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Research Article

Strong Convergence Theorem for Equilibrium Problems and Fixed Points of a Nonspreading Mapping in Hilbert Spaces

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We introduce an iterative method for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonspreading mapping in a Hilbert space. Then, we prove a strong convergence theorem which is connected with the work of S. Takahashi and W. Takahashi (2007) and Iemoto and Takahashi (2009).

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let C be a closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solution of (1.1) is denoted by $\mathrm{EP}(F)$. Given a mapping $A : C \to H$, let $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Az, y - z \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1); see, for example, [1–9] and the references therein.

A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping F is said to be *firmly nonexpansive* if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$ for all $x, y \in C$. Let E be a smooth, strictly convex and reflexive Banach space, and let J be the duality mapping of E and C a nonempty closed convex subset of E. A mapping $S : C \to C$ is said to be *nonspreading* if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x) \tag{1.2}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$; see, for instance, Kohsaka and Takahashi [10]. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. Then a nonspreading mapping $S : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2 \tag{1.3}$$

for all $x, y \in C$. Let $F(Q)$ be the set of fixed points of Q, and $F(Q)$ nonempty; a mapping $Q : C \to C$ is said to be *quasi-nonexpansive* if $\|Qx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(Q)$.

Remark 1.1. In a Hilbert space, we know that every firmly nonexpansive mapping is nonspreading and that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [10, 11].

In 1953, Mann [12] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. In an infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence (see [12, 13]). Fourteen years later, Halpern [14] introduced the following iterative scheme for approximating a fixed point of T:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \tag{1.5}$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0, 1]$. Strong convergence of this type iterative sequence has been widely studied: Wittmann [15] discussed such a sequence in a Hilbert space.

On the other hand, Kohsaka and Takahashi [10] proved an existence theorem of fixed point for nonspreading mappings in a Banach space. Recently, Lemoto and Takahashi [16] studied the approximation theorem of common fixed points for a nonexpansive mapping T of C into itself and a nonspreading mapping S of C into itself in a Hilbert space. In particular, this result reduces to approximation fixed points of a nonspreading mapping S of C into itself in a Hilbert space by using iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n. \tag{1.6}$$

Some methods have been proposed to solve the equilibrium problem and fixed point problem of nonexpansive mapping: see, for instance, [1, 2, 6, 7, 17–20] and the references therein. In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. Recently, S. Takahashi and W. Takahashi [8] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. In 2008, Plubtieng and Punpaeng [7] introduced a new iterative sequence for finding a common element of the set of solution of equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space which is the optimality condition for the minimization problem. Very recently, S. Takahashi and W. Takahashi [9] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtain that the sequence converges strongly to a common element of two sets.

In this paper, motivated by S. Takahashi and W. Takahashi [8] and Lemoto and Takahashi [16], we introduce an iterative sequence and prove a strong convergence theorem for finding solution of equilibrium problems and the set of fixed points of a nonspreading mapping in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ implies that x_n converges weakly to x and $x_n \to x$ means the strong convergence. Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C; denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \Longleftrightarrow \langle x - z, z - y \rangle \geq 0 \quad \forall y \in C. \tag{2.2}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered} \langle x - P_C x, y - P_C y \rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \end{gathered} \tag{2.3}$$

for all $x \in H$, $y \in C$. We also know that H satisfies Opial's condition [21], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H$ with $x \neq y$; see [21, 22] for more details.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (see [23]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, one has*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2. \quad (2.5)$$

Lemma 2.2 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then the following are equivalent.*

(1) *There exists $x \in C$ such that $\{S^n x\}$ is bounded;*

(2) *$F(S)$ is nonempty.*

Lemma 2.3 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

Lemma 2.4. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;

(2) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.5 (see [24]). *Let $\{a_n\}, \{b_n\} \subset [0,\infty)$, and let $\{c_n\} \subset [0,1)$ be sequences of real numbers such that*

