x^*)-Ax^*\right\|+\beta_n\|x_n-x^*\|\\ &\qquad+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\\ &\quad\le\alpha\alpha_n\gamma\|x_n-x^*\|+\alpha_n\left\|\gamma f(x^*)-Ax^*\right\|+\beta_n\|x_n-x^*\|\\ &\qquad+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\\ &\quad=\left[1-\alpha_n(\overline{\gamma}-\alpha\gamma)\right]\|x_n-x^*\|+\alpha_n\left\|\gamma f(x^*)-Ax^*\right\|.\end{aligned} \tag{2.9}$$

By simple inductions, one obtains that $\|x_n-x^*\|\le\max\{\|x_1-x^*\|,\|\gamma f(x^*)-Ax^*\|/\overline{\gamma}-\alpha\gamma\}$, which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{t_n\}$.

On the other hand, we see from the nonexpansivity of the mappings $J_{M,\lambda}$ that

$$\left\|y_{n+1}-y_n\right\|=\|J_{M,\lambda}(x_{n+1}-\lambda Bx_{n+1})-J_{M,\lambda}(x_n-\lambda Bx_n)\|\le\|x_{n+1}-x_n\|. \tag{2.10}$$

It follows that

$$\begin{aligned}\|t_{n+1}-t_n\| &= \left\|\mu S_k x_{n+1}+(1-\mu)y_{n+1}-\left[\mu S_k x_n+(1-\mu)y_n\right]\right\|\\ &\le\mu\|S_k x_{n+1}-S_k x_n\|+(1-\mu)\left\|y_{n+1}-y_n\right\|\\ &\le\|x_{n+1}-x_n\|.\end{aligned} \tag{2.11}$$

Setting

$$x_{n+1}=(1-\beta_n)e_n+\beta_n x_n,\quad\forall n\ge1, \tag{2.12}$$

we see that

$$
\begin{aligned}
e_{n+1}-e_n &\\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1})+\left[(1-\beta_{n+1})I-\alpha_{n+1}A\right]t_n}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+\left[(1-\beta_n)I-\alpha_nA\right]t_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\gamma f(x_{n+1})-At_n\right]+t_{n+1}-\frac{\alpha_n}{1-\beta_n}\left[\gamma f(x_n)-At_n\right]-t_n.
\end{aligned}
\tag{2.13}
$$

It follows that

$$
\|e_{n+1}-e_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f(x_{n+1})-At_n\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f(x_n)-At_n\right\|+\|t_{n+1}-t_n\|, \tag{2.14}
$$

which combines with (2.11) yields that

$$
\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f(x_{n+1})-At_n\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f(x_n)-At_n\right\|. \tag{2.15}
$$

It follows from the conditions (C1) and (C2) that $\limsup_{n\to\infty}(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|)\le 0$. Hence, from Lemma 1.3, one obtains $\lim_{n\to\infty}\|e_n-x_n\|=0$. From (2.12), one has $\|x_{n+1}-x_n\|=(1-\beta_n)\|e_n-x_n\|$. Thanks to the condition (C1), we see that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.16}
$$

On the other hand, we have

$$
\begin{aligned}
x_{n+1}-x_n &= \alpha_n\gamma f(x_n)+\beta_n x_n+\left[(1-\beta_n)I-\alpha_nA\right]t_n-x_n\\
&= \alpha_n\left(\gamma f(x_n)-At_n\right)+(1-\beta_n)(t_n-x_n).
\end{aligned}
\tag{2.17}
$$

It follows that

$$
(1-\beta_n)\|t_n-x_n\| \le \|x_{n+1}-x_n\|+\alpha_n\left\|\gamma f(x_n)-At_n\right\|. \tag{2.18}
$$