$a_{n+1} \le (1-c_n)a_n + b_n$, *for all* $n \in \mathbb{N}$,
$\sum_{n=1}^{\infty} c_n = \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [16]). *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \to C$ a nonspreading mapping with $F(S) \ne \emptyset$. Then S is demiclosed, that is, $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.*

Lemma 2.7 (see [16]). *Let H be a Hilbert space, C a nonempty closed convex subset of a real Hilbert space H, and let S be a nonspreading mapping of C into itself, and let $A = I - S$. Then*

$$\|Ax-Ay\|^2 \le \langle x-y, Ax-Ay\rangle + \frac{1}{2}\left(\|Ax\|^2+\|Ay\|^2\right). \quad (2.6)$$

Lemma 2.8 (see [25]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \quad n \ge 0, \quad (2.7)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [26].

Lemma 2.9 (see [26]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C. \tag{2.8}$$

The following lemma was also given in [4].

Lemma 2.10 (see [4]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{2.9}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.11 (see [27]). *Let (Γ_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $(\Gamma_{n_j})_{j \geq 0}$ of (Γ_n) which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $(\tau(n))_{n \geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}. \tag{2.10}$$

Then $(\tau(n))_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \tau(n) = \infty$, and the following properties are satisfied for all $n \geq n_0$:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \qquad \Gamma_n \leq \Gamma_{\tau(n)+1}. \tag{2.11}$$

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of fixed points of a nonspreading mapping and the set of solutions of the equilibrium problems.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), and let S be a nonspreading mapping of C into itself such that $F(S) \cap EP(F) \neq \emptyset$. Let $u \in C$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n)u_n], \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\{r_n\} \in (0,\infty)$ satisfy

$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < a \leq \beta_n \leq b < 1$,
$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$,
$\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}u$.

Proof. Let $p \in F(S) \cap \mathrm{EP}(F)$. From $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\| \tag{3.2}$$

for all $n \in \mathbb{N}$. Put $y_n = \alpha_n u + (1-\alpha_n)u_n$. We divide the proof into several steps.

Step 1. We claim that the sequences $\{x_n\}, \{u_n\}, \{y_n\}$, and $\{Sy_n\}$ are bounded. First, we note that

$$\begin{aligned}\|Sy_n - p\| &\leq \|y_n - p\| \\ &= \|\alpha_n u + (1-\alpha_n)u_n - p\| \\ &\leq \alpha_n\|u - p\| + (1-\alpha_n)\|u_n - p\| \\ &\leq \alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\|,\end{aligned} \tag{3.3}$$

and so

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n)Sy_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|Sy_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &= \beta_n\|x_n - p\| + (1-\beta_n)\|\alpha_n u + (1-\alpha_n)u_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(\alpha_n\|u - p\| + (1-\alpha_n)\|u_n - p\|) \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(\alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\|) \\ &= (1-\alpha_n(1-\beta_n))\|x_n - p\| + \alpha_n(1-\beta_n)\|u - p\|.\end{aligned} \tag{3.4}$$

Putting $M = \max\{\|x_n - p\|, \|u - p\|\}$, we note that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - p\| \leq M$. Assume that $\|x_k - p\| \leq M$ for all $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
\|x_{k+1} - p\| &\leq (1 - \alpha_k(1 - \beta_k))\|x_k - p\| + \alpha_k(1 - \beta_k)\|u - p\| \\
&\leq (1 - \alpha_k(1 - \beta_k))M + \alpha_k(1 - \beta_k)M \\
&= M.
\end{aligned} \tag{3.5}$$

By induction, we obtain that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. So, $\{x_n\}$ is bound. Hence, $\{u_n\}, \{y_n\}$, and $\{Sy_n\}$ are also bounded.

Step 2. Put $t_n = \beta_n y_n + (1 - \beta_n)Sy_n$. We claim that $\|x_{n+1} - t_n\| \to 0$ as $n \to \infty$. We note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\beta_n x_n + (1 - \beta_n)Sy_n) - (\beta_{n-1}x_{n-1} + (1 - \beta_{n-1})Sy_{n-1})\| \\
&= \|\beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} - \beta_{n-1}x_{n-1} + (1 - \beta_n)Sy_n - (1 - \beta_n)Sy_{n-1} \\
&\quad +(1 - \beta_n)Sy_{n-1} - (1 - \beta_{n-1})Sy_{n-1}\| \\
&\leq \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)\|Sy_n - Sy_{n-1}\| \\
&\quad + |(1 - \beta_n) - (1 - \beta_{n-1})|\|Sy_{n-1}\| \\
&\leq \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)\|y_n - y_{n-1}\| + |\beta_{n-1} - \beta_n|\|Sy_{n-1}\| \\
&= \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n) \\
&\quad \times \|\alpha_n u + (1 - \alpha_n)u_n - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| + |\beta_n - \beta_{n-1}|\|Sy_{n-1}\| \\
&\leq \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n) \\
&\quad \times [\|\alpha_n u - \alpha_{n-1}u\| + \|(1 - \alpha_n)u_n - (1 - \alpha_{n-1})u_{n-1}\|] + |\beta_n - \beta_{n-1}|\|Sy_{n-1}\| \\
&= \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|\|u\| \\
&\quad + (1 - \beta_n)\|(1 - \alpha_n)u_n - (1 - \alpha_n)u_{n-1} + (1 - \alpha_n)u_{n-1} - (1 - \alpha_{n-1})u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|Sy_{n-1}\| \\
&\leq \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|\|u\| \\
&\quad + (1 - \beta_n)(1 - \alpha_n)\|u_n - u_{n-1}\| + (1 - \beta_n)|(1 - \alpha_n) - (1 - \alpha_{n-1})|\|u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|Sy_{n-1}\| \\
&= \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|\|u\| \\
&\quad + (1 - \beta_n)(1 - \alpha_n)\|u_n - u_{n-1}\| + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|Sy_{n-1}\| \\
&= \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|K_1 + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1 - \beta_n)(1 - \alpha_n)\|u_n - u_{n-1}\| + (1 - \beta_n)|\alpha_n - \alpha_{n-1}|K_1 + |\beta_n - \beta_{n-1}|K_1,
\end{aligned} \tag{3.6}$$

where $K_1 = \sup\{\|x_n\| + \|Sy_n\| + \|u\| + \|u_{n-1}\| : n \in \mathbb{N}\}$. On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \tag{3.7}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \tag{3.8}$$

for all $y \in C$. Putting $y = u_{n+1}$ in (3.7) and $y = u_n$ in (3.8), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0, \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0. \end{aligned} \tag{3.9}$$

So, from (A2), we note that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.10}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.11}$$

Without loss of generality, let us assume that there exists a real number d such that $r_n > d > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.12}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| L, \end{aligned} \tag{3.13}$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. So, from (3.6), we note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \beta_n\|x_n - x_{n-1}\| + 2\left|\beta_n - \beta_{n-1}\right|K_1 + 2(1-\beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1-\beta_n)(1-\alpha_n)\left(\|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}|L\right) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - x_{n-1}\| + 2\left|\beta_n - \beta_{n-1}\right|K_1 + 2(1-\beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1-\beta_n)(1-\alpha_n)\frac{1}{d}|r_n - r_{n-1}|L \\
&= (1 - (1-\beta_n)\alpha_n)\|x_n - x_{n-1}\| + 2\left|\beta_n - \beta_{n-1}\right|K_1 + 2(1-\beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1-\beta_n)(1-\alpha_n)\frac{L}{d}|r_n - r_{n-1}|.
\end{aligned} \tag{3.14}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 \tag{3.15}$$