From the conditions (C1) and (C2) and (2.16), we see that

$$
\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{2.19}
$$

Next, we prove that $\limsup_{n\to\infty}\langle(\gamma f-A)z,x_n-z\rangle\le 0$, where $z=P_\Omega[I-(A-\gamma f)]z$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty}\left\langle(\gamma f-A)z,x_n-z\right\rangle=\lim_{i\to\infty}\left\langle(\gamma f-A)z,x_{n_i}-z\right\rangle. \tag{2.20}
$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Next, we show that $w \in F(S) \cap VI(H, M, B)$. Define a mapping D by

$$Dx = \mu S_k x + (1-\mu) J_{M,\lambda}(I - \lambda B), \quad \forall x \in H. \tag{2.21}$$

In view of Lemma 1.6, we see that D is nonexpansive such that

$$F(D) = F(S_k) \cap F(J_{M,\lambda}(I - \lambda B)) = F(S) \cap VI(H, B, M). \tag{2.22}$$

From (2.19), we obtain $\lim_{n\to\infty}\|Dx_{n_i} - x_{n_i}\| = 0$. It follows from Lemma 1.7 that $w \in F(D)$. That is, $w \in F(S) \cap VI(H, M, B)$. Thanks to (2.20), we arrive at

$$\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - A)z, x_{n_i} - z\rangle = \langle(\gamma f - A)z, w - z\rangle \le 0. \tag{2.23}$$

Finally, we show that $x_n \to z$, as $n \to \infty$. Indeed, we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad= \langle \alpha_n \gamma f(x_n) + \beta_n x_n + \left[(1-\beta_n)I - \alpha_n A\right] t_n - z, x_{n+1} - z\rangle \\
&\quad= \alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\qquad+ \langle\left[(1-\beta_n)I - \alpha_n A\right](t_n - z), x_{n+1} - z\rangle \\
&\quad\le \alpha_n\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\qquad+ \beta_n\|x_n - z\|\|x_{n+1} - z\| + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|x_n - z\|\|x_{n+1} - z\| \\
&\quad\le \frac{\gamma\alpha}{2}\alpha_n\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\qquad+ \left(1 - \alpha_n\overline{\gamma}\right)\|x_n - z\|\|x_{n+1} - z\| \\
&\quad\le \frac{\gamma\alpha}{2}\alpha_n\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\qquad+ \frac{(1 - \alpha_n\overline{\gamma})}{2}\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) \\
&\quad= \frac{1 - \alpha_n(\overline{\gamma} - \alpha\gamma)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle,
\end{aligned} \tag{2.24}$$

which implies that

$$\|x_{n+1} - z\|^2 \le \left[1 - \alpha_n(\overline{\gamma} - \alpha\gamma)\right]\|x_n - z\|^2 + 2\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle. \tag{2.25}$$

From the condition (C2), (2.23), and using Lemma 1.2, we see that $\lim_{n\to\infty}\|x_n - z\| = 0$. This completes the proof. □

Letting $\gamma = 1$ and $A = I$, the identity mapping, we can obtain from Theorem 2.1 the following result immediately.

Corollary 2.2. *Let H be a real Hilbert space and $M : H \to 2^H$ a maximal monotone mapping. Let $B : H \to H$ be a relaxed (δ, r)-cocoercive and ν-Lipschitz continuous mapping, and S a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : H \to H$ by $S_k x = kx + (1-k)Sx$. Let f be a contraction of H into itself with the contractive coefficient $\alpha(0 < \alpha < 1)$. Assume that $\Omega = F(S) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)\left[\mu S_k x_n + (1-\mu) y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{2.26}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that $\lambda \in (0, 2(r - \delta\nu^2)/\nu^2)$, $r > \delta\nu^2$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions:

(C1) $0 < a \leq \beta_n \leq b < 1$, *for all* $n \geq 1$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $z \in \Omega$.

Remark 2.3. Corollary 2.2 improves Theorem 2.1 of Zhang et al. [11] in the following sense:

(1) from nonexpansive mappings to strict pseudocontractions;

(2) the analysis technique used in this paper is different from [11]'s: the proof is also more concise than [11]'s;

(3) the restriction imposed on the parameter $\{\alpha_n\}$ is relaxed.

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