for $p \in F(S) \cup \mathrm{EP}(F)$. We note from $u_n = T_{r_n}x_n$ that

$$\begin{aligned}
\left\|u_n - p\right\|^2 &= \left\|T_{r_n}x_n - T_{r_n}p\right\|^2 \le \left\langle T_{r_n}x_n - T_{r_n}p, x_n - p\right\rangle = \left\langle u_n - p, x_n - p\right\rangle \\
&= \frac{1}{2}\left(\left\|u_n - p\right\|^2 + \left\|x_n - p\right\|^2 - \|x_n - u_n\|^2\right),
\end{aligned} \tag{3.16}$$

and hence

$$\left\|u_n - p\right\|^2 \le \left\|x_n - p\right\|^2 - \|x_n - u_n\|^2. \tag{3.17}$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &= \left\|\beta_n x_n + (1-\beta_n)Sy_n - p\right\|^2 \\
&\le \beta_n\left\|x_n - p\right\|^2 + (1-\beta_n)\left\|Sy_n - p\right\|^2 \\
&\le \beta_n\left\|x_n - p\right\|^2 + (1-\beta_n)\left\|y_n - p\right\|^2 \\
&= \beta_n\left\|x_n - p\right\|^2 + (1-\beta_n)\left\|\alpha_n u + (1-\alpha_n)u_n - p\right\|^2 \\
&\le \beta_n\left\|x_n - p\right\|^2 + \alpha_n(1-\beta_n)\left\|u - p\right\|^2 + (1-\beta_n)(1-\alpha_n)\left\|u_n - p\right\|^2 \\
&\le \beta_n\left\|x_n - p\right\|^2 + \alpha_n(1-\beta_n)\left\|u - p\right\|^2 + (1-\beta_n)(1-\alpha_n)\left(\left\|x_n - p\right\|^2 - \|x_n - u_n\|^2\right) \\
&= (1 - (1-\beta_n)\alpha_n)\left\|x_n - p\right\|^2 + \alpha_n(1-\beta_n)\left\|u - p\right\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - u_n\|^2,
\end{aligned} \tag{3.18}$$

and hence

$$\begin{aligned}
(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2 &\le \alpha_n(1-\beta_n)\|u-p\|^2-\alpha_n(1-\beta_n)\|x_n-p\|^2 \\
&\quad + \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&= \alpha_n(1-\beta_n)\|u-p\|^2-\alpha_n(1-\beta_n)\|x_n-p\|^2 \\
&\quad + \left(\|x_n-p\|-\|x_{n+1}-p\|\right)\left(\|x_n-p\|+\|x_{n+1}-p\|\right) \\
&\le \alpha_n(1-\beta_n)\|u-p\|^2-\alpha_n(1-\beta_n)\|x_n-p\|^2 \\
&\quad + \|x_n-x_{n+1}\|\left(\|x_n-p\|+\|x_{n+1}-p\|\right).
\end{aligned} \tag{3.19}$$

So, we have $\|x_n-u_n\| \to 0$. Indeed, since $y_n=\alpha_n u+(1-\alpha_n)u_n$, it follows that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-y_n\| &= \lim_{n\to\infty}\|x_n-(\alpha_n u+(1-\alpha_n)u_n)\| \\
&= \lim_{n\to\infty}\|(\alpha_n+(1-\alpha_n))x_n-(\alpha_n u+(1-\alpha_n)u_n)\| \\
&\le \lim_{n\to\infty}[\alpha_n\|x_n-u\|+(1-\alpha_n)\|x_n-u_n\|] \\
&= \lim_{n\to\infty}\alpha_n\|x_n-u\|+\lim_{n\to\infty}(1-\alpha_n)\|x_n-u_n\| \\
&= 0.
\end{aligned} \tag{3.20}$$

Then, we note that

$$\begin{aligned}
\|x_{n+1}-t_n\| &= \|(\beta_n x_n+(1-\beta_n)Sy_n)-(\beta_n y_n+(1-\beta_n)Sy_n)\| \\
&= \|\beta_n(x_n-y_n)+(1-\beta_n)(Sy_n-Sy_n)\| \\
&= \beta_n\|x_n-y_n\|.
\end{aligned} \tag{3.21}$$

Since, $0<a\le\beta_n\le b<1$ and $\|x_n-y_n\| \to 0$, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-t_n\|=0. \tag{3.22}$$

Step 3. Put $A=I-S$. From $Ap=0$, it follows by Lemma 2.7 that

$$\begin{aligned}
\|t_n-p\|^2 &= \|(\beta_n y_n+(1-\beta_n)Sy_n)-p\|^2 \\
&= \|(y_n-p)-(1-\beta_n)(y_n-Sy_n)\|^2 \\
&= \|(y_n-p)-(1-\beta_n)Ay_n\|^2 \\
&= \|(y_n-p)\|^2-2(1-\beta_n)\langle y_n-p, Ay_n-Ap\rangle+(1-\beta_n)^2\|Ay_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|(y_n-p)\|^2 - 2(1-\beta_n)\left\{\|Ay_n - Ap\|^2 - \frac{1}{2}\left(\|Ay_n\|^2 + \|Ap\|^2\right)\right\} \\
&\quad + (1-\beta_n)^2\|Ay_n\|^2 \\
&= \|\alpha_n(u-p) + (1-\alpha_n)(u_n-p)\|^2 - 2(1-\beta_n)\|Ay_n\|^2 \\
&\quad + (1-\beta_n)\|Ay_n\|^2 + (1-\beta_n)^2\|Ay_n\|^2 \\
&\leq \alpha_n\|(u-p)\|^2 + (1-\alpha_n)\|(u_n-p)\|^2 - \beta_n(1-\beta_n)\|Ay_n\|^2 \\
&\leq \alpha_n\|(u-p)\|^2 + (1-\alpha_n)\|(x_n-p)\|^2 - \beta_n(1-\beta_n)\|Ay_n\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \|x_n-p\|^2 - \beta_n(1-\beta_n)\|Ay_n\|^2.
\end{aligned}
\tag{3.23}
$$

Since $0 < a \leq \beta_n \leq b < 1$, we have $\beta_n(1-\beta_n) \geq a(1-b) := K_2$. Therefore, by (3.23), we obtain

$$
\begin{aligned}
K_2\|y_n - Sy_n\|^2 &= K_2\|Ay_n\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \|x_n-p\|^2 - \|t_n-p\|^2 \\
&\leq \alpha_n M^2 + \|x_n-p\|^2 - \|t_n-p\|^2 \\
&= \alpha_n M^2 + \|x_n-p\|^2 - \|(t_n - x_{n+1}) + (x_{n+1}-p)\|^2 \\
&= \alpha_n M^2 + \|x_n-p\|^2 - \|t_n - x_{n+1}\|^2 - 2\langle t_n - x_{n+1}, x_{n+1}-p\rangle - \|x_{n+1}-p\|^2 \\
&\leq \alpha_n M^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 - 2\langle t_n - x_{n+1}, x_{n+1}-p\rangle.
\end{aligned}
\tag{3.24}
$$

Step 4. Putting $z = P_{F(S)\cap \mathrm{EP}(F)}u$, we claim that the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)\cap \mathrm{EP}(F)}u$. Indeed, we discuss two possible cases.

Case 1. Assume that there exists n_0 such that the sequence $\{\|x_n - p\|\}$ is a nonincreasing sequence for all $n \geq n_0$. Then we have $\|x_{n+1} - p\| \leq \|x_n - p\|$ (for $n \geq n_0$), and hence $\lim_{n\to\infty}\|x_n - p\|$ exists. Therefore

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|x_{n+1} - p\|. \tag{3.25}$$

By (3.22), (3.24), and (3.25), we get

$$\|y_n - Sy_n\| \longrightarrow 0. \tag{3.26}$$

Let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, y_n - z\rangle = \lim_{n\to\infty}\langle u - z, y_{n_i} - z\rangle. \tag{3.27}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$. Since C is closed and convex, we note that C is weakly closed. So, we have $w \in C$. Since $\|Sy_n - y_n\| \to 0$, it follows by Lemma 2.6 that $w \in F(S)$. From (3.27) and the property of metric projection, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle u - z, y_n - z\rangle &= \lim_{n\to\infty} \langle u - z, y_{n_i} - z\rangle \\ &= \langle u - z, w - z\rangle \le 0. \end{aligned} \tag{3.28}$$

Finally, we prove that $x_n \to z$. In fact, since $y_n - z = \alpha_n(u - z) + (1 - \alpha_n)(u_n - z)$, it follows that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\| (\beta_n x_n + (1 - \beta_n)Sy_n) - z \right\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1 - \beta_n)\left\|Sy_n - z\right\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1 - \beta_n)\left\|y_n - z\right\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1 - \beta_n)\left[(1 - \alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle\right] \\ &\le \beta_n \|x_n - z\|^2 + (1 - \beta_n)(1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n(1 - \beta_n)\langle u - z, y_n - z\rangle \\ &= (1 - \alpha_n(1 - \beta_n))\|x_n - z\|^2 + 2\alpha_n(1 - \beta_n)\langle u - z, y_n - z\rangle. \end{aligned} \tag{3.29}$$

By (3.28) and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we immediately deduce by Lemma 2.8 that $x_n \to z$.

Case 2. Assume that for all $n \in \mathbb{N}$, there exits $m \ge n$ such that $\|x_m - p\| < \|x_{m+1} - p\|$. Put $a_m =: \|x_m - p\|$ for all $m \in \mathbb{N}$. Thus, it follows that there exists a subsequence $(a_{n_k})_{k\ge1}$ of $(a_n)_{n\ge1}$ such that $a_{n_k} < a_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let $\varphi : \mathbb{N}_1 \to \mathbb{N}$ be a mapping defined by

$$\varphi(n) = \max\{k \le n : a_k \le a_{k+1}\}, \tag{3.30}$$

where $\mathbb{N}_1 = \{n \in \mathbb{N} : n \ge n_1\}$. By Lemma 2.11, we note that $\varphi(n)$ is a nondecreasing sequence such that $\varphi(n) \to \infty$ as $n \to \infty$ and that the following properties are satisfied by all numbers $n \ge n_1$:

$$a_{\varphi(n)} \le a_{\varphi(n)+1}, \qquad a_n \le a_{\varphi(n)+1}. \tag{3.31}$$

From (3.24), we have

$$\begin{aligned} K_2\left\|y_{\varphi(n)} - Sy_{\varphi(n)}\right\|^2 &\le \alpha_{\varphi(n)}M^2 + \left\|x_{\varphi(n)} - p\right\|^2 - \left\|x_{\varphi(n)+1} - p\right\|^2 \\ &\quad - 2\langle t_{\varphi(n)} - x_{\varphi(n)+1}, x_{\varphi(n)+1} - p\rangle \\ &\le \alpha_{\varphi(n)}M^2 - 2\langle t_{\varphi(n)} - x_{\varphi(n)+1}, x_{\varphi(n)+1} - p\rangle. \end{aligned} \tag{3.32}$$

This implies that

$$\|y_{\varphi(n)} - Sy_{\varphi(n)}\| \longrightarrow 0. \tag{3.33}$$

Take a subsequence $\{y_{\varphi(n)_i}\}$ of $\{y_{\varphi(n)}\}$ such that

$$\limsup_{n\to\infty}\langle u - z, y_{\varphi(n)} - z\rangle = \lim_{n\to\infty}\langle u - z, y_{\varphi(n)_i} - z\rangle. \tag{3.34}$$

From the boundedness of $\{y_{\varphi(n)_i}\}$, we can assume that $y_{\varphi(n)_i} \rightharpoonup v$. Since C is closed and convex, it follows that C is weakly closed. So, we have $v \in C$. Since $\|Sy_{\varphi(n)} - y_{\varphi(n)}\| \to 0$, it follows by Lemma 2.6 that $v \in F(S)$. From (3.34) and the property of metric projection, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, y_{\varphi(n)} - z\rangle &= \lim_{n\to\infty}\langle u - z, y_{\varphi(n)_i} - z\rangle \\ &= \langle u - z, v - z\rangle \\ &\le 0.\end{aligned} \tag{3.35}$$

By the same argument as (3.29) in Case 1, we conclude immediately that, for all $n \ge 1$,

$$\begin{aligned}0 &\le \|x_{\varphi(n)+1} - z\|^2 - \|x_{\varphi(n)} - z\|^2 \\ &\le \beta_{\varphi(n)}\|x_{\varphi(n)} - z\|^2 + (1-\beta_{\varphi(n)})\|Sy_{\varphi(n)} - z\|^2 - \|x_{\varphi(n)} - z\|^2 \\ &\le \beta_{\varphi(n)}\|x_{\varphi(n)} - z\|^2 + (1-\beta_{\varphi(n)})\|y_{\varphi(n)} - z\|^2 - \|x_{\varphi(n)} - z\|^2 \\ &\le \beta_{\varphi(n)}\|x_{\varphi(n)} - z\|^2 + (1-\beta_{\varphi(n)}) \\ &\quad\times\left[(1-\alpha_{\varphi(n)})^2\|u_{\varphi(n)} - z\|^2 + 2\alpha_{\varphi(n)}\langle u - z, y_{\varphi(n)} - z\rangle\right] - \|x_{\varphi(n)} - z\|^2 \\ &\le \beta_{\varphi(n)}\|x_{\varphi(n)} - z\|^2 + (1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})\|u_{\varphi(n)} - z\|^2 \\ &\quad + 2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})\langle u - z, y_{\varphi(n)} - z\rangle - \|x_{\varphi(n)} - z\|^2 \\ &\le \beta_{\varphi(n)}\|x_{\varphi(n)} - z\|^2 + (1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})\|x_{\varphi(n)} - z\|^2 \\ &\quad + 2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})\langle u - z, y_{\varphi(n)} - z\rangle - \|x_{\varphi(n)} - z\|^2 \\ &= \alpha_{\varphi(n)}(1-\beta_{\varphi(n)})\left[2\langle u - z, y_{\varphi(n)} - z\rangle - \|x_{\varphi(n)} - z\|^2\right] \\ &\le 2\langle u - z, y_{\varphi(n)} - z\rangle - \|x_{\varphi(n)} - z\|^2,\end{aligned} \tag{3.36}$$

which implies that

$$\|x_{\varphi(n)} - z\|^2 \le 2\langle u - z, y_{\varphi(n)} - z\rangle. \tag{3.37}$$

By (3.35), we have

$$\lim_{n\to\infty} \left\| x_{\varphi(n)} - z \right\| = 0, \tag{3.38}$$

and hence

$$\lim_{n\to\infty} \left\| x_{\varphi(n)+1} - z \right\| = \lim_{n\to\infty} \left\| x_{\varphi(n)} - z \right\| = 0. \tag{3.39}$$

Since $\|x_n - z\| = a_n \leq a_{\varphi(n)} = \|x_{\varphi(n)} - z\|$ for all $n \geq n_1$, we have

$$\lim_{n\to\infty} \|x_n - z\| = 0. \tag{3.40}$$

This completes the proof. □

As direct consequences of Theorem 3.1, we obtain corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), and let S be a firmly nonexpansive mapping of C into itself such that $F(S) \cap EP(F) \neq \emptyset$. Let $u \in C$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) u_n], \end{gathered} \tag{3.41}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\{r_n\} \in (0,\infty)$ satisfy

$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < a \leq \beta_n \leq b < 1$,
$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$,
$\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}u$.

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